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Research Article

A DOUBLE RECEDING CONTACT PROBLEM OF A FUNCTIONALLY GRADED LAYER AND A HOMOGENEOUS ELASTIC LAYER RESTING ON A WINKLER FOUNDATION

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ABSTRACT

In this study, the plane receding contact problem of two elastic layers which one is functionally graded material (FGM) resting on a Winkler foundation is considered. The functionally graded layer is modelled as a nonhomogeneous medium with an isotropic stress-strain law. The external load is applied to the functionally graded elastic layer by means of a rigid cylindrical stamp and the homogeneous elastic layer rets on a Winkler foundation. The effect of gravity forces are neglected and only compressive normal tractions can be transmitted through the interfaces. Governing equations and mixed boundary conditions of the double receding contact problem are converted into a pair of singular integral equations by Fourier integral transforms. The system of integral equation is numerically solved by making use of appropriate Gauss-Chebyshev integration formulas for the contact pressures and contact lengths at both interfaces of contact. The main objectives of the paper are to analyze the effect of the nonhomogeneity parameter, the elastic spring constanat of Winkler foundation, the magnitude of the applied load, the radius of rigid cylindrical stamp and materials properties on the contact pressures and the contact lengths.
Keywords: Receding contact, Winkler foundation, functionally graded material.

1. INTRODUCTION

Functionally graded materials (FGMs) are increasingly used in a wide range of engineering practice and a new kind of inhomogeneous composites whose material properties over the past decade are designed for specific functions. Some potential applications of FGMs involve contact problems which is widely studied because of its applications to a great variety of importenet structures of practical interest such as foundation grillages, pavements in roads and runways, railway ballast, gas turbines, brake disks and other structures consisting of layered media.

Contact problems of two separate bodies pressed againist each other have been extensively studied previously. A contact is described to be receding that when two bodies contact each other, the applied loads cause the bodies to deform and the initial contact area changes. In this type of contact problems, the length of the contact zone and the contact pressure which is zero at the end of the contact segment are the primary unknowns.

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The receding contact problems have been widely examined during the past decades by several researchers both analytically and numerically. The numerical studies on this type contact problems include those based on the finite element method [1] – [6] or those based on the boundary element [7] – [10].

The analytical solutions of the receding contact problems between a FG layer and a rigid/ homogeneous substrate or layer are examined by El-Borgi *et al.* [11, 12], Rhimi *et al.* [13, 14], choi [15] and Jie and Xing [16]. Yan and Changwen [17] studied the receding contact problem of between a homogeneous elastic layer and a half - plane substrate coated with functionally graded materials. Comez [18] and Comez *et al.* [19] considered the frictional contact problem of a functionally graded layer and and a homogeneous orthotropic layer fully connected rigid foundation from bottom surface and loaded by a rigid cylindrical stamp from top surface. The plane frictional contact problem between a rigid cylindrical punch and a FG bilayer which the layers have different thicknesses and elastic constants is examined by Comez and Guler [20]. The contact problem of a rigid stamp and a functionally graded layer resting on a Winkler foundation is investigated by Comez [21].

As it can be seen in literature that there is limited the receding contact problems including layered media which have FG layer resting on a Winkler foundation. Therefore, the duoble receding contact problem of between a functionally graded layer and a homogeneous elastic layer resting on a Winkler foundation and loaded by a rigid cylindrical stamp is considered in this study. The functionally graded layer is modelled as a nonhomogeneous medium with an isotropic stress-strain law. The external load is applied to the functionally graded elastic layer by means of a rigid cylindrical stamp and the homogeneous elastic layer rets on a Winkler foundation. The effect of gravity forces are neglected and only compressive normal tractions can be transmitted through the interfaces. Governing equations and mixed boundary conditions of the double receding contact problem are converted into a pair of singular integral equations by Fourier integral transforms. The system of integral equation is numerically solved by making use of appropriate Gauss-Chebyshev integration formulas for the contact pressures and contact lengths at both interfaces of contact. The main objectives of the paper are to analyze the effect of the nonhomogeneity parameter, the elastic spring constanat of Winkler foundation, the magnitude of the applied load, the radius of rigid cylindrical stamp and materials properties on the contact pressures and the contact lengths.

2. FORMULATION OF THE CONTACT PROBLEM

Consider the double receding contact problem of between a functionally graded layer and a homogeneous elastic layer resting on a Winkler foundation and loaded by a rigid cylindrical stamp shown in Fig. 1. The problem is solved under the assumptions that the contact along the interfaces is frictionless, only compressive normal tractions can be transmitted through the contact interfaces and the effect of gravity forces are neglected. As it can be seen in Fig. 1 that a rigid cylindrical stamp with radius R transmits a concenrated normal force to a functionally graded layer of thickness h_1 and a homogeneous layer of thickness h_2 rests on a Winkler foundation of elastic spring constant k_0. The rigid cylindrical stamp and the FG layer are in contact with each other on the interval $(-a, a)$ and the FG layer and the homogeneous layer are in contact with each other on the interval $(-b, b)$.

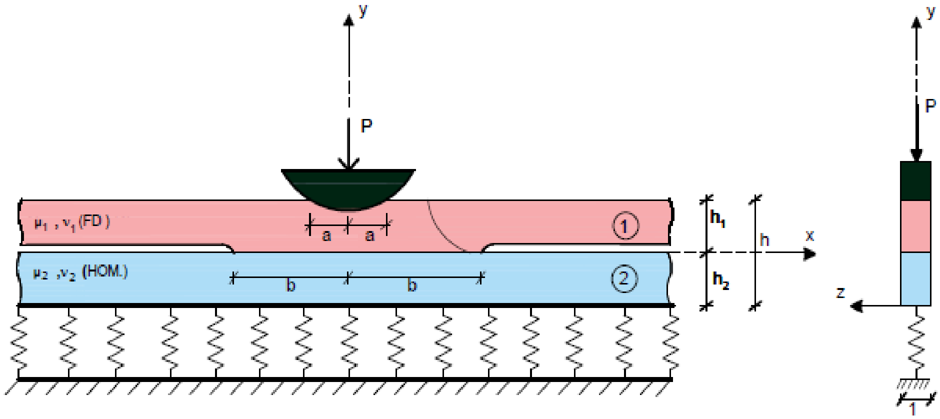

Figure 1. Geometry and load ing case of the double receding contact problem

For the graded layer, the material is modelled as a nonhomogeneous isotropic material with a gradient oriented along the y - direction. The poisson's ratio ν_1 is assumed to be a constant and the shear modulus μ_1 depends on the y - coordinate only and is expressed as the following exponential function:

$$\mu_1(y) = \mu_0 e^{\beta y}, \qquad (0 \le y \le h_1) \tag{2.1}$$

where, μ_0 is the shear modulus on the bottom surface of the FG layer (for $y = 0$) and β is nonhomogeneity parameter controlling the variation of the shear modulus in the graded medium. For the homogeneous layer, the shear modulus and the Poisson's ratio are μ_2 and ν_2, respectively.

The frictionless double receding contact problem may be solved by considering separately the graded layer and the homogeneous layer. The equations of plane problem in both domains are the equilibrium equations in the absence of body forces and the linear elastic strain law which are, respectiveley, given by

$$\frac{\partial \sigma_{ix}}{\partial x} + \frac{\partial \tau_{ixy}}{\partial y} = 0, \quad \frac{\partial \tau_{iyx}}{\partial x} + \frac{\partial \sigma_{iy}}{\partial y} = 0, \tag{2.2a,b}$$

$$\sigma_{ix} = \frac{\mu_i}{\kappa_i - 1}\left[(\kappa_i + 1)\frac{\partial u_i}{\partial x} + (3 - \kappa_i)\frac{\partial v_i}{\partial y}\right], \qquad (i = 1, 2), \tag{2.3a}$$

$$\sigma_{iy} = \frac{\mu_i}{\kappa_i - 1}\left[(3 - \kappa_i)\frac{\partial u_i}{\partial x} + (\kappa_i + 1)\frac{\partial v_i}{\partial y}\right], \qquad (i = 1, 2), \tag{2.3b}$$

$$\tau_{ixy} = \mu_i\left(\frac{\partial u_i}{\partial y} + \frac{\partial v_i}{\partial x}\right), \qquad (i = 1, 2), \tag{2.3c}$$

where, u_i and v_i $(i=1,2)$ denote the displacemet components in the x- and y-directions, respectively, $\kappa_i = 3-4\nu_i$ for plane strain and $\kappa_i = (3-\nu_i)/(1+\nu_i)$ for plane stress, in which ν_i is Poisson' ratio. σ_{ix}, σ_{iy} and τ_{ixy} represent the stress components, and μ_i is the shear modulus of the layers. The subscripts 1 and 2 denote the FG layer and the homogeneous layer, respectively.

To solve the receding contact problem between the FG layer and the homogeneous layer, the following two dimensional elasto-static Navier's equations are derived by combining Eqs. (2.1)-(2.3):

$$\left(\kappa_1+1\right)\frac{\partial^2 u_1}{\partial x^2}+\left(\kappa_1-1\right)\frac{\partial^2 u_1}{\partial y^2}+2\frac{\partial^2 v_1}{\partial x\partial y}+\beta\left(\kappa_1-1\right)\frac{\partial u_1}{\partial y}+\beta\left(\kappa_1-1\right)\frac{\partial v_1}{\partial x}=0, \quad (2.4a)$$

$$\left(\kappa_1-1\right)\frac{\partial^2 v_1}{\partial x^2}+\left(\kappa_1+1\right)\frac{\partial^2 v_1}{\partial y^2}+2\frac{\partial^2 u_1}{\partial x\partial y}+\beta\left(3-\kappa_1\right)\frac{\partial u_1}{\partial x}+\beta\left(\kappa_1+1\right)\frac{\partial v_1}{\partial y}=0, \quad (2.4b)$$

$$\left(\kappa_2+1\right)\frac{\partial^2 u_2}{\partial x^2}+\left(\kappa_2-1\right)\frac{\partial^2 u_2}{\partial y^2}+2\frac{\partial^2 v_2}{\partial x\partial y}=0,$$
$$\left(\kappa_2-1\right)\frac{\partial^2 v_2}{\partial x^2}+\left(\kappa_2+1\right)\frac{\partial^2 v_2}{\partial y^2}+2\frac{\partial^2 u_2}{\partial x\partial y}=0, \quad (2.4c,d)$$

Assuming that $x=0$ is a plane symmetry, it is sufficient to consider the problem in the region $0\le x<\infty$ only. Using Fourier transform technique and the symmetry property of the problem, the following expressions may be written, for the displacements,

$$u_i(x,y)=\frac{2}{\pi}\int_0^\infty \phi_i(\xi,y)\sin\xi x\,d\xi,\quad v_i(x,y)=\frac{2}{\pi}\int_0^\infty \psi_i(\xi,y)\cos\xi x\,d\xi,\quad (i=1,2) \quad (2.5a,b)$$

where, $\phi_i(\xi,y)$ and $\psi_i(\xi,y)$ are Fourier transform of $u_i(x,y)$ and $v_i(x,y)$ $(i=1,2)$, respectively.

By the use of Fourier integral transforms, the partial differential equations (2.4) can be reduced to a group of ordinary differential equations. If the solutions of these ordinary differential equations are sought as,

$$\phi_1=\sum_{j=1}^{4}A_j e^{n_j y},\quad \psi_1=\sum_{j=1}^{4}A_j m_j e^{n_j y} \quad (2.6a,b)$$

$$\phi_2=e^{\lambda y},\qquad \psi_2=e^{\eta y} \quad (2.6c,d)$$

the following expressions of the displacement and stress components are readily obtained as,

for the FG layer;

$$u_1(x,y)=\frac{2}{\pi}\int_0^\infty\sum_{j=1}^4 A_j e^{n_j y}\sin(\xi x)d\xi\,,\ v_1(x,y)=\frac{2}{\pi}\int_0^\infty\sum_{j=1}^4 A_j m_j e^{n_j y}\cos(\xi x)d\xi \quad (2.7a,b)$$

$$\sigma_{1x}=\frac{2\mu_0 e^{\beta y}}{\pi(\kappa_1-1)}\int_0^\infty\sum_{j=1}^4 A_j[(3-\kappa_1)m_j n_j+\xi(\kappa_1+1)]e^{n_j y}\cos(\xi x)d\xi\,, \quad (2.7c)$$

$$\sigma_{1y}=\frac{2\mu_0 e^{\beta y}}{\pi(\kappa_1-1)}\int_0^\infty\sum_{j=1}^4 A_j C_j e^{n_j y}\cos(\xi x)d\xi\,, \quad (2.7d)$$

$$\tau_{1xy}=\frac{2\mu_0 e^{\beta y}}{\pi}\int_0^\infty\sum_{j=1}^4 A_j D_j e^{n_j y}\sin(\xi x)d\xi\,, \quad (2.7e)$$

for the homogeneous layer;

$$u_2(x,y)=\frac{2}{\pi}\int_0^\infty\left[\left(B_1+B_2 y\right)e^{-\xi y}+\left(B_3+B_4 y\right)e^{\xi y}\right]\sin(\xi x)\,d\xi\,, \quad (2.8a)$$

$$v_2(x,y)=\frac{2}{\pi}\int_0^\infty\left\{\left[B_1+\left(\frac{\kappa_2}{\xi}+y\right)B_2\right]e^{-\xi y}+\left[-B_3+\left(\frac{\kappa}{\xi}-y\right)B_4\right]e^{\xi y}\right\}\cos(\xi x)\,d\xi\,, \quad (2.8b)$$

$$\frac{1}{2\mu_2}\sigma_{2x}(x,y)=\frac{2}{\pi}\int_0^\infty\left\{\left[\xi\left(B_1+B_2 y\right)-\left(\frac{3-\kappa_2}{2}\right)B_2\right]e^{-\xi y}+\left[\xi\left(B_3+B_4 y\right)+\left(\frac{3-\kappa_2}{2}\right)B_4\right]e^{\xi y}\right\}\cos(\xi x)\,d\xi\,, \quad (2.8c)$$

$$\frac{1}{2\mu_2}\sigma_{2y}(x,y)=\frac{2}{\pi}\int_0^\infty\left\{-\left[\xi\left(B_1+B_2 y\right)+\left(\frac{1+\kappa_2}{2}\right)B_2\right]e^{-\xi y}+\left[-\xi\left(B_3+B_4 y\right)+\left(\frac{1+\kappa_2}{2}\right)B_4\right]e^{\xi y}\right\}\cos(\xi x)\,d\xi\,, \quad (2.8d)$$

$$\frac{1}{2\mu_2}\tau_{2xy}(x,y)=\frac{2}{\pi}\int_0^\infty\left\{-\left[\xi\left(B_1+B_2 y\right)+\left(\frac{\kappa_2-1}{2}\right)B_2\right]e^{-\xi y}+\left[\xi\left(B_3+B_4 y\right)-\left(\frac{\kappa_2-1}{2}\right)B_4\right]e^{\xi y}\right\}\sin(\xi x)\,d\xi\,, \quad (2.8e)$$

where, A_j , B_j , C_j , and D_j $(j=1,...,4)$ are the unknown functions that will be determined from the boundary conditions of the problem. n_j $(j=1,...,4)$ satisfies the following characteristic equation,

$$n^4+2\beta n^3+(\beta^2-2\xi^2)n^2-2\xi^2\beta n+\frac{\xi^2(3\beta^2+\xi^2+(\xi^2-\beta^2)\kappa_1)}{\kappa_1+1}=0\,. \quad (2.9)$$

The roots of characteristic equation (2.9) may be given with,

$$n_1=-\frac{1}{2}\left(\beta+\sqrt{4\xi^2+\beta^2-4\xi\beta i\sqrt{\frac{3-\kappa_1}{\kappa_1+1}}}\right) \quad (2.10a)$$

$$n_2 = -\frac{1}{2}\left(\beta - \sqrt{4\xi^2 + \beta^2 - 4\xi\beta i\sqrt{\frac{3-\kappa_1}{\kappa_1+1}}}\right) \quad (2.\ 10b)$$

$$n_3 = -\frac{1}{2}\left(\beta + \sqrt{4\xi^2 + \beta^2 + 4\xi\beta i\sqrt{\frac{3-\kappa_1}{\kappa_1+1}}}\right) \quad (2.10c)$$

$$n_4 = -\frac{1}{2}\left(\beta - \sqrt{4\xi^2 + \beta^2 + 4\xi\beta i\sqrt{\frac{3-\kappa_1}{\kappa_1+1}}}\right) \quad (2.10d)$$

In equations (2.6b) and (2.7b,c), the known functions m_j may be expressed as,

$$m_j = \frac{(3\beta + 2n_j - \beta\kappa_1)\left[n_j(\beta + n_j)(\kappa_1+1) - \xi^2(\kappa_1+3)\right]}{\xi\left[4\xi^2 - \beta^2(\kappa_1-3)(\kappa_1+1)\right]},\ (j=1,...,4). \quad (2.11)$$

3. THE BOUNDARY CONDITIONS AND THE SYSTEM OF INTEGRAL EQUATIONS

The plane double receding contact problem has to be solved under the following boundary conditions:

$$\sigma_{1y}(x,h_1) = \begin{cases} -p_1(x) & ; \quad (0 \le x < a) \\ 0 & ; \quad (a \le x < \infty) \end{cases} \quad (3.1a)$$

$$\tau_{1xy}(x,h_1) = 0, \quad (0 \le x < \infty) \quad (3.1b)$$

$$\sigma_{2y}(x,0) = \begin{cases} -p_2(x) & ; \quad (0 \le x < b) \\ 0 & ; \quad (b \le x < \infty) \end{cases} \quad (3.1c)$$

$$\sigma_{1y}(x,0) = \sigma_{2y}(x,0), \quad (0 \le x < \infty) \quad (3.1d)$$

$$\tau_{1xy}(x,0) = 0, \quad (0 \le x < \infty) \quad (3.1e)$$

$$\tau_{2xy}(x,0) = 0, \quad (0 \le x < \infty) \quad (3.1f)$$

$$\sigma_{2y}(x,-h_2) = k_0 v_2(x,-h_2), (0 \le x < \infty) \quad (3.1g)$$

$$\tau_{2xy}(x,-h_2) = 0, \quad (0 \le x < \infty) \quad (3.1h)$$

$$\frac{\partial}{\partial x}\left[v_1(x,h_1)\right] = \frac{\partial F(x)}{\partial x} = f(x), (0 \le x < a) \quad (3.1i)$$

$$\frac{\partial}{\partial x}\left[v_1(x,0)-v_2(x,0)\right]=0,\ (0\leq x<b) \tag{3.1j}$$

where, a and b are the half contact lengths between the rigid stamp and the FG layer, and between the FG layer and the homogeneous layer, respectively. $p_1(x)$ and $p_2(x)$ are the primary unknown contact pressures on the contact surfaces, respectively. $F(x)$ is a known function giving the profile of the rigid stamp and k_0 is the elastic spring constant representing of the Winkler foundation.

By making use of the boundary conditions (3.1a - h), the unknwon functions A_j , B_j , C_j , and D_j appearing in equations (2.7) and (2.8) may be determined in terms of the primary unknwon contact pressures $p_1(x)$ and $p_2(x)$. Thus, the stresses and the displacements can be expressed depending on the unknown contact pressures which have not yet determined.

The solution (2.7) and (2.8) with A_j , B_j , C_j , and D_j satisfies all of the boundary conditions given by equations (3.1a - h) except the mixed conditions (3.1i) and (3.1j). The primary unknown functions $p_1(x)$ and $p_2(x)$ are determined from the mixed conditions which have not yet satisfied. After some routine manipulations and using the symmetry consideration, these mixed conditions give the following system of the integral equations depending on $p_1(x)$ and $p_2(x)$.

$$\frac{2}{\pi}\int_{-a}^{+a}p_1(t_1)\left\{M_1\left(x_1,t_1\right)-\frac{\kappa_1+1}{8}\left[\frac{1}{t_1-x_1}\right]\frac{1}{\mu_0 e^{\beta h_1}}\right\}dt_1+\frac{2}{\pi}\int_{-b}^{+b}p_2(t_2)M_2(x_1,t_2)dt_2=\frac{x}{R}, \tag{3.2a}$$

$$\frac{2}{\pi}\int_{-a}^{+a}p_1(t_1)N_1(x_2,t_1)dt_1+\frac{2}{\pi}\int_{-b}^{+b}p_2(t_2)\left\{N_2(x_2,t_2)+\frac{\kappa_1+1}{4\mu_1}\left(\frac{1}{t_2-x_2}\right)+L_1(x_2,t_2)\right\}dt_2=0, \tag{3.2b}$$

where, R is the radius of the rigid cylindrical stamp and,

$$M_1\left(x_1,t_1\right)=\int_0^\infty\left[\frac{\xi}{2\Delta_1}\left(\sum_{j=1}^{4}A_{j1}m_j e^{n_j h_1}\right)+\frac{\kappa_1+1}{8}\frac{1}{\mu_0 e^{\beta h_1}}\right]\sin\left(\xi\left(t_1-x_1\right)\right)d\xi, \tag{3.3a}$$

$$M_2\left(x_1,t_2\right)=\int_0^\infty\left(\frac{\xi}{2\Delta_1}\sum_{j=1}^{4}A_{j2}m_j e^{n_j h_1}\right)\sin\left(\xi\left(t_2-x_1\right)\right)d\xi \tag{3.3b}$$

$$N_1(x_2,t_1)=\int_0^\infty\frac{\xi}{2\mu_0 e^{\beta h_1}\Delta_1}\sum_{j=1}^{4}A_{j1}m_j\sin\left(\xi\left(t_1-x_2\right)\right)d\xi \tag{3.3c}$$

$$N_2(x_2,t_2)=\int_0^{\infty}\left[\frac{\xi}{2\mu_0\Delta_1}\sum_{j=1}^{4}A_j m_{j2}-\frac{\kappa_1+1}{4\mu_1}\right]\sin\left(\xi\left(t_2-x_2\right)\right)d\xi \tag{3.3d}$$

$$L_1(x_2,t_2)=\int_0^{\infty}\frac{1}{8\mu_2\Delta_2}\left\{-4\xi+k-4\kappa_2\xi+2\kappa_2 k+\kappa_2^2 k+4\kappa_2\xi e^{4\xi h_2}-4\kappa_2 k e^{2\xi h_2}+2\kappa_2 k e^{4\xi h_2}\right.$$
$$+16\kappa_2\xi^2 h_2 e^{2\xi h_2}-2\kappa_2^2 k e^{2\xi h_2}+\kappa_2^2 k e^{4\xi h_2}-2ke^{2\xi h_2}$$
$$\left.+ke^{4\xi h_2}+4\xi e^{4\xi h_2}+16\xi^2 h_2 e^{2\xi h_2}\right\}\sin\left(\xi\left(t_2-x_2\right)\right)d\xi . \tag{3.3e}$$

In equations (3.3a) - (3.3e), Δ_1 and Δ_2 are given in Eyuboglu [22], and

$$k=k_0/\mu_2. \tag{3.3f}$$

In addition to the contact pressures $p_1(x)$ and $p_2(x)$, the half contact lengths a and b are also unknown in the system of the integral equations (3.2a) and (3.2b). These unknowns a and b are determined from the equilibrium conditions expressed as following equations:

$$\int_{-a}^{a}p_1(t)dt=P, \qquad \int_{-b}^{b}p_2(t)dt=P. \tag{3.4a,b}$$

4. THE NUMERICAL SOLUTION OF THE SYSTEM OF INTEGRAL EQUATIONS

Designating the variables (x,t) on $y=0$ and $y=h_1$ by (x_1,t_1) and (x_2,t_2), respectively, and defining following dimensionless quantities,

$$x_1=as_1, \qquad x_2=bs_2, \tag{4.1a,b}$$

$$t_1=ar_1,\ t_2=br_2,\ \text{(4.1c,d)}\ G_1(r_1)=\frac{h_1}{P}p_1(t_1),\ G_2(r_2)=\frac{h_1}{P}p_2(t_2) \tag{4.1e,f}$$

the normalized form of the integral equations (3.2a, b) and the equilibrium conditions (3.4a, b) may be written as follows:

$$\int_{-1}^{+1}G_1(r_1)\left\{\frac{a}{h_1}M_1(s_1,r_1)-\frac{\kappa_1+1}{8e^{\beta h_1}}\left(\frac{1}{r_1-s_1}\right)\right\}dr_1+$$
$$\frac{b}{h_1}\int_{-1}^{+1}G_2(r_2)M_2(s_1,r_2)dr_2=\frac{\pi}{2}\frac{\mu_0}{P/h_1}\frac{a/h_1}{R/h_1}s_1, \tag{4.2a}$$

$$\frac{a}{h_1}\int_{-1}^{+1}G_1(r_1)N_1(s_2,r_1)dr_1+\int_{-1}^{+1}G_2(r_2)\left[\frac{b}{h_1}N_2(s_2,r_2)+\frac{\kappa_1+1}{8}\left(\frac{1}{r_2-s_2}\right)+\frac{b}{h_1}L_1(s_2,r_2)\right]dr_2=0, \tag{4.2b}$$

$$\frac{a}{h_1}\int_{-1}^{+1} G_1(r_1)dr_1 = 1, \quad \frac{b}{h_1}\int_{-1}^{+1} G_2(r_2)dr_2 = 1, \tag{4.3a,b}$$

Because of the smooth contact at the end points a and b, the contact pressures $p_1(x)$ and $p_2(x)$ are zero at the edges. Thus, the integral equations (3.2a) and (3.2b) have index -1 [23]. To insure smooth contact at the end points, let

$$G_1(r_{1i}) = w_1(r_{1i})\, g_1(r_{1i}), \quad w_1(r_{1i}) = \left(1 - r_{1i}^2\right)^{1/2}, \quad (i = 1,\dots,N), \tag{4.4a}$$

$$G_2(r_{2i}) = w_2(r_{2i})\, g_2(r_{2i}), \quad w_2(r_{2i}) = \left(1 - r_{2i}^2\right)^{1/2}, \quad (i = 1,\dots,N), \tag{4.4b}$$

where, $g_1(r_1)$ and $g_2(r_2)$ are continuous and bounded functions in the interval [-1, 1]. Using Gauss-Chebyshev integration formulas [23], equations (4.2) and (4.3) can be converted to a system of algebraic equations as follows:

$$\sum_{i=1}^{N} W_{1i}\, g_{1i}(r_{1i}) \left[\frac{a}{h_1} M_1(s_{1k}, r_{1i}) - \frac{\kappa_1 + 1}{8\, e^{\beta h_1}} \left(\frac{1}{r_1 - s_1} \right) \right]$$
$$+ \frac{b}{h_1} \sum_{i=1}^{N} W_{2i}\, g_{2i}(r_{2i}) M_2(s_{1k}, r_{2i}) = \frac{\pi}{2} \frac{\mu_0}{P/h_1} \frac{a/h_1}{R/h_1} s_{1k}, \quad (k = 1,\dots,N+1), \tag{4.5a}$$

$$\frac{a}{h_1} \sum_{i=1}^{N} W_{1i}\, g_1(r_{1i}) N_1(s_{2k}, r_{1i}) + \sum_{i=1}^{N} W_{2i}\, g_2(r_{2i}) \left[\frac{b}{h_1} N_2(s_{2k}, r_{2i}) \right.$$
$$\left. + \frac{\kappa_1 + 1}{4} \left(\frac{1}{r_2 - s_2} \right) + \frac{b}{h_1} L_1(s_{2k}, r_{2k}) \right] = 0, \quad (k = 1,\dots,N+1), \tag{4.5b}$$

$$\frac{a}{h_1} \sum_{i=1}^{N} W_{1i}\, g_1(r_{1i}) = 1, \qquad \frac{b}{h_1} \sum_{i=1}^{N} W_{2i}\, g_2(r_{2i}) = 1, \tag{4.6a,b}$$

where, r_i and s_k are the zeros of the related Chebyshev polynomials and W_i^N is the weighting constant:

$$W_i^N = \pi \left(\frac{1 - r_i^2}{N + 1} \right), \qquad (i = 1,\dots,N), \tag{4.7a}$$

$$r_i = \cos\left(\frac{i\pi}{N+1}\right), \qquad (i=1,...,N), \tag{4.7b}$$

$$s_k = \cos\left(\frac{\pi}{2}\frac{2k-1}{N+1}\right), \qquad (k=1,...,N+1). \tag{4.7c}$$

It can be demonstrated that the $(N/2+1)$-th equations in (4.5a) and (4.5b) are automatically satisfied. Thus, equations (4.5) and (4.6) give $2N+2$ algebraic equations to determine the $2N+2$ unknowns $g_1(r_i)$, $g_2(r_i)$ $(i=1,...,N)$, a and b. The system of equations are linear in $g_1(r_i)$ and $g_2(r_i)$, but nonlinear in a and b. Therefore, an iteration scheme has to be used to determine these two unknowns.

5. NUMERICAL RESULTS AND DISCUSSION

This section presents numarical results and discussion for the contact pressures and contact lengths at both interfaces of contact. Therefore, the effects of the nonhomogeneity parameter, the elastic spring constanat, the magnitude of the applied load, the radius of rigid cylindrical stamp and materials properties on the contact pressures and the contact lengths are examined in Figs. 2-9 and Tables 1-4 for various dimonsionless quantities such as R/h_1, βh_1, μ_0/μ_2 and $k = k_0/\mu_2$.

Tables 1-3 and Figures 2-5 show the variations of the half contact lengths a/h_1 and b/h_1 for various dimensionless quantities such as R/h_1, βh_1, μ_0/μ_2 and $k = k_0/\mu_2$. As it can be seen in Table 1 that both the half contact lengths a/h_1 and b/h_1 between the rigid stamp and the FG layer, and between the FG layer and the homogeneous layer, respectively, increases with increasing R/h_1 ratio.

Table 1. Variations of half contact lengths a/h_1 and b/h_1 with β for various values of rigid stamp radius ($\kappa_1 = \kappa_2 = 2,\ \mu_0/\mu_2 = 1,\ k = 1$).

βh_1	$R/h_1 = 10$		$R/h_1 = 50$		$R/h_1 = 100$	
	a/h_1	b/h_1	a/h_1	b/h_1	a/h_1	b/h_1
-1	0.151739	1.336927	0.324858	1.361852	0.450813	1.390786
0,01	0.097590	1.445485	0.221070	1.455382	0.317386	1.468406
1	0.061266	1.598321	0.144055	1.601851	0.212656	1.606983

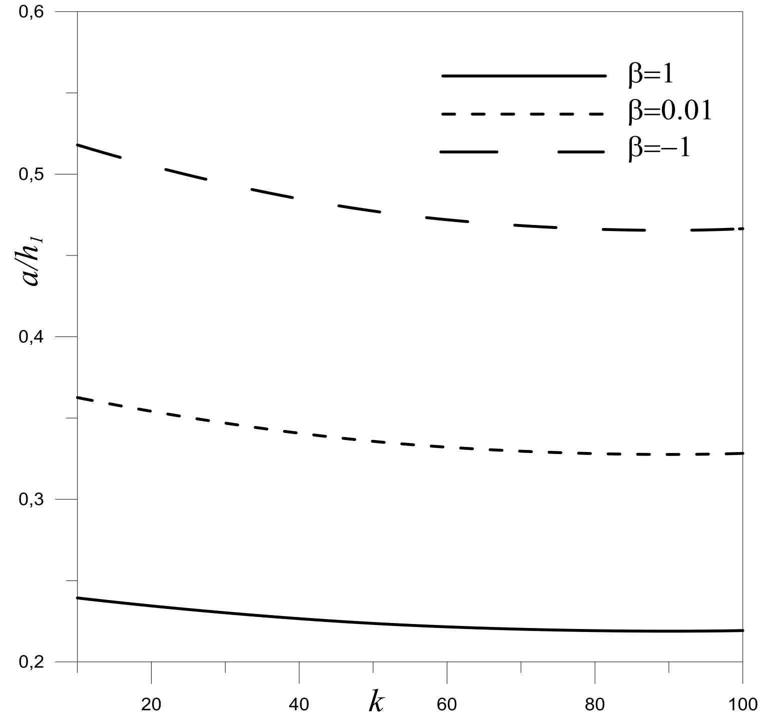

Figure 2. Variations of half contact length a/h_1 between the rigid stamp and the FG layer with elastic spring constant ratio k for various values of the nonhomogeneity parameter β ($\kappa_1=\kappa_2=2,\ \mu_0/\mu_2=1,\ R/h_1=100$).

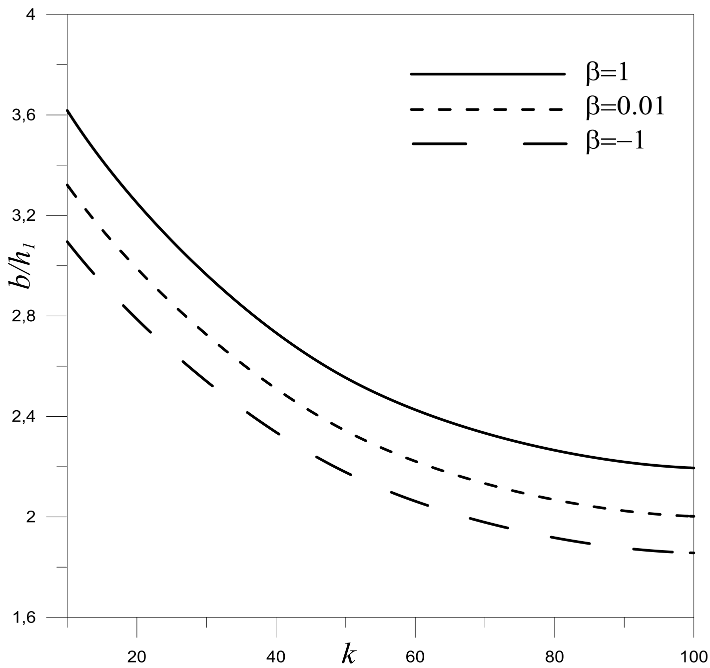

Figure 3. Variations of half contact length b/h_1 between the FG layer and the homogeneous layer with elastic spring constant ratio k for various values of nonhomogeneity parameter β ($\kappa_1=\kappa_2=2,\ \mu_0/\mu_2=1,\ R/h_1=100$).

$\beta h_1>0$ indicates that the rigidity of the top surface is higher than the bottom surface of the FG layer whereas $\beta h_1\to 0$ corresponds to a special case where the layer is homogeneous. Tables 1-3 and Figures 2-5 show that the effect of nonhomogeneity parameter βh_1 on the half contact lengths. It is clearly evident from the tables and the figures that as nonhomogeneity parameter βh_1 increases, the half contact length a/h_1 between the rigid stamp and the FG layer decreases, but the half contact length b/h_1 between the FG layer and the homogeneous layer increases. Figures 2-5 and Tables 2-3 illustrate that increasing elastic spring constant k, the half contact lengths a/h_1 and b/h_1 decrease. That increasing elastic spring constant k increases the rigidity of the foundation. That's why both the contact lengths are decreased. As it can be seen in Table 3 and Figures 4-5 that the half contact length a/h_1 between the rigid stamp and the FG layer decreases, but the half contact length b/h_1 between the FG layer and the homogeneous layer increases as shear modules ratio μ_0/μ_2 increases.

Table 2. Variations of the half contact lengths a/h_1 and b/h_1 with β for various values of k ($\kappa_1 = \kappa_2 = 2,\ \mu_0/\mu_2 = 1,\ R/h_1 = 100$).

βh_1	*k=0,02*		*k=0,1*		*k=0,2*	
	a/h_1	b/h_1	a/h_1	b/h_1	a/h_1	b/h_1
-1	0.517983	3.095658	0.477351	2.177473	0.466473	1.856735
0,01	0.362626	3.321718	0.335652	2.342216	0.328249	2.002576
1	0.239327	3.618156	0.223658	2.554162	0.219241	2.194941

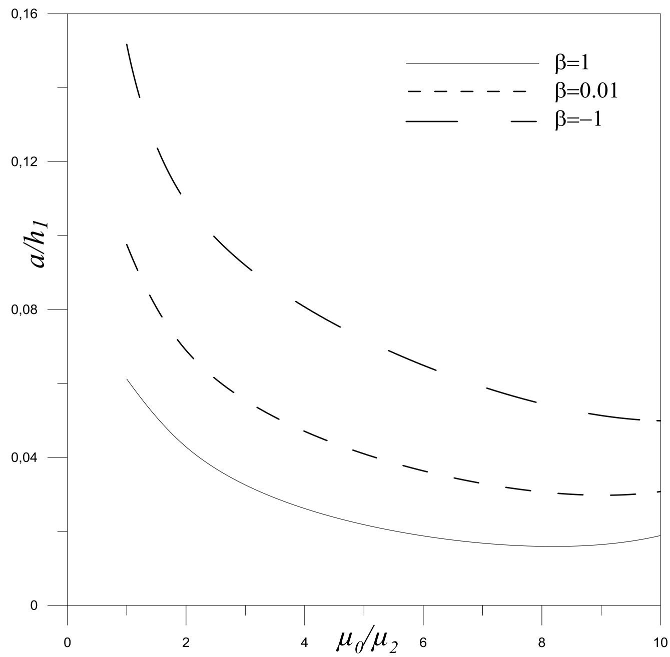

Figure 4. Variations of the half contact length a/h_1 between the rigid stamp and the FG layer with the shear modules ratio μ_0/μ_2 for various values of the nonhomogeneity parameter β ($\kappa_1 = \kappa_2 = 2,\ k = 1,\ R/h_1 = 10$).

Table 3. Variations of the half contact lengths a/h_1 and b/h_1 with β for various values of the shear modulus ratios μ_0/μ_2 ($\kappa_1=\kappa_2=2,\ k=1,\ R/h_1=10$).

βh_1	$\mu_0/\mu_2=1$		$\mu_0/\mu_2=2$		$\mu_0/\mu_2=10$	
	a/h_1	b/h_1	a/h_1	b/h_1	a/h_1	b/h_1
-1	0.151739	1.336927	0.109098	1.569336	0.049939	2.305771
0,01	0.097590	1.445485	0.068995	1.726751	0.030802	2.604380
1	0.061266	1.598321	0.042878	1.921335	0.018902	2.944393

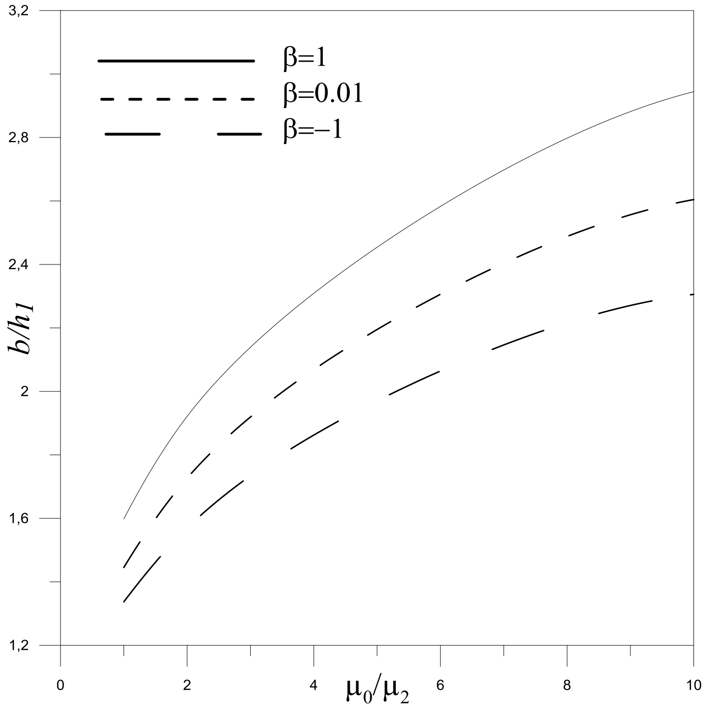

Figure 5. Variations of the half contact length b/h_1 between the FG layer and the homogeneous layer with the shear modules ratio μ_0/μ_2 for various values of the nonhomogeneity parameter β ($\kappa_1=\kappa_2=2,\ k=1,\ R/h_1=10$).

Figures 6-9 depict the distributions of the normalized contact pressures $p_1(x)/(P/h_1)$ and $p_2(x)/(P/h_1)$ for various values of the nonhomogeneity parameter β and the elastic spring constant ratio k. As it can be seen the figures that the contact pressures along the contact surfaces are always maximum at the symmetry plane. Figures 6 and 7 show that the effect of nonhomogeneity parameter β on the contact pressures $p_1(x)/(P/h_1)$ and $p_2(x)/(P/h_1)$. It is clearly evident from the figures that as the nonhomogeneity parameter β increases, i.e. the rigidity of the top surface is higher than the bottom surface of the FG layer, the maximum values of the contact pressures between the rigid stamp and the FG layer increase, but the maximum values of the contact pressures between the FG layer and the homogeneous layer decrease. Figures 8 and 9 illustrate that as the elastic spring constant increases, i.e. as the foundation becomes more rigid, the maximum values of the contact pressures between the rigid stamp and the FG layer decrease, but the maximum values of the contact pressures between the FG layer and the homogeneous layer increase. These results are consistent with Figures 2-3 and Table 2.

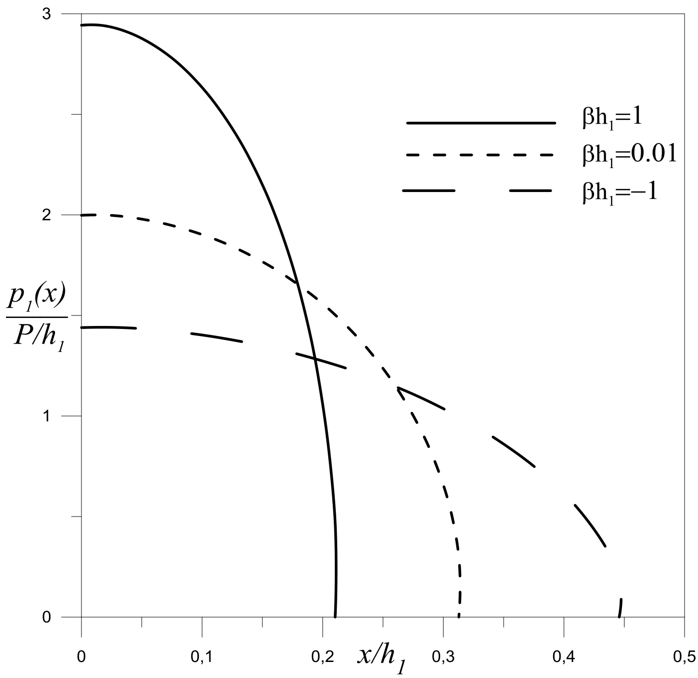

Figure 6. Contact pressures distributions under the rigid stamp for various values of nonhomogeneity parameter β

($\kappa_1 = \kappa_2 = 2,\ \mu_0/\mu_2 = 1,\ k = 1,\ R/h_1 = 100$).

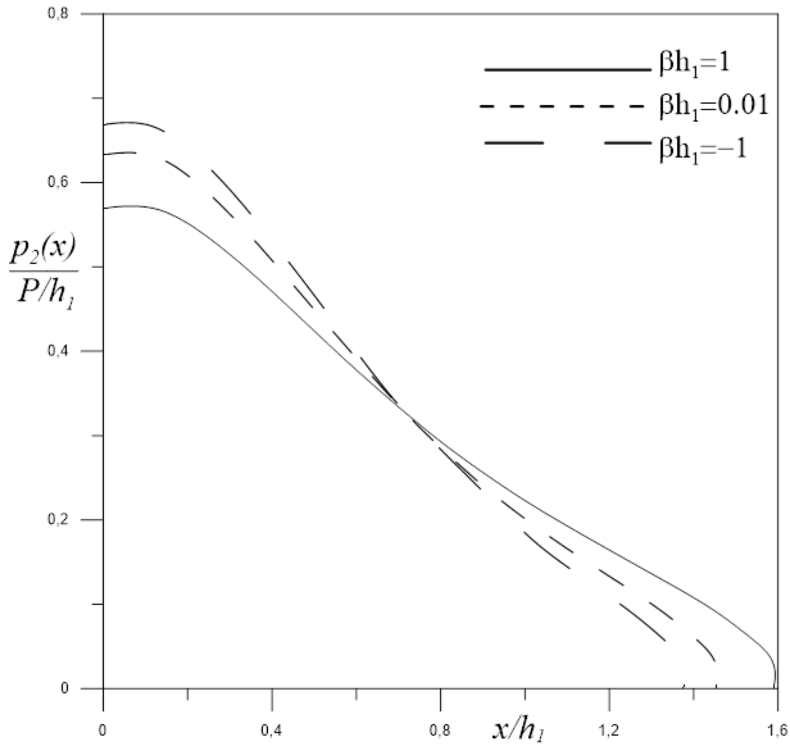

Figure 7. Contact pressures distributions between the FG layer and the homogeneous layer for various values of nonhomogeneity parameter β ($\kappa_1 = \kappa_2 = 2,\ \mu_0/\mu_2 = 1,\ k = 1,\ R/h_1 = 100$).

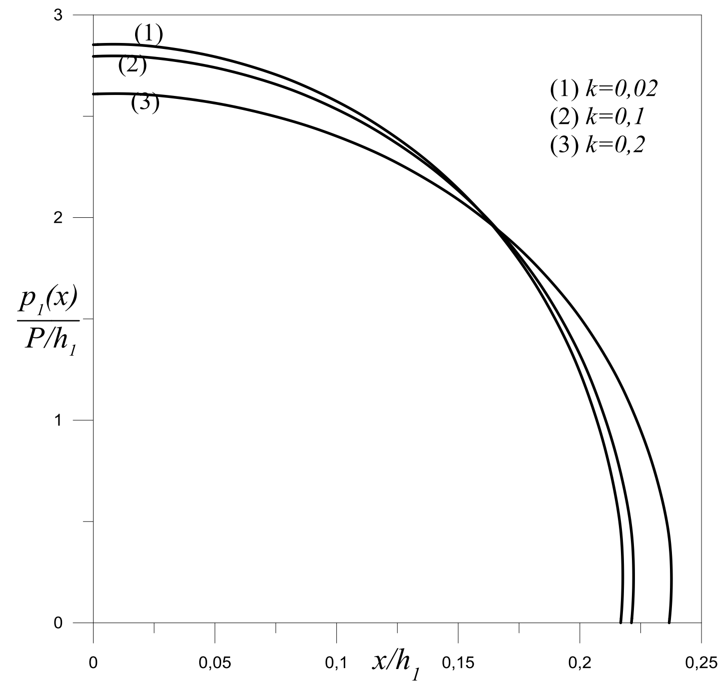

Figure 8. Contact pressures distributions under the rigid stamp for various values of the elastic spring constant ratio k ($\kappa_1 = \kappa_2 = 2,\ \mu_0 / \mu_2 = 1,\ R / h_1 = 100,\ \beta = 1$).

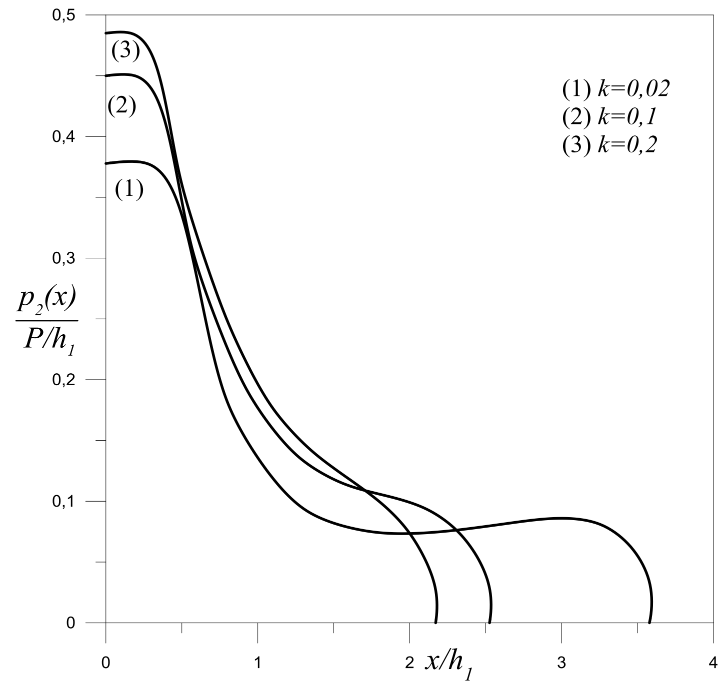

Figure 9. Contact pressures distributions between the FG layer and the homogeneous layer for various values of elastic spring constant ratio k ($\kappa_1 = \kappa_2 = 2,\ \mu_0/\mu_2 = 1,\ R/h_1 = 100,\ \beta = 1$).

Table 4. Comparison of the half contact lengths obtained from present study with Comez et al. (2003) for various dimensionless values of $\mu_2/(P/h_1)$ and R/h_1 ($\kappa_1 = \kappa_2 = 2,\ k \to \infty\ \beta h_1 \to 0$).

$\frac{\mu_2}{P/h_1}$	$R/h_1 = 10$				$R/h_1 = 500$				$R/h_1 = 1000$			
	Present study		Comez *et al.* [24]		Present study		Comez *et al.* [24]		Present study		Comez *et al.* [24]	
	a/h_1	b/h_1	a/h_1	b/h_1	a/h_1	b/h_1	a/h_1	b/h_1	a/h_1	b/h_1	a/h_1	b/h_1
100	0.2207	1.2586	0.2202	1.2534	1.5890	1.9432	1.5873	1.9547	2.1678	2.4479	2.1352	2.4081
250	0.1384	1.2490	0.1386	1.2446	1.0304	1.5839	1.0312	1.5626	1.4504	1.8606	1.4354	1.8379
500	0.0974	1.2557	0.0978	1.2421	0.7306	1.4185	0.7243	1.4025	1.0304	1.5840	1.0312	1.5641
750	0.0795	1.2450	0.0798	1.2407	0.5932	1.3584	0.5862	1.3457	0.8432	1.4719	0.8484	1.4559
1000	0.0688	1.2445	0.0690	1.2395	0.5105	1.3282	0.5044	1.3172	0.7258	1.4162	0.7243	1.4024

The double receding contact problem of two homogeneous elastic layers resting on a rigid foundation which the external load is applied to upper elastic layer by means of a rigid cylindrical stamp is studied by Comez *et al.* [24]. In the present study, when $k \to \infty$ and $\beta h_1 \to 0$, the problem becomes to the study of Comez *et al.* [24]. Comparison of the half contact lengths obtained from present study with Comez *et al.* [24] for various dimensionless values of $\mu_2 / (P / h_1)$ and R / h_1 is given in Table 4. As it can be seen from the table that the results obtained from the present study are compatible with the results given in Comez *et al.* [24].

6. CONCLUSIONS

In the present study, the double receding contact problem of between a functionally graded layer and a homogeneous elastic layer resting on a Winkler foundation and loaded by a rigid cylindrical stamp is considered. The nonhomogeneity parameter, the elastic spring constant, the radius of rigid cylindrical stamp and materials properties have an important effect on the contact pressures and the contact lengths. The contact lengths increase with the increasing the radius of rigid cylindrical stamp. The contact length between the rigid stamp and the FG layer decreases, but the contact length between the FG layer and the homogeneous layer increases as the rigidity of the FG layer decreases from the loaded top surface to the bottom surface, i.e. nonhomogeneity parameter βh_1 increases. The contact lengths at the contact surfaces decrease with increasing the elastic spring constant. As the shear modules ratio μ_0 / μ_2 increases, the contact length between the rigid stamp and the FG layer decreases, but the contact length between the FG layer and the homogeneous layer increases. The contact pressures along the contact surfaces are always maximum at the symmetry plane. The maximum values of the contact pressures between the rigid stamp and the FG layer increase, but the maximum values of the contact pressures between the FG layer and the homogeneous layer decrease as the rigidity of the top surface is higher than the bottom surface of the FG layer. If the foundation becomes more rigid, the maximum values of the contact pressures between the rigid stamp and the FG layer decrease, but the maximum values of the contact pressures between the FG layer and the homogeneous layer increase. These results are quite compatible with the literature.

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