Sigma J Eng Nat Sci, Vol. 39, No. 2, pp. 148-158, June, 2021

Sigma Journal of Engineering and Natural Sciences –
Sigma Mühendislik ve Fen Bilimleri Dergisi
Web page info: https://sigma.yildiz.edu.tr DOI: 10.14744/sigma.2021.00004

Research Article

Estimation procedures on Type-II censored data from a scaled Muth distribution

Hayrinisa Demirci BIÇER[1], Berkay ÖZTÜRKER[2], Cenker BIÇER[1,*]

[1]*Department of Statistics, Kırıkkale University, Kırıkkale Turkey*
[2]*Turkish Employment Agency, Ankara, Turkey*

ARTICLE INFO

Article history
Received: 13 March 2020
Accepted: 04 March 2021

Key words:
Lifetime model, Muth distribution, censored data, maximum spacing, maximum likelihood, least-squares

ABSTRACT

In the present paper, we consider the estimation problem for the scaled Muth distribution under Type-II censoring scheme. In order to estimate the model parameters α and β, the maximum likelihood, the least-squares, and the maximum spacing estimators are derived. To show estimation efficiencies of the estimators obtained with this paper, we present an extensive Monte-Carlo simulation study in which the estimators are compared according to bias and mean squared error criteria. Furthermore, we evaluate the applicability of the scaled Muth distribution by taking into account both full and Type-II censored data situations by an analysis conducted on a real-life dataset.

Cite this article as: Biçer HD, Öztürker B, Biçer C. Estimation procedures on type-II censored data from a scaled muth distribution Sigma J Eng Nat Sci 2021;39(2):148–158.

INTRODUCTION

The Muth distribution (MD) was firstly introduced on the positive real interval by Muth [1] as the statistical lifetime model with increasing, decreasing, and bathtub failure rates. Moreover, Jodra et al. [2] comprehensively studied some statistical features of the MD. In a wide variety of lifetime data observed in many fields from engineering and natural sciences to health and social sciences, the MD has got the potential of a good option for analyzing datasets with such failure rates. In this regard, it is a powerful alternative to popular lifetime models such as exponential, gamma, Weibull, log-normal, Rayleigh, and inverse Gaussian, with a weak probability mass property in the tail [2]. The probability density function of the MD is

$$f(x,\alpha)=\left(e^{\alpha x}-\alpha\right)\exp\left(\alpha x-\frac{1}{\alpha}\left(e^{\alpha x}-1\right)\right), x>0, \quad (1)$$

and the corresponding cumulative distribution function is

$$F(x,\alpha)=1-\exp\left(\alpha x-\frac{1}{\alpha}\left(e^{\alpha x}-1\right)\right),\ x>0, \quad (2)$$

where, $\alpha \in (0,1)$ is a parameter that plays a vital role in the behavior of the distribution. The expected value of the

*Corresponding author.
*E-mail address: cbicer@kku.edu.tr

This paper was recommended for publication in revised form by Section Editor Gülhayat Gölbaşı Şimşek

Published by Yıldız Technical University Press, İstanbul, Turkey

MD with the single parameter is $E(X) = 1$. This is a strong constraint at the real data modeling stage. Jodra et al. [2] derived a new two-parameter form of MD called scaled Muth distribution (SMD) by adding a scale parameter to the distribution in their study. By this new form, a strong restriction on the expected value of the MD is removed. The pdf and cdf of the SMD are, respectively,

$$f(x,\alpha,\beta) = \frac{1}{\beta}\left(e^{\frac{\alpha x}{\beta}} - \alpha\right)\exp\left(\frac{\alpha x}{\beta} - \frac{1}{\alpha}\left(e^{\frac{\alpha x}{\beta}} - 1\right)\right),\ x > 0, \quad (3)$$

and

$$F(x,\alpha,\beta) = 1 - \exp\left(\frac{\alpha x}{\beta} - \frac{1}{\alpha}\left(e^{\frac{\alpha x}{\beta}} - 1\right)\right),\ x > 0, \quad (4)$$

where $\beta > 0$ is the scale parameter of the SMD. We present Figure 1 to illustrate the pdf behavior of the SMD for different values of the parameters α and β.

In the reliability and lifetime data analysis, it is well known that a complete dataset should be used to optimally analyze a phenomenon. However, sometimes the dataset may not be completely obtained or time cost constraints may be encountered. In such cases, the researcher may consider the censoring of the data. Censoring of the data will mostly enable significant cost savings in reliability engineering while time-saving in the modeling of lifetimes. There are various censoring schemes in the literature and commonly used of these are Type-I and Type-II. The main opinion in the Type-I censoring scheme, the experiment continues till a predetermined time T. The base idea in the Type-II censoring scheme, an experiment ends when records a predetermined number of data (failure times). So far, many researchers have made valuable studies on the solution of the statistical inference problem under the Type-II censoring scheme for specific probability distribution models, see [3–9]. This paper mainly motivated to examine different estimation procedures under the Type-II censoring scheme for the SMD. As far as we know, no attempt has yet been made to discuss different estimation procedures under the Type-II censoring scheme of SMD. This is quite an important task for areas where censored data is encountered, such as reliability engineering and lifetime analyzing, since optimally estimating model parameters have significant effects on determining a suitable model to data and modeling performance.

The rest of the paper is organized as follows. In the section Inference on SMD Parameters, we investigate the parameter estimators of the SMD under the Type -II censoring scheme by considering the different estimation methods, such as maximum likelihood (ML), least-squares (LS), and maximum spacing (MSP). The section Simulation Experiments includes a comprehensive Monte-Carlo simulation study in which compares the efficiencies of the estimators obtained in the section Inference on SMD Parameters according to bias and mean-squared error criteria. In the section Data Analysis, we present an illustrative data example to show the usefulness of SMD in modeling the Type-II censored data. Finally, the section Conclusion concludes the study.

INFERENCE ON THE SMD PARAMETERS

Assume that $X_1, X_2, \ldots X_n$ is a random sample from the SMD with parameters α and β, where $X_1, X_2, \ldots X_n$ random variables imply the failure times for the n independent unit. We also denote the order statistics of the random sample $X_1, X_2, \ldots X_n$ by $X_{(1)}, X_{(2)}, \ldots X_{(n)}$. Note that we only observe the first r (before pre-decided $r < n$) order statistics of sample $X_1, X_2, \ldots X_n$ under the Type-II censoring scheme.

Now, we obtain the ML, LS, and MSP estimators of SMD parameters for Type-II censored data, which are commononly used estimators in the literature.

ML Estimators

Let $X_1, X_2, \ldots X_n$ be a random sample from SMD, and the $(n - r)$ of n observations be censored according to the Type-II censoring scheme. In this stuation, by considering

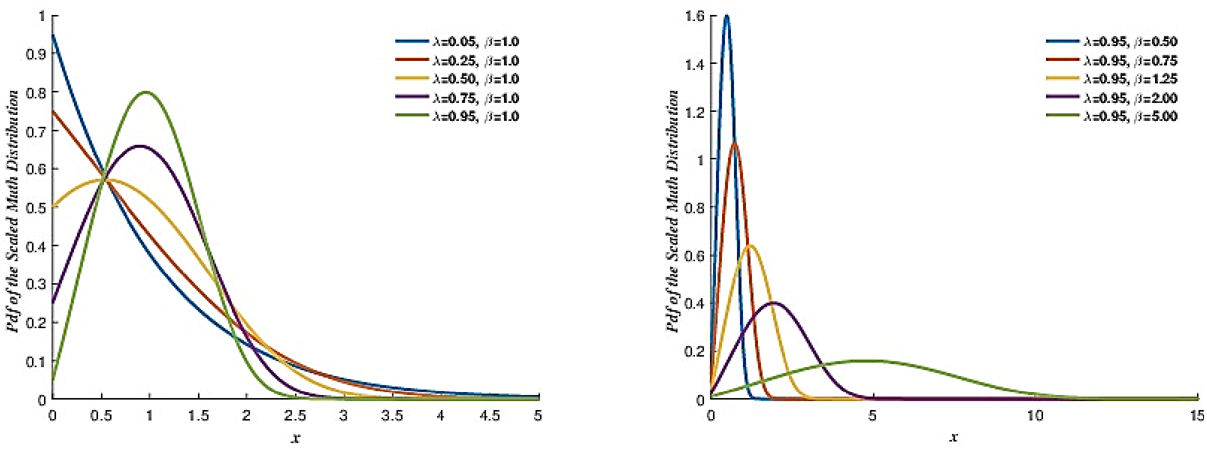

Figure 1. Pdf of the SMD for different values of the parameters λ and β.

the pdf (3) and the cdf (4), the likelihood function of the SMD with parameters α and β is immediately written as

$$L(\alpha,\beta)=\frac{n!}{(n-r)!}\prod_{i=1}^{r}\frac{1}{\beta}\left(e^{\frac{\alpha x_{(i)}}{\beta}}-\alpha\right)\exp\left(\frac{\alpha x_{(i)}}{\beta}-\frac{1}{\alpha}\left(e^{\frac{\alpha x_{(i)}}{\beta}}-1\right)\right)$$
$$\times\left[\exp\left(\frac{\alpha x_{(r)}}{\beta}-\frac{1}{\alpha}\left(e^{\frac{\alpha x_{(r)}}{\beta}}-1\right)\right)\right]^{n-r} \tag{5}$$

and the corresponding log-likelihood function is

$$ln\mathrm{L}(\alpha,\beta)=\ln\left(\frac{n!}{(n-r)!}\right)-r\ln\beta+\sum_{i=1}^{r}\ln\left(e^{\frac{\alpha x_{(i)}}{\beta}}-\alpha\right)$$
$$+\sum_{i=1}^{r}\frac{\alpha x_{(i)}}{\beta}-\frac{1}{\alpha}\sum_{i=1}^{r}\left(e^{\frac{\alpha x_{(i)}}{\beta}}-1\right) \tag{6}$$
$$+(n-r)\left[\frac{\alpha x_{(r)}}{\beta}-\frac{1}{\alpha}\left(e^{\frac{\alpha x_{(r)}}{\beta}}-1\right)\right]$$

Derivating the log-likelihood function given by (6) with respect to the parameters α and β, we have the following score functions:

$$\frac{\partial\ln L(\alpha,\beta)}{\partial\alpha}=(n-r)\left(\frac{e^{\frac{\alpha x_{(r)}}{\beta}}-1}{\alpha^2}-\frac{x_{(r)}e^{\frac{\alpha x_{(r)}}{\beta}}}{\alpha\beta}+\frac{x_{(r)}}{\beta}\right)$$
$$-\sum_{i=1}^{r}\frac{x_{(i)}\left(e^{\frac{\alpha x_{(i)}}{\beta}}\right)}{\alpha\beta}+\sum_{i=1}^{r}\frac{\left(e^{\frac{\alpha x_{(i)}}{\beta}}-1\right)}{\alpha^2} \tag{7}$$
$$+\sum_{i=1}^{r}\frac{x_{(i)}}{\beta}+\sum_{i=1}^{r}\frac{\left(x_{(i)}e^{\frac{\alpha x_{(i)}}{\beta}}-\beta\right)}{\beta\left(e^{\frac{\alpha x_{(i)}}{\beta}}-\alpha\right)}=0$$

and

$$\frac{\partial lnL(\alpha,\beta)}{\partial\beta}=-\frac{r}{\beta}-\sum_{i=1}^{r}\frac{\alpha x_{(i)}e^{\frac{\alpha x_{(i)}}{\beta}}}{\beta^2\left(e^{\frac{\alpha x_{(i)}}{\beta}}-\alpha\right)}-\sum_{i=1}^{r}\frac{\alpha x_{(i)}}{\beta^2}+\sum_{i=1}^{r}\frac{\alpha x_{(i)}e^{\frac{\alpha x_{(i)}}{\beta}}}{\alpha\beta^2}$$
$$+(n-r)\left(\frac{x_{(r)}e^{\frac{\alpha x_{(r)}}{\beta}}}{\beta^2}-\frac{\alpha x_{(r)}}{\beta^2}\right)=0 \tag{8}$$

The ML estimators of the parameters α and β are obtained from solution of the nonlinear system given by equations (7) and (8). Unfortunately, this nonlinear system cannot be solved with respect to the parameters α and β analytically. But, we can use a numerical method to obtain ML estimates of the parameters. Newton-Raphson is an iterative approach to derive the root(s) of a real-valued function using its derivative and is widely used in the literature to obtain the numerical solution of likelihood equations. The main iterative formula of the Newton-Raphson is

$$\hat{\theta}_{j+1}=\hat{\theta}_j-H^{-1}\left(\hat{\theta}_j\right)\nabla\left(\hat{\theta}_j\right), \tag{9}$$

where j shows the iteration number, $\hat{\theta}_j$ shows the estimates of parameter vector at step j, $\nabla(.)$ and $H(.)$ imply the first and second derivatives of the likelihood equations with respec to parameters, respectively. By using these notations, in here, we can easily write the statements needed to run the Newton-Raphson iterative formula as follows:

Q1

$$\hat{\theta}_j=\begin{bmatrix}\hat{\alpha}_j\\ \hat{\beta}_j\end{bmatrix}, \tag{10}$$

$$\nabla\left(\hat{\theta}_j\right)=\begin{bmatrix}\dfrac{\partial\ln L(\alpha,\beta)}{\partial\alpha}\\ \dfrac{\partial\ln L(\alpha,\beta)}{\partial\beta}\end{bmatrix}_{\alpha=\hat{\alpha}_j,\beta=\hat{\beta}_j} \tag{11}$$

$$H\left(\hat{\theta}_j\right)=\begin{bmatrix}\dfrac{\partial^2\ln L(\alpha,\beta)}{\partial\alpha^2} & \dfrac{\partial^2\ln L(\alpha,\beta)}{\partial\alpha\partial\beta}\\ \dfrac{\partial^2\ln L(\alpha,\lambda,\beta)}{\partial\alpha\partial\beta} & \dfrac{\partial^2\ln L(\alpha,\lambda,\beta)}{\partial\beta^2}\end{bmatrix}_{\alpha=\hat{\alpha}_j,\beta=\hat{\beta}_j}, \tag{12}$$

The elements of the gradient vector $\nabla(\theta)$ are as given in equations (7)–(8). The elements of the matrix $H(\theta)$, say h_{ij} $(i,j=1,2)$, are obtained as below

$$h_{11}=\frac{\partial^2 lnL}{\partial\alpha^2}=-\frac{2\sum_{i=1}^{r}\left(e^{\frac{\alpha x_{(i)}}{\beta}}-1\right)}{\alpha^3}+\frac{2\sum_{i=1}^{r}\frac{x_{(i)}e^{\frac{\alpha x_{(i)}}{\beta}}}{\beta}}{\alpha^2}$$
$$+\sum_{i=1}^{r}\left(\frac{x_{(i)}^2e^{\frac{\alpha x_{(i)}}{\beta}}}{\beta^2\left(e^{\frac{\alpha x_{(i)}}{\beta}}-\alpha\right)}-\frac{\left(\frac{x_{(i)}e^{\frac{\alpha x_{(i)}}{\beta}}}{\beta}-1\right)^2}{\left(e^{\frac{\alpha x_{(i)}}{\beta}}-\alpha\right)^2}\right)^2-\frac{\sum_{i=1}^{r}\frac{x_{(i)}^2e^{\frac{\alpha x_{(i)}}{\beta}}}{\beta^2}}{\alpha} \tag{13}$$
$$+(n-r)\left(-\frac{2\left(e^{\frac{\alpha x_{(r)}}{\beta}}-1\right)}{\alpha^3}+\frac{2x_{(r)}e^{\frac{\alpha x_{(r)}}{\beta}}}{\alpha^2\beta}-\frac{x_{(r)}^2e^{\frac{\alpha x_{(r)}}{\beta}}}{\alpha\beta^2}\right)$$

$$h_{12}=h_{21}=\frac{\partial^2 lnL}{\partial a\partial\beta}=\frac{\sum_{i=1}^{r}-\frac{\alpha x_{(i)}e^{\frac{\alpha x_{(i)}}{\beta}}}{\beta^2}}{\alpha^2}+\sum_{i=1}^{r}\left(\frac{\alpha x_{(i)}e^{\frac{\alpha x_{(i)}}{\beta}}\left(\frac{x_{(i)}e^{\frac{\alpha x_{(i)}}{\beta}}}{\beta}-1\right)}{\beta^2\left(e^{\frac{\alpha x_{(i)}}{\beta}}-\alpha\right)^2}+\frac{-\frac{\alpha x_{(i)}^2e^{\frac{\alpha x_{(i)}}{\beta}}}{\beta^3}-\frac{x_{(i)}e^{\frac{\alpha x_{(i)}}{\beta}}}{\beta^2}}{e^{\frac{\alpha x_{(i)}}{\beta}}-\alpha}\right)$$
$$-\frac{\sum_{i=1}^{r}\left(-\frac{\alpha x_{(i)}^2e^{\frac{\alpha x_{(i)}}{\beta}}}{\beta^3}-\frac{x_{(i)}e^{\frac{\alpha x_{(i)}}{\beta}}}{\beta^2}\right)}{\alpha}-\sum_{i=1}^{r}\frac{x_{(i)}}{\beta^2}+(n-r)\left(\frac{x_{(r)}^2e^{\frac{\alpha x_{(r)}}{\beta}}}{\beta^3}-\frac{x_{(r)}}{\beta^2}\right) \tag{14}$$

$$h_{22}=\frac{\partial^2 ln\mathrm{L}}{\partial\beta^2}=\sum_{i=1}^{r}\left(\frac{\alpha^2x_{(i)}^2\mathrm{e}^{\frac{\alpha x_{(i)}}{\beta}}}{\beta^4\left(\mathrm{e}^{\frac{\alpha x_{(i)}}{\beta}}-\alpha\right)}-\frac{\alpha^2x_{(i)}^2\mathrm{e}^{\frac{2\alpha x_{(i)}}{\beta}}}{\beta^4\left(\mathrm{e}^{\frac{\alpha x_{(i)}}{\beta}}-\alpha\right)^2}+\frac{2\alpha x_{(i)}e^{\frac{\alpha x_{(i)}}{\beta}}}{\beta^3\left(\mathrm{e}^{\frac{\alpha x_{(i)}}{\beta}}-\alpha\right)}\right)-\frac{\sum_{i=1}^{r}\left(\frac{\alpha^2x_{(i)}^2\mathrm{e}^{\frac{\alpha x_{(i)}}{\beta}}}{\beta^4}+\frac{2\alpha x_{(i)}\mathrm{e}^{\frac{\alpha x_{(i)}}{\beta}}}{\beta^3}\right)}{\alpha}$$
$$+\sum_{i=1}^{r}\frac{2\alpha x_{(i)}}{\beta^3}+(n-r)\left(-\frac{\alpha x_{(r)}^2\mathrm{e}^{\frac{\alpha x_{(r)}}{\beta}}}{\beta^4}-\frac{2x_{(r)}\mathrm{e}^{\frac{\alpha x_{(r)}}{\beta}}}{\beta^3}+\frac{2\alpha x_{(r)}}{\beta^3}\right)+\frac{r}{\beta^2} \tag{15}$$

Thus, by starting with an initial estimation $\hat{\theta}_0$ of the parameter vector θ, the method is repeated until the root(s) is obtained according to a predetermined convergence criterion. Then, we have the ML estimates of the parameters α and β, say $\hat{\alpha}_{\mathrm{ML}}$ and $\hat{\beta}_{\mathrm{ML}}$, from the corresponding elements of the $\hat{\theta}$ vector obtained at the last stage of the iteration.

LS Estimators

We assign this section of the paper to investigate the LS estimators of the SMD under the Type-II censored scheme. The LS estimation method was first introduced in 1988 by Swain et al. [10] for estimating the parameters of the Beta distribution at the complete data situations. Suppose X_1, X_2, $\ldots X_n$ be a random sample from any continuous distribution with cdf $F(X,.)$, and also $x_{(1)}$, $x_{(2)}$, $\ldots x_{(n)}$ be the ordered observations. In the complete data case, the LS estimators of the distributional parameters are obtained by minimizing the given below quadratic function with respect to distribution parameters.

$$Q_{complete}=\sum_{i=1}^{n}\left(F\left(x_{(i)},.\right)-P_i\right)^2, \tag{16}$$

where, $P_i=\dfrac{i}{n+1}$ is the value of the empirical cumulative distribution function corresponding to i-th observation. Thus, considering the cdf of SMD given by the equation (4) the LS estimators of the parameters α and β are obtained numerically minimizing the quadratic function $Q_{complete}$ given in the equation (16) with respect to the parameters α and β.

Now, we investigate the LS estimations of the SMD parameters for the case of Type-II censored data. Let $X_{(1)}$, $X_{(2)}$, $\ldots X_{(n)}$ be an ordered random sample from SMD, and the last $(n-r)$ of n observations be censored, namely, $X_{(1)}$, $X_{(2)}$, $\ldots X_{(r)}$ can be observed as $X_{(1)}$, $X_{(2)}$, $\ldots X_{(r)}$ and $X_{(r+1)}$, $X_{(r+2)}$, $\ldots X_{(n)}$ cannot be observed. By considering this assumption and Kaplan-Meier estimator of the emprical distribution function, we can obtain the LS estimates of the parameters α and β of SMD under the Type-II censoring scheme by numericaly minimizing the quadratic function $Q_{censored}$ given as below with respect to the α and β.

$$Q_{censored}=\sum_{i=1}^{r}\left(1-\exp\left(\frac{\alpha x_{(i)}}{\beta}-\frac{1}{\alpha}\left(e^{\frac{\alpha x_{(i)}}{\beta}}-1\right)\right)-P_i^{\star}\right)^2, \tag{17}$$

where $P_i^{\star}$ is the Kaplan-Meier estimator of the empirical distribution function. For further information about Kaplan-Meier estimator, we refer the readers to [11]. Under the Type-II censoring scheme, $P_i^{\star}$ can be easily calculate as follow:

$$P_i^{*}=1-\prod_{j\le i}\left(1-\frac{1}{n-j+1}\right), i=1,2,...,r. \tag{18}$$

MSP Estimators

In this subsection of the paper, we investigate the MSP estimator of the SMD parameters for Type-II censored data. The MSP estimators were originally studied by [9]. It is a strong alternative to the ML estimators and has got useful features such as consistency and asymptotically unbiasedness. For advanced information about MSP estimation method, we refer the readers to [12, 13, 14].

Let $X_{(1)}, X_{(2)}, \ldots X_{(n)}$ be ordered sample from the SMD with parameters α and β. By considering the notations of the [13], in the case of complete data, the MSP estimators of the SMD can be easily obtained by maximizing the utility function S given as

$$\begin{aligned} S &= \sum_{j=1}^{n+1} \ln\left[F\left(X_{(j)},\alpha,\beta\right) - F\left(X_{(j-1)},\alpha,\beta\right)\right] \\ &= \sum_{j=1}^{n+1} \ln\left[\exp\left(\frac{\alpha x_{(j-1)}}{\beta} - \frac{1}{\alpha}\left(e^{\frac{\alpha x_{(j-1)}}{\beta}} - 1\right)\right) - \exp\left(\frac{\alpha x_{(j)}}{\beta} - \frac{1}{\alpha}\left(e^{\frac{\alpha x_{(j)}}{\beta}} - 1\right)\right)\right] \end{aligned} \quad (19)$$

with respect to parameters α and β, where $F(., \alpha, \beta)$ is the cdf of SMD given by equation (4), $F(X_{(0)}, \alpha, \beta) \equiv 0$, and $F(X_{(n+1)}, \alpha, \beta) \equiv 1$.

Now, we consider the Type-II censored data case. Let $X_{(1)}, X_{(2)}, \ldots X_{(n)}$ be ordered sample taken from the SMD with parameters α and β, and and the last $(n - r)$ of n observations be censored according to Type-II censoring scheme. For obtaining the MSP estimators under censoring case, Ng et al. [15] modified the utility function S given in [13] as follow

$$\begin{aligned} S_{censored} &= \sum_{j=1}^{r+1} \ln\left[F\left(X_{(j)},\alpha,\beta\right) - F\left(X_{(j-1)},\alpha,\beta\right)\right] \\ &\quad + (n-r)\ln\left[1 - F\left(X_{(r)},\alpha,\beta\right)\right]. \end{aligned} \quad (20)$$

Hence, by considering the $S_{censored}$ utility function and cdf of the SMD, the MSP estimators of the SMD parameters for the Type-II censored data are easily obtained by maximizing the

$$\begin{aligned} S_{censored} = \sum_{j=1}^{r+1} \ln &\left[\exp\left(\frac{\alpha x_{(j-1)}}{\beta} - \frac{1}{\alpha}\left(e^{\frac{\alpha x_{(j-1)}}{\beta}} - 1\right)\right)\right. \\ &\left. - \exp\left(\frac{\alpha x_{(j)}}{\beta} - \frac{1}{\alpha}\left(e^{\frac{\alpha x_{(j)}}{\beta}} - 1\right)\right)\right] \\ &+ (n-r)\left[\frac{\alpha x_{(r)}}{\beta} - \frac{1}{\alpha}\left(e^{\frac{\alpha x_{(r)}}{\beta}} - 1\right)\right] \end{aligned} \quad (21)$$

with respect to parameters α and β.

SIMULATION EXPERIMENTS

In this section, we conduct Monte-Carlo simulation studies to investigate the estimation performances of the ML, LS and MSP estimators obtained in the previous section for both complete and Type-II censored data situations. Throughout the Monte-Carlo simulation study, we set the values of the SMD parameters to α = (0.25, 0.50, 0.75) and β = (0.5, 2.0). These randomly selected values of the α and β exemplify the various formal behaviors of SMD pdf. In each combination of the parameter values, we generate random samples of different sizes n = 30, 60, 100, 200 considering various censoring proportions, p = 0, 0.1, 0.2, 0.3, (where $p = 1 - \frac{r}{n}$, $r \le n$), from SMD distribution, and estimate the α and β parameters using the ML, LS and MSP estimators. In addition, the biases and the mean square error (MSE) values of these estimators are also calculated to clarify the estimation performances of them. The simulated results are given by Tables 1–6.

As can be seen from the simulated results given by Tables 1-6, all estimators produce quite gratifying estimations in all the combinations of the parameter values, sample sizes, and censoring proportions. One can also see from Tables 1–6 that both the biases and MSE values of all estimators gradually increase to an acceptable level as the censoring proportion p increases for all sample sizes n. Furthermore, we can conclude from the simulated results that all estimators are asymptotically unbiased and consistent because of both biases and MSE values decrease when the sample of size n increases, and that the ML and MSP estimators outperform the LS estimators with smaller biases and MSE values.

DATA ANALYSIS

In this section, we give an illustrative application on a practical dataset called the Air-condition system dataset to show data modeling with SMD for both complete and Type-II censored data situations, considering various censoring plans. The Air-condition system dataset contains 27 observations deal with times between successive failures (in hours) of the air-conditioning system of an airplane (airplane number 7913) [16]. The sorted complete data are as follows: 1, 4, 11, 16, 18, 18, 18, 24, 31, 39, 46, 51, 54, 63, 68, 77, 80, 82, 97, 106, 111, 141, 142, 163, 191, 206, 216.

Before analysis, we investigate the underlying distribution of the data. We first draw the Total Time on Test (TTT) plot of data, see [17], to decide the suitable distribution families with consistent hazard rate function. Figure 2 shows the TTT plot of the data.

From Figure 2, it is lucid that the underlying distribution of the Air-condition system dataset has an increasing hazard rate function. Thus we can propose the SMD, Weibull, gamma and log-normal distributions for modeling the data.

Table 7 shows the calculated Kolmogorov-Smirnov (K-S) statistics and the corresponding p-values considering probability distribution models SMD, Weibull, gamma, and log-normal.

According to K-S test results given by Table 7, the proposed distributions are suitable for fitting the dataset.

Table 1. Simulated results for $\alpha = 0.25$ and $\beta = 0.5$

p	n	Method	$\alpha = 0.25$		$\beta = 0.5$		p	n	Method	$\alpha = 0.25$		$\beta = 0.5$	
			Bias	MSE	Bias	MSE				Bias	MSE	Bias	MSE
0	30	ML	0.0832	0.0401	0.0004	0.0051	0.2	30	ML	0.0709	0.0544	−0.0040	0.0081
		LS	0.0410	0.0542	−0.0200	0.0057			LS	0.0036	0.0503	−0.0122	0.0084
		MSP	0.0761	0.0429	0.0009	0.0052			MSP	0.0037	0.0491	0.0179	0.0091
	60	ML	0.0252	0.0133	−0.0012	0.0027		60	ML	0.0240	0.0332	0.0002	0.0025
		LS	0.0036	0.0286	−0.0108	0.0029			LS	−0.0331	0.0405	0.0041	0.0030
		MSP	0.0240	0.0155	−0.0013	0.0028			MSP	−0.0202	0.0350	0.0137	0.0028
	100	ML	0.0129	0.0091	−0.0040	0.0015		100	ML	0.0204	0.0195	−0.0024	0.0023
		LS	−0.0090	0.0183	−0.0072	0.0019			LS	−0.0135	0.0234	−0.0010	0.0027
		MSP	0.0117	0.0097	−0.0040	0.0015			MSP	−0.0096	0.0205	0.0061	0.0025
	200	ML	0.0094	0.0066	0.0001	0.0011		200	ML	0.0000	0.0159	0.0042	0.0013
		LS	0.0033	0.0124	−0.0036	0.0013			LS	−0.0214	0.0193	0.0046	0.0016
		MSP	0.0081	0.0072	0.0001	0.0011			MSP	−0.0202	0.0167	0.0100	0.0015
0.1	30	ML	0.0641	0.0481	0.0047	0.0056	0.3	30	ML	0.0831	0.0690	−0.0081	0.0109
		LS	0.0171	0.0590	−0.0099	0.0065			LS	−0.0230	0.0675	0.0004	0.0130
		MSP	0.0100	0.0457	0.0181	0.0063			MSP	0.0061	0.0652	0.0246	0.0135
	60	ML	0.0484	0.0231	0.0045	0.0030		60	ML	0.0500	0.0405	−0.0024	0.0050
		LS	−0.0044	0.0300	0.0040	0.0033			LS	−0.0369	0.0425	0.0110	0.0053
		MSP	0.0121	0.0228	0.0123	0.0032			MSP	−0.0008	0.0398	0.0165	0.0058
	100	ML	0.0275	0.0164	−0.0044	0.0014		100	ML	0.0474	0.0276	−0.0042	0.0028
		LS	0.0048	0.0204	−0.0071	0.0017			LS	0.0033	0.0325	0.0032	0.0037
		MSP	0.0052	0.0161	0.0002	0.0014			MSP	0.0108	0.0281	0.0083	0.0032
	200	ML	0.0065	0.0098	0.0002	0.0011		200	ML	0.0230	0.0212	0.0037	0.0019
		LS	−0.0135	0.0123	0.0005	0.0013			LS	−0.0120	0.0240	0.0101	0.0024
		MSP	−0.0106	0.0104	0.0036	0.0011			MSP	−0.0020	0.0217	0.0123	0.0022

Table 2. Simulated results for $\alpha = 0.25$ and $\beta = 2$

p	n	Method	$\alpha = 0.25$		$\beta = 2$		p	n	Method	$\alpha = 0.25$		$\beta = 2$	
			Bias	MSE	Bias	MSE				Bias	MSE	Bias	MSE
0	30	ML	0.0582	0.0268	−0.0198	0.0897	0.2	30	ML	0.0520	0.0478	0.0000	0.0889
		LS	0.0234	0.0437	−0.1017	0.1030			LS	−0.0184	0.0468	−0.0295	0.1034
		MSP	0.0515	0.0304	−0.0193	0.0898			MSP	−0.0160	0.0454	0.0931	0.1095
	60	ML	0.0444	0.0163	−0.0049	0.0355		60	ML	0.0402	0.0298	0.0133	0.0616
		LS	0.0230	0.0207	−0.0368	0.0458			LS	0.0000	0.0302	0.0098	0.0679
		MSP	0.0417	0.0169	−0.0049	0.0358			MSP	−0.0041	0.0301	0.0672	0.0726
	100	ML	0.0172	0.0067	−0.0047	0.0295		100	ML	0.0214	0.0202	0.0052	0.0351
		LS	−0.0058	0.0118	−0.0224	0.0337			LS	−0.0179	0.0245	0.0185	0.0395
		MSP	0.0128	0.0070	−0.0037	0.0296			MSP	−0.0083	0.0211	0.0389	0.0393
	200	ML	0.0089	0.0061	−0.0154	0.0210		200	ML	0.0039	0.0164	−0.0205	0.0234
		LS	0.0060	0.0136	−0.0318	0.0230			LS	−0.0174	0.0193	−0.0159	0.0290
		MSP	0.0095	0.0064	−0.0158	0.0212			MSP	−0.0163	0.0171	0.0020	0.0238

(*continued*)

Table 2. Simulated results for $\alpha = 0.25$ and $\beta = 2$ (cont.)

p	n	Method	α = 0.25		β = 2		p	n	Method	α = 0.25		β = 2	
			Bias	MSE	Bias	MSE				Bias	MSE	Bias	MSE
0.1	30	ML	0.0624	0.0405	0.0013	0.0903	0.3	30	ML	0.1136	0.0645	–0.0695	0.1499
		LS	–0.0174	0.0445	–0.0292	0.1142			LS	0.0074	0.0489	–0.0400	0.1655
		MSP	0.0047	0.0378	0.0570	0.0971			MSP	0.0300	0.0547	0.0578	0.1794
	60	ML	0.0361	0.0216	0.0003	0.0423		60	ML	0.0419	0.0353	–0.0393	0.0981
		LS	0.0008	0.0265	–0.0159	0.0458			LS	–0.0105	0.0346	–0.0315	0.1003
		MSP	–0.0008	0.0221	0.0317	0.0447			MSP	–0.0106	0.0343	0.0365	0.1090
	100	ML	0.0309	0.0167	0.0119	0.0260		100	ML	0.0007	0.0316	0.0157	0.0616
		LS	0.0126	0.0192	–0.0015	0.0296			LS	–0.0440	0.0327	0.0356	0.0610
		MSP	0.0086	0.0162	0.0316	0.0266			MSP	–0.0308	0.0322	0.0624	0.0692
	200	ML	0.0039	0.0101	0.0007	0.0179		200	ML	0.0207	0.0209	0.0153	0.0366
		LS	–0.0208	0.0147	0.0043	0.0197			LS	–0.0204	0.0258	0.0489	0.0496
		MSP	–0.0133	0.0108	0.0156	0.0177			MSP	–0.0036	0.0211	0.0489	0.0409

Table 3. Simulated results for $\alpha = 0.5$ and $\beta = 0.5$

p	n	Method	α = 0.5		β = 0.5		p	n	Method	α = 0.5		β = 0.5	
			Bias	MSE	Bias	MSE				Bias	MSE	Bias	MSE
0	30	ML	0.0380	0.0233	–0.0034	0.0035	0.2	30	ML	0.0570	0.0458	–0.0032	0.0053
		LS	–0.0041	0.0361	–0.0147	0.0039			LS	–0.0282	0.0630	–0.0017	0.0059
		MSP	0.0376	0.0247	–0.0037	0.0035			MSP	0.0007	0.0552	0.0128	0.0060
	60	ML	0.0099	0.0124	0.0003	0.0018		60	ML	0.0155	0.0240	–0.0039	0.0019
		LS	–0.0147	0.0200	–0.0055	0.0021			LS	–0.0437	0.0391	–0.0010	0.0024
		MSP	0.0082	0.0133	0.0004	0.0018			MSP	–0.0171	0.0283	0.0042	0.0019
	100	ML	0.0181	0.0114	–0.0003	0.0011		100	ML	0.0105	0.0113	0.0001	0.0011
		LS	0.0062	0.0147	–0.0050	0.0012			LS	–0.0160	0.0141	–0.0013	0.0012
		MSP	0.0198	0.0112	–0.0004	0.0012			MSP	–0.0089	0.0125	0.0047	0.0012
	200	ML	–0.0049	0.0051	–0.0023	0.0007		200	ML	0.0160	0.0096	–0.0010	0.0008
		LS	–0.0081	0.0073	–0.0060	0.0008			LS	0.0001	0.0120	–0.0007	0.0009
		MSP	–0.0058	0.0051	–0.0022	0.0007			MSP	0.0034	0.0102	0.0020	0.0008
0.1	30	ML	0.0169	0.0353	–0.0039	0.0034	0.3	30	ML	0.0547	0.0638	0.0012	0.0053
		LS	–0.0474	0.0561	–0.0090	0.0043			LS	–0.0482	0.0805	0.0082	0.0057
		MSP	–0.0281	0.0418	0.0052	0.0035			MSP	–0.0150	0.0839	0.0280	0.0075
	60	ML	0.0251	0.0206	0.0053	0.0018		60	ML	0.0084	0.0324	0.0008	0.0024
		LS	–0.0114	0.0262	0.0022	0.0019			LS	–0.0626	0.0490	0.0099	0.0031
		MSP	0.0005	0.0219	0.0099	0.0019			MSP	–0.0318	0.0400	0.0141	0.0028
	100	ML	0.0232	0.0110	0.0015	0.0010		100	ML	0.0286	0.0190	0.0008	0.0014
		LS	0.0090	0.0135	–0.0025	0.0011			LS	–0.0212	0.0289	0.0092	0.0020
		MSP	0.0080	0.0112	0.0041	0.0010			MSP	0.0057	0.0212	0.0080	0.0015
	200	ML	0.0078	0.0083	0.0024	0.0009		200	ML	0.0012	0.0125	–0.0004	0.0010
		LS	–0.0053	0.0099	0.0014	0.0010			LS	–0.0202	0.0163	0.0008	0.0011
		MSP	–0.0028	0.0087	0.0043	0.0009			MSP	–0.0165	0.0149	0.0048	0.0011

Table 4. Simulated results for $\alpha = 0.5$ and $\beta = 2$

p	n	Method	$\alpha = 0.5$		$\beta = 2$		p	n	Method	$\alpha = 0.5$		$\beta = 2$	
			Bias	MSE	Bias	MSE				Bias	MSE	Bias	MSE
0	30	ML	0.0394	0.0309	0.0068	0.0419	0.2	30	ML	0.0571	0.0444	−0.0211	0.0390
		LS	0.0054	0.0380	−0.0451	0.0448			LS	−0.0306	0.0645	−0.0166	0.0468
		MSP	0.0367	0.0327	0.0065	0.0425			MSP	−0.0018	0.0556	0.0271	0.0410
	60	ML	0.0210	0.0154	0.0249	0.0200		60	ML	0.0176	0.0222	−0.0197	0.0196
		LS	0.0022	0.0233	0.0033	0.0207			LS	−0.0323	0.0272	−0.0180	0.0223
		MSP	0.0186	0.0162	0.0258	0.0205			MSP	−0.0145	0.0266	0.0046	0.0200
	100	ML	0.0148	0.0087	−0.0004	0.0105		100	ML	0.0105	0.0128	0.0012	0.0124
		LS	0.0041	0.0127	−0.0119	0.0116			LS	−0.0179	0.0156	0.0046	0.0140
		MSP	0.0141	0.0094	0.0003	0.0108			MSP	−0.0083	0.0150	0.0159	0.0131
	200	ML	0.0053	0.0051	0.0026	0.0066		200	ML	0.0046	0.0107	0.0050	0.0074
		LS	−0.0040	0.0072	−0.0050	0.0075			LS	−0.0102	0.0153	0.0050	0.0086
		MSP	0.0047	0.0055	0.0031	0.0067			MSP	−0.0077	0.0122	0.0151	0.0077
0.1	30	ML	0.0463	0.0379	−0.0165	0.0323	0.3	30	ML	0.0431	0.0530	−0.0043	0.0489
		LS	−0.0123	0.0423	−0.0453	0.0383			LS	−0.0750	0.0698	0.0306	0.0650
		MSP	0.0012	0.0414	0.0089	0.0323			MSP	−0.0304	0.0683	0.0738	0.0624
	60	ML	0.0098	0.0143	−0.0014	0.0186		60	ML	0.0414	0.0287	−0.0013	0.0259
		LS	−0.0158	0.0198	−0.0166	0.0214			LS	−0.0173	0.0406	0.0176	0.0361
		MSP	−0.0155	0.0169	0.0128	0.0189			MSP	0.0032	0.0349	0.0372	0.0310
	100	ML	0.0236	0.0104	−0.0051	0.0102		100	ML	0.0234	0.0155	−0.0261	0.0153
		LS	0.0142	0.0124	−0.0171	0.0122			LS	−0.0115	0.0232	−0.0105	0.0184
		MSP	0.0086	0.0111	0.0035	0.0103			MSP	0.0002	0.0185	−0.0038	0.0153
	200	ML	0.0088	0.0075	−0.0115	0.0072		200	ML	0.0129	0.0125	−0.0053	0.0074
		LS	−0.0029	0.0105	−0.0184	0.0080			LS	−0.0139	0.0195	0.0054	0.0109
		MSP	−0.0019	0.0084	−0.0058	0.0072			MSP	−0.0038	0.0153	0.0107	0.0078

Table 5. Simulated results for $\alpha = 0.75$ and $\beta = 0.5$

p	n	Method	$\alpha = 0.75$		$\beta = 0.5$		p	n	Method	$\alpha = 0.75$		$\beta = 0.5$	
			Bias	MSE	Bias	MSE				Bias	MSE	Bias	MSE
0	30	ML	0.0348	0.0239	0.0002	0.0023	0.2	30	ML	0.0432	0.0266	−0.0011	0.0029
		LS	0.0014	0.0293	−0.0133	0.0027			LS	−0.0123	0.0351	−0.0019	0.0033
		MSP	0.0331	0.0243	−0.0001	0.0023			MSP	0.0137	0.0299	0.0084	0.0030
	60	ML	0.0012	0.0133	0.0002	0.0013		60	ML	0.0130	0.0173	0.0006	0.0013
		LS	−0.0046	0.0151	−0.0056	0.0015			LS	−0.0222	0.0230	0.0004	0.0014
		MSP	0.0016	0.0133	0.0002	0.0013			MSP	−0.0057	0.0182	0.0054	0.0013
	100	ML	0.0091	0.0081	0.0013	0.0006		100	ML	0.0144	0.0083	0.0005	0.0008
		LS	0.0033	0.0100	−0.0032	0.0007			LS	−0.0005	0.0103	−0.0003	0.0009
		MSP	0.0075	0.0080	0.0014	0.0006			MSP	0.0041	0.0085	0.0034	0.0008
	200	ML	0.0097	0.0045	0.0010	0.0004		200	ML	0.0090	0.0059	0.0040	0.0006
		LS	0.0099	0.0060	−0.0017	0.0005			LS	0.0015	0.0068	0.0029	0.0006
		MSP	0.0102	0.0047	0.0010	0.0004			MSP	0.0025	0.0061	0.0060	0.0006

(*continued*)

Table 5. Simulated results for $\alpha = 0.75$ and $\beta = 0.5$ (cont.)

p	n	Method	$\alpha = 0.75$		$\beta = 0.5$		p	n	Method	$\alpha = 0.75$		$\beta = 0.5$	
			Bias	MSE	Bias	MSE				Bias	MSE	Bias	MSE
0.1	30	ML	0.0227	0.0292	–0.0034	0.0027	0.3	30	ML	–0.0053	0.0315	–0.0067	0.0037
		LS	–0.0233	0.0362	–0.0119	0.0030			LS	–0.0862	0.0542	–0.0041	0.0040
		MSP	–0.0015	0.0340	0.0030	0.0027			MSP	–0.0481	0.0453	0.0093	0.0042
	60	ML	0.0030	0.0142	–0.0012	0.0016		60	ML	0.0101	0.0217	0.0002	0.0014
		LS	–0.0213	0.0169	–0.0041	0.0018			LS	–0.0213	0.0282	0.0001	0.0016
		MSP	–0.0120	0.0152	0.0021	0.0017			MSP	–0.0091	0.0255	0.0076	0.0015
	100	ML	0.0110	0.0086	0.0018	0.0007		100	ML	0.0050	0.0103	0.0015	0.0009
		LS	–0.0036	0.0108	0.0011	0.0008			LS	–0.0148	0.0129	0.0024	0.0011
		MSP	0.0019	0.0089	0.0038	0.0007			MSP	–0.0068	0.0112	0.0057	0.0009
	200	ML	0.0035	0.0061	0.0002	0.0005		200	ML	0.0008	0.0071	0.0000	0.0007
		LS	–0.0058	0.0071	–0.0020	0.0005			LS	–0.0090	0.0086	0.0000	0.0007
		MSP	–0.0030	0.0063	0.0015	0.0005			MSP	–0.0074	0.0075	0.0028	0.0007

Table 6. Simulated results for $\alpha = 0.75$ and $\beta = 2$

p	n	Method	$\alpha = 0.75$		$\beta = 2$		p	n	Method	$\alpha = 0.75$		$\beta = 2$	
			Bias	MSE	Bias	MSE				Bias	MSE	Bias	MSE
0	30	ML	0.0313	0.0201	–0.0050	0.0360	0.2	30	ML	0.0007	0.0256	–0.0224	0.0439
		LS	0.0052	0.0253	–0.0494	0.0456			LS	–0.0650	0.0453	–0.0349	0.0480
		MSP	0.0323	0.0223	–0.0030	0.0371			MSP	–0.0323	0.0326	0.0177	0.0447
	60	ML	0.0241	0.0120	0.0096	0.0224		60	ML	0.0067	0.0165	–0.0112	0.0244
		LS	0.0027	0.0151	–0.0058	0.0257			LS	–0.0236	0.0222	–0.0251	0.0274
		MSP	0.0231	0.0123	0.0106	0.0225			MSP	–0.0103	0.0186	0.0087	0.0245
	100	ML	0.0181	0.0082	0.0044	0.0109		100	ML	0.0094	0.0093	–0.0081	0.0139
		LS	0.0129	0.0108	–0.0127	0.0136			LS	–0.0111	0.0118	–0.0119	0.0152
		MSP	0.0194	0.0081	0.0041	0.0111			MSP	–0.0014	0.0096	0.0035	0.0139
	200	ML	0.0012	0.0044	–0.0057	0.0068		200	ML	0.0052	0.0057	–0.0063	0.0097
		LS	–0.0096	0.0058	–0.0157	0.0076			LS	–0.0092	0.0077	–0.0084	0.0101
		MSP	0.0008	0.0045	–0.0054	0.0068			MSP	–0.0020	0.0059	0.0015	0.0097
0.1	30	ML	0.0288	0.0295	–0.0121	0.0393	0.3	30	ML	0.0317	0.0344	0.0197	0.0631
		LS	–0.0186	0.0365	–0.0496	0.0465			LS	–0.0566	0.0544	0.0326	0.0726
		MSP	0.0016	0.0324	0.0118	0.0397			MSP	–0.0089	0.0432	0.0786	0.0721
	60	ML	0.0228	0.0168	0.0043	0.0209		60	ML	0.0177	0.0205	0.0189	0.0335
		LS	–0.0063	0.0208	–0.0134	0.0198			LS	–0.0142	0.0228	0.0211	0.0355
		MSP	0.0060	0.0168	0.0164	0.0211			MSP	–0.0013	0.0229	0.0473	0.0360
	100	ML	0.0172	0.0094	–0.0034	0.0097		100	ML	0.0085	0.0138	0.0058	0.0208
		LS	0.0073	0.0112	–0.0119	0.0111			LS	–0.0158	0.0153	0.0049	0.0215
		MSP	0.0079	0.0095	0.0047	0.0097			MSP	–0.0043	0.0146	0.0225	0.0216
	200	ML	0.0114	0.0054	0.0014	0.0074		200	ML	0.0125	0.0086	0.0099	0.0105
		LS	0.0027	0.0079	–0.0095	0.0081			LS	0.0006	0.0109	0.0104	0.0123
		MSP	0.0048	0.0055	0.0067	0.0075			MSP	0.0045	0.0089	0.0208	0.0109

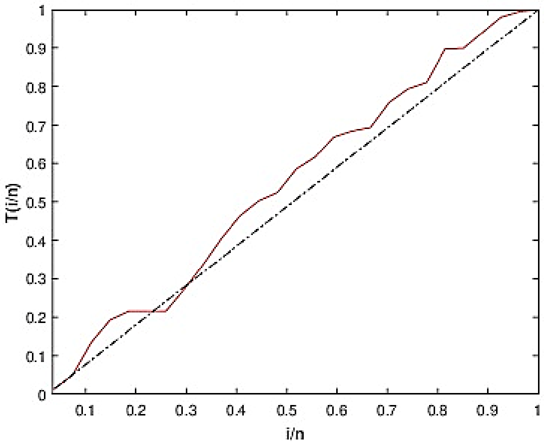

Figure 2. TTT plot of the air-condition system data.

For evaluated models, Table 8 presents the Akaike information criterion (AIC), negative log-likelihood (N.Log-lik) values, and ML estimates of the parameters for both complete and censored data cases, where we assume that the largest $n - r$, $(n - r = 0, 2, 4, 8, 10)$ observations of the data are censored.

According to the analysis results provided by Table 8, the SMD is a more appropriate model than other models for modeling the dataset with minimum AIC and N.Log-lik.

Table 7. K-S Test results for the air-conditioning system data

	Model			
	SMD	Weibull	Gamma	Log-Normal
K-S Test	0.0944	0.0883	0.0770	0.1264
p-value	0.9513	0.9721	0.9933	0.7350

values in all cases except one which the censored observation number is 2.

CONCLUSION

In this paper, the problem of estimating the parameters of the SMD under the Type II censoring scheme have been considered. We have obtained various estimators for the unknown parameters of the SMD based on the most frequently used estimation methodologies in the literature such as ML, LS, and MSE. We have also compared the estimation efficiencies of these estimators via a comprehensive simulation study. The results of the simulation study carried out on the different sample-sizes small, medium and large have revealed that all estimators obtained with the study are able to quite satisfactorily estimate the unknown parameters α and β, and that these estimators are asymptotically

Table 8. Model Comparisions for the air-conditioning system data

Model	Number of Censored Observations $(n - r)$	AIC	N.Log-lik.	ML Estimates	
SMD	0	290.89829	143.44914	$\hat{\alpha} = 0.26191$	$\hat{\beta} = 76.86239$
Weibull		291.91248	143.95624	$\hat{\theta} = 79.92387$	$\hat{\lambda} = 1.12314$
Gamma		292.18017	144.09008	$\hat{\kappa} = 1.13257$	$\hat{\eta} = 67.82347$
Log_Normal		299.24989	147.62494	$\hat{\mu} = 3.83887$	$\hat{\sigma} = 1.25646$
SMD	2	273.95361	125.76762	$\hat{\alpha} = 2.832E{-7}$	$\hat{\beta} = 82.96066$
Weibull		273.87707	125.57470	$\hat{\theta} = 83.58582$	$\hat{\lambda} = 1.02924$
Gamma		273.91031	125.64802	$\hat{\kappa} = 1.03455$	$\hat{\eta} = 80.02561$
Log_Normal		278.42077	129.60015	$\hat{\mu} = 3.88775$	$\hat{\sigma} = 1.31213$
SMD	4	252.46701	111.63635	$\hat{\alpha} = 1.016E{-7}$	$\hat{\beta} = 90.17912$
Weibull		252.72729	111.96940	$\hat{\theta} = 89.31712$	$\hat{\lambda} = 0.94426$
Gamma		252.61474	111.85511	$\hat{\kappa} = 0.94115$	$\hat{\eta} = 96.57372$
Log_Normal		256.07815	114.52497	$\hat{\mu} = 3.95097$	$\hat{\sigma} = 1.40410$
SMD	8	211.16566	89.36563	$\hat{\alpha} = 2.262E{-8}$	$\hat{\beta} = 109.16357$
Weibull		212.01071	89.63544	$\hat{\theta} = 110.88645$	$\hat{\lambda} = 0.80310$
Gamma		211.85849	89.77056	$\hat{\kappa} = 0.77727$	$\hat{\eta} = 151.76537$
Log_Normal		212.76451	89.56906	$\hat{\mu} = 4.14427$	$\hat{\sigma} = 1.63551$
SMD	10	190.59905	80.18482	$\hat{\alpha} = 9.619E{-8}$	$\hat{\beta} = 122.00044$
Weibull		191.47973	79.82207	$\hat{\theta} = 130.37211$	$\hat{\lambda} = 0.74240$
Gamma		191.45780	80.22046	$\hat{\kappa} = 0.70684$	$\hat{\eta} = 199.75125$
Log_Normal		191.32442	78.96027	$\hat{\mu} = 4.29017$	$\hat{\sigma} = 1.78316$

unbiased and consistent. Furthermore, even if the censoring proportion increases, the ML and MSP estimators satisfactorily estimate the model parameters. In addition to these results, we give an illustrative example performed on actual data to exemplify the data modeling with SMD under complete and Type-II censored data cases. The application results have shown that SMD is a possible alternative to the famous lifetime models such as Weibull, Gamma, Log-Normal, etc. Hence, we can say that the SMD is a helpful probability model for modeling the lifetime datasets in both complete and censored data cases.

ACKNOWLEDGMENTS

The authors are sincerely grateful to the handling editor Professor Gülhayat Gölbaşı Şimşek and referees for their comments and suggestions that greatly improved an early version of this paper.

AUTHORSHIP CONTRIBUTIONS

Concept: H.D.B., B.O.,C B.; Design: H.D.B., B.O., C B.; Supervision: H.D.B.; Materials: H.D.B., B.O., C B.; Data: -; Analysis: H.D.B., B.O., C B.; Literature search: H.D.B., B.O., C B.; Writing: H.D.B., B.O., C B.; Critical revision: H.D.B., B.O., C B.

DATA AVAILABILITY STATEMENT

No new data were created in this study. The published publication includes all graphics collected or developed during the study.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

REFERENCES

[1] Muth E. J. Reliability models with positive memory derived from the mean residual life function. Theory and applications of reliability 1977; 2:401–436.

[2] Jodra, P., Jimenez-Gamero, M. D., Alba-Fernandez, M. V. On the Muth distribution. Mathematical Modelling and Analysis 2015; 20(3): 291–310.

[3] Balakrishnan, N. Approximate maximum likelihood estimation of the mean and standard deviation of the normal distribution based on type II censored samples. Journal of Statistical Computation and Simulation 1989; 32(3): 137–148.

[4] Tiku, M. L., Gill, P. S. Modified maximum likelihood estimators for the bivariate normal based on Type II censored samples. Communications in Statistics-Theory and Methods 1989; 18(9): 3505–3518.

[5] Balakrishnan, N., Cutler, C. D. Maximum likelihood estimation of Laplace parameters based on Type-II censored samples. In Statistical Theory and Applications, Springer, New York, 1996; p. 145–151.

[6] Ng, H. K. T., Kundu, D., Balakrishnan, N. Point and interval estimation for the two-parameter Birnbaum–Saunders distribution based on Type-II censored samples. Computational statistics & data analysis 2006; 50(11): 3222–3242.

[7] Kundu, D., Howlader, H. Bayesian inference and prediction of the inverse Weibull distribution for Type-II censored data. Computational Statistics & Data Analysis 2010; 54(6): 1547–1558.

[8] Kundu, D., Raqab, M. Z. Bayesian inference and prediction of order statistics for a Type-II censored Weibull distribution. Journal of statistical planning and inference 2012; 142(1): 41–47.

[9] Okasha, H. M. E-Bayesian estimation for the Lomax distribution based on type-II censored data. Journal of the Egyptian Mathematical Society 2014;22(3): 489–495.

[10] Swain, J. J., Venkatraman, S., Wilson, J. R. Least-squares estimation of distribution functions in Johnson's translation system. Journal of Statistical Computation and Simulation 1988; 29(4); 271–297.

[11] Lawless, J. F. Statistical models and methods for lifetime data. John Wiley & Sons, New-York, 2011.

[12] Cheng, R. C. H., Amin, N. A. K. Estimating parameters in continuous univariate distributions with a shifted origin. Journal of the Royal Statistical Society: Series B (Methodological) 1983; 45(3): 394–403.

[13] Ranneby, B. The maximum spacing method. An estimation method related to the maximum likelihood method. Scandinavian Journal of Statistics 1984; 11(2): 93–112.

[14] Ekström, M. Alternatives to maximum likelihood estimation based on spacings and the Kullback–Leibler divergence. Journal of Statistical Planning and Inference 2008; 138(6): 1778–1791.

[15] Ng, H. K. T., Luo, L., Hu, Y., Duan, F. Parameter estimation of three-parameter Weibull distribution based on progressively type-II censored samples. Journal of Statistical Computation and Simulation 2012; 82(11): 1661–1678.

[16] Proschan F. Theoretical explanation of observed decreasing failure rate. Technometrics 1963; 5(3): 375–383.

[17] Aarset, M. V. How to identify a bathtub hazard rate. IEEE Transactions on Reliability 1987; 36(1): 106–108.