Sigma J Eng Nat Sci, Vol. 42, No. 1, pp. 245−251, February, 2024

Sigma Journal of Engineering and Natural Sciences
Web page info: https://sigma.yildiz.edu.tr
DOI: 10.14744/sigma.2024.00020

Research Article

Solutions of linear Fredholm integral equations with the three-step iteration method

Lale CONA[1,*], **Kadir ŞENGÜL**[1]

[1]*Department of Mathematical Engineering, Gumushane University, Gumushane, 29000, Türkiye*

ARTICLE INFO

Article history
Received: 20 January 2022
Revised: 10 April 2022
Accepted: 28 May 2022

Keywords:
Banach Fixed Point Theorem; Data Dependency; Fredholm Integral Equations; Three-Step Iteration Method

ABSTRACT

In this article, the solution of the second type of nonhomogeneous linear Fredholm integral equations is investigated using a three-step iteration algorithm. It has been shown that the sequences obtained from this algorithm converge to the solution of the mentioned equations. Morever, data dependency is obtained for the second type of nonhomogeneous linear Fredholm integral equations. This result is supported by an example.

Cite this article as: Cona L, Şengül K. Solutions of linear Fredholm integral equations with the three-step iteration method. Sigma J Eng Nat Sci 2024;42(1):245−251.

INTRODUCTION

Fixed point theory is a research subject in many branches of science such as mathematics, physics, chemistry, biology, engineering, economics etc. The origin of the fixed point theory goes back to the Liouville, Cauchy, Lipschitz, Peano and Picard approximation methods, which were used extensively especially in the theory of differential equations towards the end of the 19th century [5,6,11,13]. In 1922, Stephan Banach introduced the Banach fixed point theorem, which proved the existence and uniqueness of the fixed point under various conditions. One of the important results he obtained is that the sequence obtained with the Picard iteration converges to the fixed point [9]. Banach fixed point theorem, especially in the second half of the 20th century, made it possible to define many fixed point theorems with iteration models developed by many researchers by taking different spaces and mapping classes. In time, these developments caused the subject to settle in the literature as fixed point theory [4,6,10,18,19,25,41].

In parallel with the studies on fixed point theory, many contraction maps have been defined depending on the problem and space studied. For example; Lipschitzian map, contraction map, contraction-like map, non-expanding map, pseudo contraction, quasi contraction map, almost contraction map are some of them [6,10,19].

The concept of iteration method, whose origin goes back centuries, was introduced by Liouville and used by Cauchy. In 1890, Picard showed a theoretical approach to the fixed point with the iteration he defined and started the development process of the iteration method concept, which is still studied today [37]. Some iteration methods used in fixed point theory are listed below according to their development in the literature: Mann iteration method [29], Krasnosel'skii iteration method [28], Schaefer iteration

***Corresponding author.**
*E-mail address: lalecona@gumushane.edu.tr

This paper was recommended for publication in revised form by Regional Editor Ahmet Selim Dalkilic

method [39], Halpern iteration method [19], Kirk iteration method [27], Ishikawa iteration method [21], Noor iteration method [30], Multistep iteration method [35], S iteration method [4], Two Steps Mann iteration method [41], Kirk-Ishikawa and Kirk-Mann iteration methods [30], SP iteration method [35], CR and Kirk-Noor iteration methods [12], Kirk-SP and Kirk-CR iteration methods [20], S* iteration method [23], Picard-Mann iteration method [26], Abbas-Nazir iteration method [1], Picard-S iteration method [18], New three-step iteration method [25].

With these developments, the fixed point theory is basically grouped under three headings: topological, metric and discrete fixed point theory. In addition, the theory has been studied in relation to many mathematical structures such as metric spaces, normed spaces, real analysis, linear algebra, ordinary differential equations theory, integral equations theory. Also, for many iteration methods developed, there are many studies on convergence equivalence, convergence speed, strong convergence of newly defined transformations to fixed points and data dependency [1,8,12-15,32].

The interest in integral equations is increasing day by day due to the relationship of integral equations with differential equations and their widespread use in differential equation techniques. Thus, the theory of integral equations has become one of the most common areas of applied mathematics. Topics have been updated gradually and many solution methods have been developed [2,3,16,17,22,31-34,42].

The organization of this paper is as follows. In Section 2, we provide necessary background. In Section 3, strong convergence of the second type of nonhomogeneous linear Fredholm integral equations is investigated by using the three-step iteration algorithm. Finally, data dependence is obtained for the second type of nonhomogeneous linear Fredholm integral equations and this result is supported by an example.

KNOWN RESULTS

Definition 1 Let (X, d) be a metric space and $T : X \to X$ be a mapping. T is called a Lipschitzian mapping, if there is a $\lambda > 0$ number such that

$$d(Tx, Ty) \leq \lambda d(x, y)$$

for all $x, y \in X$ [10].

Definition 2 Let (X, d) be a metric space, $T : X \to X$ be a Lipschitzian mapping. If there is at least one $\lambda \in (0,1)$ real number such that

$$d(Tx, Ty) \leq \lambda d(x, y)$$

for all $x, y \in X$, T is called a contraction mapping. λ is called the contraction ratio [10].

Geometrically, this definition can be interpreted as Tx and Ty, which are images of any x and y points, are closer together than x and y [10].

Definition of contraction mapping in X normed space by

$$\|Tx - Ty\| \leq \lambda \|x - y\|$$

where $\lambda \in (0,1)$ the contraction ratio.

Below is the statement of Banach fixed point theorem [9].

Theorem 1 If (X, d) is a complete metric space and $T : X \to X$ is a contraction mapping,

- T has one and only one fixed point $x \in X$.
- For any $x_0 \in X$, iteration sequence $(T^n x_0)$ (ie iteration sequence (x_n) defined by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$) converges to unique fixed point of T.

Let us give the definition of the following three-step iteration algorithm, which was shown to be faster than many iteration algorithms such as Picard, Mann, Ishikawa, Noor, SP, S, CR and Picard-S by Karakaya et al. in 2017.

Definition 3 The iteration method

$$\left.\begin{aligned} x_{n+1} &= Ty_n \\ y_n &= (1-\beta_n)z_n + \beta_n Tz_n \\ z_n &= Tx_n \end{aligned}\right\} \tag{1}$$

for $x_0 \in X$, where X is a Banach space, $T : X \to X$ is an operator, and $\{\beta_n\}_{n=0}^{\infty} \subset [0,1]$ is a sequence satisfying certain conditions, is called the three-step iteration method [25].

Definition 4 Let $f(t)$ be a continuous function given on $[a, b]$ and $x(t)$ be a function on $[a, b]$ whose solution is desired. The integral equations in the form of

$$x(t) = f(t) + \lambda \int_a^b k(t,s)x(s)ds \tag{2}$$

where λ is a parameter and a continuous function given over the region $k(t, s)$, are called second type of nonhomogeneous linear Fredholm integral equations. Here k is called the kernel of the equation [5].

Lemma 1 Since the kernel of the integral equation defined in equation (2) is k, bounded and continuous over the region $D = \{(t, s) : a \leq t, s \leq b\}$, there is a M such that

$$M = \max_{a \leq t, s \leq b} \int_a^b |k(t,s)| ds \tag{3}$$

for all $(t, s) \in D$ [5].

Let us now state the existence uniqueness theorem for nonhomogeneous linear Fredholm integral equations.

Theorem 2 Consider the operator $T : (C[a, b], \|\cdot\|) \to (C[a, b], \|\cdot\|)$ defined by

$$Tx(t) = f(t) + \lambda \int_a^b k(t,s)x(s)ds \tag{4}$$

with $f(t) \in C([a, b])$. Since,

$$\|Tx_1 - Tx_2\| \leq |\lambda| M \|x_1 - x_2\|$$

for M in equation (3), T is a contraction mapping if $\alpha = |\lambda|M < 1$. In this case, the solution of the Fredholm integral equation is the fixed point of the operator T, and when $\alpha = |\lambda|M < 1$ it is obtained from the Banach fixed point theorem that there is only one fixed point in $C[a, b]$. The solution is the limit of the iteration

$$x_{n+1}(t) = f(t) + \lambda \int_a^b k(t,s)x_n(s)ds \tag{5}$$

for all $n \in \mathbb{N}$ with a starting point x_0 on $[a, b]$ [5].

Definition 5 Let $A_1, A_2 : C \to C$ be operators. If $\|A_1x - A_2x\| \leq \varepsilon$ for each $x \in C$ and constant $\varepsilon > 0$, then A_2 is called the approximation operator of A_1 [40].

Lemma 2 Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be two non-negative real sequences satisfting the following condition:

$$a_{n+1} \leq (1-\mu_n)a_n + b_n,$$

where $\mu_n \in (0,1)$ for each $n \geq n_0, \sum_{n=0}^{\infty} \mu_n = \infty$ and $\frac{b_n}{\mu_n} \to 0$ as $n \to \infty$. Then $\lim_{n\to\infty} a_n = 0$ [43].

Lemma 3 Let $\{a_n\}_{n=0}^{\infty}$ be a non-negative real sequence and there exists $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$ satisfting the following condition:

$$a_{n+1} \leq (1-\mu_n)a_n + \mu_n\gamma_n\,,$$

where $\mu_n \in (0,1)$ such that $\sum_{n=0}^{\infty} \mu_n = \infty$ and $\gamma_n \geq 0$. Then,

$$0 \leq \limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} \gamma_n$$

the inequality holds [40].

MAIN RESULTS

Theorem 3 Let $T : (C[a, b], \|\cdot\|) \to (C[a, b], \|\cdot\|)$ be an operator and $\{\beta_n\}^{\infty} \subset [0,1]$ be a sequence satisfying certain conditions. In this case, the integral equation given by equation (2) has a unique solution in the form of $x^* \in C[a, b]$ and the sequence $\{x_n\}^{\infty}$ obtained from the iteration algorithm given by equation (1) converges to this solution.

Proof Consider the sequence $\{x_n\}^{\infty}$ obtained from the iteration algorithm given by equation (1) constructed with the operator $T : (C[a, b], \|\cdot\|) \to (C[a, b], \|\cdot\|)$. It will be shown that for $n \to \infty$ is $x_n \to x^*$. Using equation (1), equation (4) and conditions of Theorem 2, we are obtained the following inequality.

$$\begin{aligned}|x_{n+1}(t) - x^*(t)| &= |Ty_n(t) - Tx^*(t)| \\ &= \left|f(t) + \lambda\int_a^b k(t,s)y_n(s)ds - f(t) - \lambda\int_a^b k(t,s)x^*(s)ds\right| \\ &= |\lambda|\left|\int_a^b k(t,s)\big(y_n(s) - x^*(s)\big)ds\right| \\ &\leq |\lambda|\int_a^b |k(t,s)||y_n(s) - x^*(s)|ds \\ &\leq |\lambda|M\|y_n - x^*\| \\ &= \alpha\|y_n - x^*\|\end{aligned}$$

Then,

$$\|x_{n+1}(t) - x^*(t)\| \leq \alpha\|y_n - x^*\| \tag{6}$$

is found. Similarly, the following inequalities are obtained.

$$\begin{aligned}\|y_n - x^*\| &= \|(1-\beta_n)z_n + \beta_n Tz_n - Tx^*\| \\ &= \|z_n - x^* + \beta_n(Tz_n - z_n)\| \\ &\leq \|z_n - x^*\|\end{aligned} \tag{7}$$

$$\begin{aligned}\|z_n - x^*\| &= \|Tx_n - Tx^*\| \\ &= \left\|f(t) + \lambda\int_a^b k(t,s)x_n(s)ds - f(t) - \lambda\int_a^b k(t,s)x^*(s)ds\right\| \\ &\leq |\lambda|M\|x_n - x^*\| \\ &= \alpha\|x_n - x^*\|\end{aligned} \tag{8}$$

If inequalities (8) and (7) are written in inequality (6),

$$\|x_{n+1}(t) - x^*(t)\| \leq \alpha^2\|x_n - x^*\|$$

is found. By applying induction to the last inequality, the following inequality is obtained.

$$\|x_{n+1}(t) - x^*(t)\| \leq \alpha^{2(n+1)}\|x_0 - x^*\|$$

Thus,

$$\lim_{n\to\infty}\|x_{n+1}(t) - x^*(t)\| = 0$$

and the proof is completed.

Now, let us examine the data dependency of the solution of the integral equation given by equation (2) using the iteration algorithm given in equation (1). Consider the following integral equation, the operator

$$S : (C[a,b], \|\cdot\|) \to (C[a,b], \|\cdot\|)$$

$$S\big(u(t)\big) = g(t) + \lambda_1 \int_a^b h(t,s)u(s)ds, \tag{9}$$

where $g(t)$ is continuous on $[a, b]$, $h(t, s)$ is a continuous function given over the region $D = \{(t, s) : a \leq t, s \leq b\}$ and λ_1 is a parameter.

If the iteration algorithm given in equation (1) is reconstructed with operators $T(4)$ and $S(9)$, respectively, the following iteration algorithms can be written.

$$\left.\begin{aligned}x_{n+1}(t) &= f(t) + \lambda\int_a^b k(t,s)y_n(s)ds \\ y_n(t) &= (1-\beta_n)z_n(t) + \beta_n\left[f(t) + \lambda\int_a^b k(t,s)z_n(s)ds\right] \\ z_n(t) &= f(t) + \lambda\int_a^b k(t,s)x_n(s)ds\end{aligned}\right\} \tag{10}$$

$$\left.\begin{aligned}u_{n+1}(t) &= g(t) + \lambda_1\int_a^b h(t,s)v_n(s)ds \\ v_n(t) &= (1-\beta_n)w_n(t) + \beta_n\left[g(t) + \lambda_1\int_a^b h(t,s)w_n(s)ds\right] \\ w_n(t) &= g(t) + \lambda_1\int_a^b h(t,s)u_n(s)ds\end{aligned}\right\} \tag{11}$$

Theorem 4 Let the sequence $\{\beta_n\}_{n=0}^{\infty} \subset [0,1]$ satisfy the condition $\beta_n \geq \frac{1}{2}$ for each $n \in \mathbb{N}$. Consider the sequence $\{x_n\}_{n=0}^{\infty}$ obtained from equation (10) and the sequence $\{u_n\}_{n=0}^{\infty}$ obtained from equation (11). Let the solutions of the integral equations (4) and (9) be x^* and u^*, respectively, with the conditions of Theorem 3.

- Let the constants N and ε_1 exist such that $\int_a^b \|k(t,s) - h(t,s)\| ds \leq N$ and $\|f(t,s) - g(t,s)\| \leq \varepsilon_1$ for each $(t,s) \in [a,b]$.
- Let the constant ε_2 exists such that $N \leq \frac{\varepsilon_2}{\|u_n\|+\|v_n\|+\|w_n\|}$ for each $n \in \mathbb{N}$.(Let at least one of $\|u_n\|, \|v_n\|, \|w_n\|$ be nonzero.)

If $x_n \to x^*$ and $u_n \to u^*$ as $n \to \infty$, then the inequality

$$\|x^* - u^*\| \leq \frac{5\varepsilon_1 + 2\lambda_0\varepsilon_2}{1 - \lambda_0 M}$$

is valid, with $\lambda_0 = \max\{|\lambda|, |\lambda_1|\}$.

Proof The following inequality is obtained using the hypotheses of the theorem.

$$\begin{aligned}
\|x_{n+1} - u_{n+1}\| &= \left\| f(t) + \lambda \int_a^b k(t,s) y_n(s) ds - g(t) - \lambda_1 \int_a^b h(t,s) v_n(s) ds \right\| \\
&\leq \|f(t) - g(t)\| + \left\| \lambda_0 \int_a^b \begin{pmatrix} k(t,s)y_n(s) - k(t,s)v_n(s) \\ +k(t,s)v_n(s) - \ h(t,s)v_n(s) \end{pmatrix} ds \right\| \\
&\leq \|f(t) - g(t)\| + \lambda_0 \begin{pmatrix} \int_a^b \|k(t,s)\| \|y_n(s) - v_n(s)\| ds \\ + \int_a^b \|v_n(s)\| \|k(t,s) - h(t,s)\| ds \end{pmatrix} \\
&\leq \varepsilon_1 + \lambda_0 (M \|y_n - v_n\| + N \|v_n\|)
\end{aligned} \tag{12}$$

Similarly,

$$\begin{aligned}
\|y_n - v_n\| &\leq (1-\beta_n)\|z_n - w_n\| + \beta_n \left\| f(t) + \lambda \int_a^b k(t,s) z_n(s) ds - g(t) - \lambda_1 \int_a^b h(t,s) w_n(s) ds \right\| \\
&\leq (1-\beta_n)\|z_n - w_n\| + \beta_n \|f(t) - g(t)\| + \beta_n \left\| \lambda \int_a^b k(t,s) z_n(s) ds - \lambda_1 \int_a^b h(t,s) w_n(s) ds \right\| \\
&\leq (1-\beta_n)\|z_n - w_n\| + \beta_n\varepsilon_1 + \beta_n\lambda_0 \underbrace{\int_a^b \|k(t,s)z_n(s) - h(t,s)w_n(s)\| ds}_{I}.
\end{aligned} \tag{13}$$

$$\begin{aligned}
I &= \int_a^b \|k(t,s)z_n(s) - k(t,s)w_n(s) + k(t,s)w_n(s) - h(t,s)w_n(s)\| ds \\
&\leq \int_a^b \|k(t,s)\| \|z_n(s) - w_n(s)\| ds + \int_a^b \|w_n(s)\| \|k(t,s) - h(t,s)\| ds \\
&\leq M\|z_n - w_n\| + N\|w_n\|.
\end{aligned} \tag{14}$$

If inequality (14) is written in inequality (13), the following inequality is obtained.

$$\begin{aligned}
\|y_n - v_n\| &\leq (1-\beta_n)\|z_n - w_n\| + \beta_n\varepsilon_1 + \beta_n\lambda_0(M\|z_n - w_n\| + N\|w_n\|) \\
&= \|z_n - w_n\|(1 - \beta_n + \beta_n\lambda_0 M) + \beta_n\varepsilon_1 + \beta_n\lambda_0 N\|w_n\|
\end{aligned} \tag{15}$$

Finally, the following inequality is found.

$$\begin{aligned}
\|z_n - w_n\| &= \left\| f(t) + \lambda \int_a^b k(t,s) x_n(s) ds - g(t) - \lambda_1 \int_a^b h(t,s) u_n(s) ds \right\| \\
&\leq \|f(t) - g(t)\| + \lambda_0 \left\| \int_a^b k(t,s) x_n(s) - h(t,s) u_n(s) ds \right\| \\
&\leq \varepsilon_1 + \lambda_0 \int_a^b \|k(t,s)x_n(s) - k(t,s)u_n(s) + k(t,s)u_n(s) - h(t,s)u_n(s)\| ds \\
&\leq \varepsilon_1 + \lambda_0 \left(\int_a^b \|k(t,s)\| \|x_n(s) - u_n(s)\| ds + \int_a^b \|u_n(s)\| \|k(t,s) - h(t,s)\| ds \right) \\
&= \varepsilon_1 + \lambda_0 (M\|x_n - u_n\| + N\|u_n\|) \\
&= \varepsilon_1 + \lambda_0 M\|x_n - u_n\| + \lambda_0 N\|u_n\|
\end{aligned} \tag{16}$$

Using the following hypotheses:

- $|\lambda| M < 1$
- $|\lambda_1| M < 1$
- $\lambda_0 = \max\{|\lambda|, |\lambda_1|\}$
- $\lambda_0 M < 1$
- $\frac{1}{2} \leq \beta_n \leq 1$
- $1 - \beta_n \leq \beta_n$
- $\beta_n \lambda_0 M < 1$
- $L = 1 - \beta_n + \beta_n\lambda_0 M = 1 - (\beta_n - \beta_n\lambda_0 M) < 1$,

if inequality (16) is written in inequality (15),

$$\begin{aligned}
\|y_n - v_n\| &\leq L(\varepsilon_1 + \lambda_0 M\|x_n - u_n\| + \lambda_0 N\|u_n\|) + \beta_n\varepsilon_1 + \beta_n\lambda_0 N\|w_n\| \\
&\leq L(\varepsilon_1 + \|x_n - u_n\| + \lambda_0 N\|u_n\|) + \beta_n\varepsilon_1 + \beta_n\lambda_0 N\|w_n\| \\
&= L\varepsilon_1 + L\|x_n - u_n\| + L\lambda_0 N\|u_n\| + \beta_n\varepsilon_1 + \beta_n\lambda_0 N\|w_n\| \\
&\leq L\|x_n - u_n\| + \varepsilon_1 + \beta_n\varepsilon_1 + \lambda_0 N(\|u_n\| + \|w_n\|)
\end{aligned}$$

is obtained. If this last inequality is written in inequality (12),

$$\begin{aligned}
\|x_{n+1} - u_{n+1}\| &\leq \varepsilon_1 + \lambda_0\{M[L\|x_n - u_n\| + \varepsilon_1 + \beta_n\varepsilon_1 + \lambda_0 N(\|u_n\| + \|w_n\|)] + N\|v_n\|\} \\
&= \varepsilon_1 + \lambda_0[ML\|x_n - u_n\| + M\varepsilon_1 + M\beta_n\varepsilon_1 + M\lambda_0 N(\|u_n\| + \|w_n\|) + N\|v_n\|] \\
&\leq \varepsilon_1 + \lambda_0[ML\|x_n - u_n\| + M\varepsilon_1 + M\beta_n\varepsilon_1 + N(\|u_n\| + \|w_n\|) + N\|v_n\|] \\
&= \varepsilon_1 + \lambda_0 ML\|x_n - u_n\| + \lambda_0 M\varepsilon_1 + \lambda_0 M\beta_n\varepsilon_1 + \lambda_0 N(\|u_n\| + \|w_n\| + \|v_n\|) \\
&\leq L\|x_n - u_n\| + 2\varepsilon_1 + \beta_n\varepsilon_1 + \lambda_0 N(\|u_n\| + \|w_n\| + \|v_n\|)
\end{aligned}$$

and

$$\begin{aligned}
\|x_{n+1} - u_{n+1}\| &\leq [1 - (\beta_n - \beta_n\lambda_0 M)]\|x_n - u_n\| \\
&\quad + 2\varepsilon_1 + \beta_n\varepsilon_1 + \lambda_0 N(\|u_n\| + \|w_n\| + \|v_n\|)
\end{aligned}$$

are found. Using $N \leq \frac{\varepsilon_2}{\|u_n\|+\|v_n\|+\|w_n\|}$,

$$\begin{aligned}
\|x_{n+1} - u_{n+1}\| &\leq [1 - (\beta_n - \beta_n\lambda_0 M)]\|x_n - u_n\| \\
&\quad + \beta_n\varepsilon_1 + 2\varepsilon_1 + \lambda_0 \ \varepsilon_2
\end{aligned}$$

is obtained. Using $\beta_n \geq \frac{1}{2}$,

$$\begin{aligned}
\|x_{n+1} - u_{n+1}\| &\leq [1 - (\beta_n - \beta_n\lambda_0 M)]\|x_n - u_n\| + \beta_n\varepsilon_1 + 4\beta_n\varepsilon_1 + 2\lambda_0\beta_n\varepsilon_2 \\
&= [1 - \beta_n(1 - \lambda_0 M)]\|x_n - u_n\| + \beta_n(5\varepsilon_1 + 2\lambda_0\varepsilon_2) \\
&= [1 - \beta_n(1 - \lambda_0 M)]\|x_n - u_n\| + \beta_n(1 - \lambda_0 M)\tfrac{5\varepsilon_1 + 2\lambda_0\varepsilon_2}{1 - \lambda_0 M}
\end{aligned} \tag{17}$$

is found from the last inequality. Let

$$\begin{aligned}
a_n &= \|x_n - u_n\|, \\
\mu_n &= \beta_n(1 - \lambda_0 M) \in (0,1), \\
\gamma_n &= \frac{5\varepsilon_1 + 2\lambda_0\varepsilon_2}{1 - \lambda_0 M} \geq 0.
\end{aligned}$$

$\beta_n \geq \frac{1}{2}$ requires $\sum_{n=0}^{\infty} \beta_n = \infty$ for each $n \in \mathbb{N}$. Hence, the inequality given by inequality (17) satisfies the conditions of Lemma 3. Then,

$$0 \leq \limsup_{n\to\infty} \|x_n - u_n\| \leq \limsup_{n\to\infty} \gamma_n = \limsup_{n\to\infty} \frac{5\varepsilon_1 + 2\lambda_0\varepsilon_2}{1-\lambda_0 M}$$

is obtained. Since $x_n \to x^*$ and $u_n \to u^*$ as $n \to \infty$.

$$\|x^* - u^*\| \leq \frac{5\varepsilon_1+2\lambda_0\varepsilon_2}{1-\lambda_0 M} \tag{18}$$

is found.

Example 1 $x(t) = t + \int_0^1 tsx(s)ds$

where $k(t, s) = ts$ is a continuous function given over the region $D = \{(t, s) : 0 \leq t, s \leq 1\}$.

$$M = \max_{0\leq t,s\leq 1} \int_0^1 |ts|ds = \max_{0\leq t\leq 1} \frac{t}{2} = \frac{1}{2}$$

for each $0 \leq t, s \leq 1$. Since $|\lambda| = 1, \alpha = |\lambda|M = \frac{1}{2} < 1$. The equation in question has only one continuous solution x^* on $[0,1]$. Let's define the following algorithm with the operator

$$Tx_n(t) = t + \int_0^1 tsx_n(s)ds$$

for the solution.

$$x_{n+1}(t) = Ty_n(t) = t + \int_0^1 tsy_n(s)ds$$

$$y_n(t) = (1-\beta_n)z_n(t) + \beta_n Tz_n(t) = (1-\beta_n)z_n(t) + \beta_n\left(t + \int_0^1 tsz_n(s)ds\right)$$

$$z_n(t) = Tx_n(t) = t + \int_0^1 tsx_n(s)ds$$

On the other hand,

$$u(t) = t + \frac{1}{2}\int_0^1 u(s)ds$$

where $h(t, s) = 1$ is a continuous function given over the region $G = \{(t, s) : 0 \leq t, s \leq 1\}$.

$$M_1 = \max_{0\leq t,s\leq 1} \int_0^1 ds = 1$$

for each $0 \leq t, s \leq 1$. Since $|\lambda_1| = \frac{1}{2}, \alpha_1 = |\lambda_1|M_1 = \frac{1}{2} < 1$. The equation in question has only one continuous solution u^* on $[0,1]$. Let's define the following algorithm with the operator

$$Su_n(t) = t + \frac{1}{2}\int_0^1 u_n(s)ds$$

for the solution.

$$u_{n+1}(t) = Sv_n(t) = t + \frac{1}{2}\int_0^1 v_n(s)ds$$

$$v_n(t) = (1-\beta_n)w_n(t) + \beta_n Sw_n(t) = (1-\beta_n)w_n(t) + \beta_n\left(t + \frac{1}{2}\int_0^1 w_n(s)ds\right)$$

$$w_n(t) = Su_n(t) = t + \frac{1}{2}\int_0^1 u_n(s)ds$$

Thus,

$$\lambda_0 = \max\{|\lambda|, |\lambda_1|\} = \max\left\{1, \frac{1}{2}\right\} = 1$$

is found. Let the constants N, ε_1 and ε_2 exist such that

$$\int_0^1 \|k(t,s) - h(t,s)\|ds = \int_0^1 \|ts - 1\|ds \leq \frac{1}{2} = N$$

$$\|f(t,s) - g(t,s)\| = \|t - t\| = 0 \leq \varepsilon_1$$

$$N = \frac{1}{2} \leq \varepsilon_2$$

for $n \in \mathbb{N}$ and each $(t, s) \in [0,1]$. So, all the conditions of Theorem 4 are satisfied. If the fixed points are written in inequality (18),

$$\|x^* - u^*\| = \frac{5\varepsilon_1 + 2\lambda_0\varepsilon_2}{1-\lambda_0 M} \leq 2$$

is obtained. Indeed, $x^* = \frac{3}{2}t$ and $u^* = t + \frac{1}{2}$ are found. Thus,

$$\|x^* - u^*\| \leq \left\|\frac{3t}{2} - t - \frac{1}{2}\right\| \leq \left\|\frac{t-1}{2}\right\| \leq \frac{1}{2} \leq 2$$

is provided.

CONCLUSION

In this article, the solution of the second type of nonhomogeneous linear Fredholm integral equations is investigated using a three-step iteration algorithm. The aim of this article is to show that the sequence obtained from equation (1) iteration method converges strongly to the solution of equation (2). That is, it has been shown that the sequences obtained from this algorithm converge to the solution of the mentioned equations. Morever, data dependence is obtained for the second type of nonhomogeneous linear Fredholm integral equations and this result is supported by an example. Interested researchers can obtain new results on strong convergence and data dependence by using different iteration methods and different integral equations.

AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

REFERENCES

[1] Abbas M, Nazir T. A new faster iteration process applied to constrained minimization and feasibility problems. Mat Vesnik 2014;66:223−234.

[2] Adebisi AF, Ojurongbe TA, Okunlola KA, Peter OJ. Application of Chebyshev polynomial basis function on the solution of Volterra integro-differential equations using Galerkin method. Math Comput Sci 2021;2:41−51.

[3] Adebisi AF, Okunola KA, Raji MT, Adedeji JA, Peter OJ. Galerkin and perturbed collocation methods for solving a class of linear fractional integro-differential equations. Aligarh Bull Math 2020;40:45−57.

[4] Agarwal RP, O'Regan D, Sahu DR. Iterative construction of fixed points of nearly asymptotically nonexpansive mappings. J Nonlinear Convex Anal 2007;8:61−79.

[5] Akbulut A. Sabit nokta teoremlerinin Cauchy problemine ve integral denklemlere uygulanması [Master's thesis]. Ankara: Gazi Üniversitesi; 2007.

[6] Atalan Y. Yeni bir iterasyon yöntemi için hemen-hemen büzülme dönüşümleri altında bazı sabit nokta teoremleri. Marmara Fen Bilim Derg 2018;30:276–285. [Turkish] [CrossRef]

[7] Atalan Y. İteratif yaklaşım altında bir fonksiyonel-integral denklem sınıfının çözümünün incelenmesi. J Inst Sci Technol 2019;9:1622–1632. [Turkish]

[8] Atalan Y, Maldar S. A new note on convergence and data dependence concept for a Volterra integral equation by fixed point iterative algorithm. J Math Ext 2023;17:1−18.

[9] Banach S. Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. Fundam Math 1922;3:133−181. [CrossRef]

[10] Berinde V, Takens F. Iterative approximation of fixed points. Berlin: Springer; 2007. [CrossRef]

[11] Brouwer LEJ. Über abbildung von mannigfaltigkeiten. Math Ann 1911;71:97−115. [CrossRef]

[12] Chugh R, Kumar V, Kumar S. Strong convergence of a new three-step iterative scheme in Banach spaces. Am J Comput Math 2012;2:345−357. [CrossRef]

[13] Cona L, Şengül K. On data dependency and solutions of nonlinear Fredholm integral equations with the three-step iteration method. Ikonion J Math 2023;2:53−64. [CrossRef]

[14] Cona L, Tuna T. Solutions of linear Fredholm integral equations with the Picard's three-step iteration method. In: Proceedings of the 11th International Summit Scientific Research Congress; 2023 Dec 15−17; Gaziantep, Turkey.

[15] Cona L, Tuna T. Data dependence with Picard's three-step iteration method for linear Fredholm integral equations. In: Proceedings of the 11th International Summit Scientific Research Congress; 2023 Dec 15−17; Gaziantep, Turkey.

[16] Cona L. Solutions of some fractional order integro-differential equation by the three-step iteration method. In: Proceedings of the Bilsel International Ahlat Scientific Researches Congress; 2023 Dec 9−10; Bitlis, Turkey.

[17] Cona L. Data dependency for some fractional order integro-differential equation by the three-step iteration method. In: Proceedings of the Bilsel International Ahlat Scientific Researches Congress; 2023 Dec 9−10; Bitlis, Turkey.

[18] Gürsoy F. A Picard-S iterative method for approximating fixed point of weak-contraction mappings. Filomat 2016;30:2829−2845. [CrossRef]

[19] Halpern B. Fixed points of nonexpanding maps. Bull Am Math Soc 1967;73:957−961. [CrossRef]

[20] Hussain N, Chugh R, Kumar V, Rafiq A. On the rate of convergence of Kirk-type iterative schemes. J Appl Math 2012;2012:526503. [CrossRef]

[21] Ishikawa S. Fixed points by a new iteration method. Proc Am Math Soc 1974;44:147−150. [CrossRef]

[22] Ishola CY, Taiwo OA, Adebisi AF, Peter OJ. Numerical solution of two-dimensional Fredholm integro-differential equations by Chebyshev integral operational matrix method. J Appl Math Comput Mech 2022;21:29−40. [CrossRef]

[23] Karahan İ, Özdemir M. A general iterative method for approximation of fixed points and their applications. Adv Fixed Point Theory 2013;3:510−526. [CrossRef]

[24] Karakaya V, Atalan Y, Doğan K, Bouzara NEH. Some fixed point results for a new three steps iteration process in Banach spaces. Fixed Point Theory 2017;18:625−640. [CrossRef]

[25] Khan SH. A Picard-Mann hybrid iterative process. Fixed Point Theory Appl 2013;2013:69. [CrossRef]

[26] Kirk WA. On successive approximations for nonexpansive mappings in Banach spaces. Glasgow Math J 1971;12:6–9. [CrossRef]
[27] Krasnosel'skii MA. Two comments on the method of successive approximations. Usp Mat Nauk 1955;10:123–127.
[28] Mann WR. Mean value methods in iteration. Proc Am Math Soc 1953;4:506–510. [CrossRef]
[29] Noor MA. New approximation schemes for general variational inequalities. J Math Anal Appl 2000;251:217–229. [CrossRef]
[30] Olatinwo MO. Some stability results for two hybrid fixed point iterative algorithms in normed linear space. Mat Vesnik 2009;61:247–256.
[31] Oyedepo T, Adebisi AF, Tayo RM, Adedeji JA, Ayinde MA, Peter OJ. Perturbed least squares technique for solving Volterra fractional integro-differential equations based on constructed orthogonal polynomials. J Math Comput Sci 2020;11:203–218.
[32] Oyedepo T, Uwaheren OA, Okperhie EP. Solution of fractional integro-differential equation using modified homotopy perturbation technique and constructed orthogonal polynomials as basis functions. ATBU J Sci Technol Educ. 2019;7:157–164.
[33] Peter OJ. Transmission dynamics of fractional order Brucellosis model using Caputo-Fabrizio operator. Int J Differ Equ 2020;2020:2791380. [CrossRef]
[34] Peter OJ, Adebisi AF, Oguntolu FA, Bitrus S, Akpan CE. Multi-step homotopy analysis method for solving Malaria model. Malays J Appl Sc 2018;3:34–45.
[35] Phuengrattana W, Suantai S. On the rate of convergence of Mann, Ishikawa, Noor and SP-iterations for continuous functions on an arbitrary interval. J Comput Appl Math 2011;235:3006–3014. [CrossRef]
[36] Picard E. Memoire sur la theorie des equations aux derivees partielles et la methode des approximations successives. J Math Pures Appl 1890;6:145–210.
[37] Rhoades BE, Soltuz SM. The equivalence between Mann-Ishikawa iterations and multistep iteration. Nonlinear Anal Theory Methods Appl. 2004;58:219–228. [CrossRef]
[38] Schaefer H. Über die methode sukzessiver approximationen. Jahresber Dtsch Math-Ver 1957;59:131–140.
[39] Şoltuz ŞM, Grosan T. Data dependence for Ishikawa iteration when dealing with contractive-like operators. Fixed Point Theory Appl 2008;2008:1–7. [CrossRef]
[40] Thianwan S. Common fixed points of new iterations for two asymptotically nonexpansive nonself-mappings in a Banach space. J Comput Appl Math 2009;224:688–695. [CrossRef]
[41] Uwaheren OA, Adebisi AF, Olotu OT, Etuk MO, Peter OJ. Legendre Galerkin method for solving fractional integro-differential equations of Fredholm type. Aligarh Bull Math 2021;40:15–27.
[42] Weng X. Fixed point iteration for local strictly pseudo-contractive mapping. Proc Am Math Soc 1991;113:727–731. [CrossRef]