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Research Article

SPECTRA AND PSEUDOSPECTRA OF THE DIRECT SUM OPERATORS

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ABSTRACT

In this paper, the relationships between spectrum and pseudospectrum of the direct sum of linear operators in the direct sum of Hilbert spaces and its coordinate operators have been researched. Also, the analogous relations of some numerical characteristics (spectral and pseudospectral radii) of such operators have been investigated.
Keywords: Direct sum of operators, spectrum, pseudospectrum, spectral radii.
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1. INTRODUCTION

The importance of determining spectra of linear operators does not need much explanation as such spectra are essential in quantum mechanics, both relativistic and nonrelativistic, and in general in mathematical physics. Although, we should draw attention to the importance of non-selfadjoint operators and their spectra. The growing interest in non-Hermitian quantum mechanics, nonselfadjoint differential operators and in general nonnormal phenomena has emphasized the importance of nonselfadjoint operators and pseudospectral theory. In this article, we aim to show that there are ways of computing spectrum of a linear operator. Thus, we fill in the long standing gap in the computational spectral theory. The arithmetic operations carried out are not exact, when we should do a computation of the spectrum on a computer. Hence, we may get the true solution to a slightly perturbed problem. The problem above does not occur, when we are considering the pseudospectrum in bounded case.

In 1994 [1], according to William Arveson's view on the situation of the computational spectral problem, there is a dearth of literature on this basic problem and there are no proven techniques. This is related to the general problem. If we have more structure available (for example, self-adjointness), we can say much more. On the other hand, the non-normal operators and their spectra have become increasingly important during the last two decades. The interest in non-selfadjoint differential operators [2,3], in non-Hermitian quantum mechanics [4,5] and in general non-normal phenomena [6,7] has increased. Therefore, non-selfadjoint operators and pseudospectral theory have become inevitable. This points to the importance of the general question. Moreover, it poses a slightly philosophical problem: could there be operators whose

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spectra we can never determine? Since these operators lead to serious restrictions to our possible understanding of some physical systems, they are inevitable in areas of mathematical physics. Fortunately, there are the recent developments on this subject. We can be optimistic for the future according to the results in [8]. We can refer to [9,10] for other papers related to the ideas presented in this paper.

In [11], the infinite direct sum of Hilbert spaces $H_n, n \in \mathbb{Z}$ in sense of $l_2(\mathbb{Z})$ and the direct sum of linear densely defined closed operators A_n in $H_n, n \in \mathbb{Z}$ are defined as

$$H = \overset{\infty}{\underset{n=-\infty}{\oplus}} H_n = \left\{ x = (x_n): x_n \in H_n, \quad n \in \mathbb{Z}, \quad \|x\| = \left(\sum_{n=-\infty}^{\infty} \|x_n\|^2 \right)^{1/2} < \infty \right\}$$

and

$$A: D(A) \subset H \to H,$$

$$A = \overset{\infty}{\underset{n=-\infty}{\oplus}} A_n, D(A) = \{x = (x_n) \in H: x_n \in D(A_n),\ n \in \mathbb{Z}, Ax = (A_n x_n) \in H\},$$

respectively. Recall that H is a Hilbert space with the norm induced by the inner product

$$(x, y) = \sum_{n=-\infty}^{\infty} (x_n, y_n)_n,\ x = (x_n), \quad y = (y_n) \in H,$$

where $(\cdot\,,\ \cdot)_n$ is an inner product in $H_n, n \in \mathbb{Z}$.

In the mathematical literature, it is known that the spectral theory of linear operators in direct sum of Hilbert (or Banach) spaces should be examined in order to solve many physical problems in life sciences. These and other similar reasons led to the emergence of the topic examined in current paper and the need for review.

There are numerous physical problems arising in the modelling of proceses the physics of rigid bodies, multiparticle quantum mechanics and quantum field theory. These problems support to study the theory of linear operators in the direct sum of Hilbert spaces (see [12, 13, 14, 15] and references in them).

Let $\mathcal{H}$ be a Hilbert space, $C_\infty(\mathcal{H})$ be a class of linear compact operators in $\mathcal{H}$ and $T \in C_\infty(\mathcal{H})$. The eigenvalues of the operator $(T^*T)^{1/2} \in C_\infty(\mathcal{H})$ are called the $s - numbers$ of the operator T. We shall enumerate the nonzero $s - numbers$ in decreasing order, taking account of their multiplicities, so that

$$s_n(T) = \lambda_n\big((T^*T)^{1/2}\big), n = 1,2, \ldots$$

(see [16]).

The Schatten-von Neumann operator ideals are defined as

$$C_p(\mathcal{H}) = \left\{T \in C_\infty(\mathcal{H}): \textstyle\sum_{n=1}^{\infty} s_n^p(T) < \infty\right\}, 1 \le p < \infty$$

in [17, 18].

Throughout this paper, the set of linear bounded operators and the identity operator in a Hilbert space $\mathcal{H}$ are denoted by $B(\mathcal{H})$ and I, respectively.

Our aim in this paper is to investigate the spectrum, $\epsilon -$pseudospectrum and $(n, \epsilon) -$pseudospectrum of the infinite direct sum of operators. Then, some examples are provided as an application of our results.

2. SPECTRA AND PSEUDOSPECTRA OF THE DIRECT SUM OPERATORS

In this section, we will investigate the spectrum, $\epsilon -$pseudospectrum and $(n, \epsilon) -$pseudospectrum of the infinite direct sum of operators.

The following result can be easily proved using the similar method in [19, 20, 21, 22].

Theorem 1. *In order to $A \in B(H)$ the necessary and sufficient condition is $sup_{n\in\mathbb{Z}}\|A_n\| < \infty$. Moreover, in the case that $A \in B(H)$, the norm of A is in the form $\|A\|=sup_{n\in\mathbb{Z}}\|A_n\|$.*

By the definition of compactness of an operator in [11], we have that if $A \in C_\infty(H)$, then $A_n \in C_\infty(H_n)$ for $n \in \mathbb{Z}$.

Let us prove the following result.

Theorem 2. *Let $A_n \in C_\infty(H_n)$ for $n \in \mathbb{Z}$. $A \in C_\infty(H)$ if and only if*

$$lim_{n\to\pm\infty}\|A_n\| = 0.$$

Proof. If $A \in C_\infty(H)$, the restrictions $A|_{H_-} = \bigoplus_{n=-\infty}^{1} A_n$ and $A|_{H_+} = \bigoplus_{n=1}^{\infty} A_n$ of the operator A to the linear subspaces $H_- = \bigoplus_{n=-\infty}^{1} H_n$ and $H_+ = \bigoplus_{n=1}^{\infty} H_n$ are compact operators, too. Hence, using the presented methods in [20, 21] we can obtain the following results

$$\lim_{n\to-\infty}\|A_n\| = 0 \text{ and } \lim_{n\to\infty}\|A_n\| = 0.$$

Conversely, assumed that $\lim_{n\to\pm\infty}\|A_n\| = 0$. In case when $A_n \in C_\infty(H_n)$, $n \in \mathbb{Z}$ the operator defined as

$$B_n := \bigoplus_{m=-n}^{n} A_m, B_n: H \to H, n \geq 1$$

is a compact operator. On the other hand, for each $x \in H$ and $n \geq 1$ we have

$$\|(A - B_n)x\| = \left\|\sum_{m=-\infty}^{-(n+1)} A_m x_m + \sum_{m=n+1}^{\infty} A_m x_m\right\| \leq$$
$$\left(\sup_{m\leq-(n+1)}\|A_m\| + \sup_{m\geq n+1}\|A_m\|\right)\|x\|.$$

Consequently, for any $n \geq 1$ we get

$$\|A - B_n\| \leq \sup_{m\leq-(n+1)}\|A_m\| + \sup_{m\geq n+1}\|A_m\|.$$

Since $\lim_{n\to\pm\infty}\|A_n\| = 0$, then $B_n \xrightarrow{strong} A$ as $n \to \infty$. Thus, by the important theorem of compact operators theory in [11], the operator A belongs to the class $C_\infty(H)$.

Hence, the proof of theorem is completed.

With the use of the Theorem 4.8, Theorem 4.9 and Corollary 4.11 in [21], for the singular numbers of the direct sum operator A in H we can easily obtain the following result.

Theorem 3. *Let $A_n \in C_{p_n}(H_n), 1 \leq p_n < \infty$ for $n \in \mathbb{Z}$ and $p = \sup_{n\in\mathbb{Z}} p_n < \infty$. $A \in C_p(H)$ if and only if the series*

$$\sum_{n=-\infty}^{\infty}\sum_{m=1}^{\infty} s_m^p(A_n)$$

is convergent.

Definition 1. [11] *The spectrum and resolvent sets of a linear densely defined closed operator $T \in B(\mathcal{H})$ in any Hilbert space $\mathcal{H}$ are defined as*

$$\sigma(T) := \{z \in \mathbb{C}: zI - T \text{ does not have an inverse in } B(\mathcal{H})\}$$

and

$$\rho(T) := \mathbb{C} \setminus \sigma(T),$$

respectively.

Let us prove the following assertion.

Theorem 4. *The resolvent and spectrum sets of the operator A are of the forms*

$$\rho(A) = \{z \in \cap_{n=-\infty}^{\infty} \rho(A_n): sup_{n\in\mathbb{Z}}\|R_z(A_n)\| < \infty\}$$

and

$$\sigma(A) = \cup_{n=-\infty}^{\infty} \sigma(A_n) \cup \{z \in \cap_{n=-\infty}^{\infty} \rho(A_n): sup_{n\in\mathbb{Z}}\|R_z(A_n)\| = \infty\},$$

respectively, where $R_z(\cdot)$ is the resolvent operator of an operator.

Proof. Let $z \in \rho(A)$. Then, the operator $A_z: D(A) \subset H \to H, A_z \coloneqq zI - A$ is one to one from H onto H. Hence, the restriction operator $Az|_{H_n}$ to the linear subspace $H_n, n \in \mathbb{Z}$ is one to one from H_n onto H_n, too. This means that the operator $Az|_{H_n}: D(A_n) \subset H_n \to H_n$, $Az|_{H_n} = zI - A_n, n \in \mathbb{Z}$ is boundedly solvable, i.e. $z \in \rho(A_n), n \in \mathbb{Z}$. On the other hand, since $(zI - A)^{-1} \in B(H)$ and $(zI - A)^{-1} = \bigoplus_{n=-\infty}^{\infty} (zI - A_n)^{-1}$, then $sup_{n\in\mathbb{Z}}\|R_z(A_n)\| < \infty$. Hence, we have

$$\rho(A) \subset \{z \in \cap_{n=-\infty}^{\infty} \rho(A_n): sup_{n\in\mathbb{Z}}\|R_z(A_n)\| < \infty\}.$$

On the contrary, let $z \in \cap_{n=-\infty}^{\infty} \rho(A_n)$ with the condition $sup_{n\in\mathbb{Z}}\|R_z(A_n)\| < \infty$. Thus, we get $z \in \rho(A)$. Consequently, we have

$$\rho(A) = \{z \in \cap_{n=-\infty}^{\infty} \rho(A_n): sup_{n\in\mathbb{Z}}\|R_z(A_n)\| < \infty\}.$$

Finally, we obtain

$$\sigma(A) = \cup_{n=-\infty}^{\infty} \sigma(A_n) \cup \{z \in \cap_{n=-\infty}^{\infty} \rho(A_n): sup_{n\in\mathbb{Z}}\|R_z(A_n)\| = \infty\}.$$

Remark 1. *The analogous result for the direct sum of operators in the direct sum of Hilbert (Banach) spaces in the case of $l_2(\mathbb{N})$ has been obtained by using the another methods in* [19, 20].

Let us give the definitions of ϵ −pseudospectrum and (n, ϵ) −pseudospectrum of an operator.

Definition 2. [7, 23, 24, 25] Let T be a densely defined closed operator on a Hilbert space $\mathcal{H}$ such that $\sigma(T) \neq \mathbb{C}$ and $\epsilon > 0$.

(1) The ϵ −pseudospectrum set of the operator T is defined as

$$\sigma_\epsilon(T) \coloneqq \sigma(T) \cup \{z \notin \sigma(T): \|(zI - T)^{-1}\| > 1/\epsilon\}$$

or

$$\sigma_\epsilon(T) \coloneqq \{z \in \mathbb{C}: \|(zI - T)^{-1}\| > 1/\epsilon\}.$$

(2) Let $n \in \mathbb{Z}_+$. The (n, ϵ) −pseudospectrum set of the operator T is defined as

$$\sigma_{n,\epsilon}(T) \coloneqq \sigma(T) \cup \left\{z \notin \sigma(T): \left\|(zI - T)^{-2^n}\right\|^{1/2^n} > 1/\epsilon\right\}$$

or

$$\sigma_{n,\epsilon}(T) \coloneqq \left\{z \in \mathbb{C}: \left\|(zI - T)^{-2^n}\right\|^{1/2^n} > 1/\epsilon\right\}.$$

Let us prove the following assertion.

Theorem 5. *The ϵ −pseudospectrum of the direct sum operator A in the direct sum of Hilbert spaces is of the form*

$$\sigma_\epsilon(A) = \cup_{n=-\infty}^{\infty} \sigma_\epsilon(A_n), \epsilon > 0.$$

Proof. Let $z \in \sigma_\epsilon(A)$. Hence, $\|(zI - A)^{-1}\| > 1/\epsilon$. On the other hand, by Theorem 1 we have

$$\sup_{n\in\mathbb{Z}}\|(zI - A_n)^{-1}\| > 1/\epsilon.$$

This means that there is an integer number $n_0 \in \mathbb{Z}$ such that

$$\left\|\left(zI - A_{n_0}\right)^{-1}\right\| > 1/\epsilon.$$

Thus, we have $z \in \sigma_\epsilon\left(A_{n_0}\right)$. Therefore, $z \in \cup_{n=-\infty}^{\infty} \sigma_\epsilon(A_n)$. Consequently,

$$\sigma_\epsilon(A) \subset \bigcup_{n=-\infty}^{\infty} \sigma_\epsilon(A_n).$$

On the contrary, let $z \in \bigcup_{n=-\infty}^{\infty} \sigma_\epsilon(A_n)$. Hence, there is at least one $n_0 \in \mathbb{Z}$ such that $z \in \sigma_\epsilon(A_{n_0})$.
Namely,

$$\left\|\left(zI - A_{n_0}\right)^{-1}\right\| > 1/\epsilon.$$

From this realation, we obtain

$$\|(zI - A)^{-1}\| = \sup_{n\in\mathbb{Z}}\|(zI - A_n)^{-1}\| > 1/\epsilon.$$

Hence, we get $z \in \sigma_\epsilon(A)$ and

$$\bigcup_{n=-\infty}^{\infty} \sigma_\epsilon(A_n) \subset \sigma_\epsilon(A).$$

Finally, we have

$$\sigma_\epsilon(A) = \bigcup_{n=-\infty}^{\infty} \sigma_\epsilon(A_n).$$

Hence, the proof of theorem is completed.

Let us prove the following assertion.

Theorem 6. *Let $n \in \mathbb{Z}_+$ and $\epsilon > 0$. The (n, ϵ) −pseudospectrum of the direct sum operator A in the direct sum of Hilbert spaces is of the form*

$$\sigma_{n,\epsilon}(A) = \bigcup_{m=-\infty}^{\infty} \sigma_{(n,\epsilon)}(A_m).$$

Proof. One can easily notice that for any $n \in \mathbb{Z}_+$

$$(zI - A)^{-1} = \bigoplus_{m=-\infty}^{\infty} (zI - A_m)^{-1}, z \in \rho(A).$$

From this relation, for any $n \in \mathbb{Z}_+$ we have

$$(zI - A)^{-2^n} = \bigoplus_{m=-\infty}^{\infty} (zI - A_m)^{-2^n}.$$

Therefore, for any $n \in \mathbb{Z}_+$ we get

$$\left\|(zI - A_n)^{-2^n}\right\| = \sup_{m\in\mathbb{Z}}\left\|(zI - A_n)^{-2^n}\right\|.$$

Then, for any $n \in \mathbb{Z}_+$ and $z \in \rho(A)$ we obtain

$$\left\|(zI - A)^{-2^n}\right\|^{1/2^n} = \left(\sup_{m\in\mathbb{Z}}\left\|(zI - A_m)^{-2^n}\right\|\right)^{1/2^n} = \sup_{m\in\mathbb{Z}}\left\|(zI - A_m)^{-2^n}\right\|^{1/2^n}.$$

Thus, with the use of the method in the proof of Theorem 5 we have

$$\sigma_{n,\epsilon}(A) = \bigcup_{m=-\infty}^{\infty} \sigma_{n,\epsilon}(A_m).$$

Definition 3. [11] *The spectral radius $r(T)$, the ϵ −pseudospectral radius $r_\epsilon(T), \epsilon > 0$ and the (n, ϵ) −pseudospectral radius $r_{n,\epsilon}(T), n \in \mathbb{Z}_+,\ \epsilon > 0$ of the operator $T \in B(\mathcal{H})$ in any Hilbert space $\mathcal{H}$ are defined as*

$$r(T) = \sup\{|z| : z \in \sigma(T)\},$$
$$r_\epsilon(T) = \sup\{|z| : z \in \sigma_\epsilon(T)\}$$

and

$$r_{n,\epsilon}(T) = \sup\{|z| : z \in \sigma_{n,\epsilon}(T)\},$$

respectively.

Let us prove the following assertion.

Theorem 7. *The spectral radius, the ϵ −pseudospectral radius and the (n,ϵ) −pseudospectral radius of the direct sum operator $A \in B(H)$ in the direct sum of Hilbert spaces are of the form*

$$r(A) = \sup_{n\in\mathbb{Z}} r(A_n),$$
$$r_\epsilon(A) = \sup_{n\in\mathbb{Z}} r_\epsilon(A_n)$$

and

$$r_{n,\epsilon}(A) = \sup_{m\in\mathbb{Z}} r_{n,\epsilon}(A_m),$$

respectively.

Proof. Let us prove this theorem for the ϵ −pseudospectral radius. By Theorem 5, the ϵ −pseudospectrum of the operator A is of the form

$$\sigma_\epsilon(A) = \cup_{n=-\infty}^{\infty} \sigma_\epsilon(A_n).$$

Since $\sigma_\epsilon(A_n) \subset \sigma_\epsilon(A), n \in \mathbb{Z}$, we have

$$r_\epsilon(A_n) \le r_\epsilon(A).$$

Consequently,

$$\sup_{n\in\mathbb{Z}} r_\epsilon(A_n) \le r_\epsilon(A).$$

On the contrary, in the case of $\sup_{n\in\mathbb{Z}} r_\epsilon(A_n) < r_\epsilon(A)$, we must obtain at least one element $z_* \in \sigma_\epsilon(A)$ such that

$$\sup_{n\in\mathbb{Z}} r_\epsilon(A_n) < |z_*| \le r_\epsilon(A).$$

In this case, there is an integer number $n_* \in \mathbb{Z}$ such that $z_* \in \sigma_\epsilon\left(A_{n_*}\right)$. Hence, we have

$$r_\epsilon\left(A_{n_*}\right) \le |z_*|.$$

However, this is a contradiction. Thus, it seems that

$$r_\epsilon(A) = \sup_{n\in\mathbb{Z}} r_\epsilon(A_n).$$

Similary, one can obtain that

$$r(A) = \sup_{n\in\mathbb{Z}} r(A_n) \text{ and } r_{n,\epsilon}(A) = \sup_{m\in\mathbb{Z}} r_{n,\epsilon}(A_m).$$

3. APPLICATIONS

In this section, we will provide some examples as an application of our results. Throughout this section, the circle with radius r and center z, the open region in a plane bounded by this circle are denoted by $\boldsymbol{C}(z,r)$ and $\boldsymbol{D}(z,r)$, respectively.

Example 1. In the direct sum $H \coloneqq \bigoplus_{n=-\infty}^{\infty} \mathbb{C}^2$, consider the direct sum operator $A \coloneqq \bigoplus_{n=-\infty}^{\infty} A_n$, where $A_n x_n := \lambda_n x_n, x_n, \lambda_n \in \mathbb{C}$. By Theorem 4, we obtain

$$\sigma(A) = \overline{\cup_{n=-\infty}^{\infty}\{\lambda_n\}}.$$

On the other hand, it is easy to see that

$$\|R_z(A_n)\| = \frac{1}{|z-\lambda_n|}, n \in \mathbb{Z}, z \in \rho(A_n).$$

Hence, for any $\epsilon > 0$ the inequality

$$\|R_z(A_n)\| > \frac{1}{\epsilon}$$

is equivalent to the following relation

$$|z - \lambda_n| < \epsilon.$$

Then, we have

$$\sigma_\epsilon(A_n) = \{z \in \mathbb{C}: |z - \lambda_n| < \epsilon\} = \boldsymbol{D}(\lambda_n, \epsilon)$$

and

$$\sigma_{m,\epsilon}(A_n) = \boldsymbol{D}(\lambda_n, \epsilon),\ n \in \mathbb{Z}, m \in \mathbb{Z}_+.$$

Thus, by Theorem 5 and 6 we have

$$\sigma_\epsilon(A) = \cup_{n=-\infty}^{\infty} \boldsymbol{D}(\lambda_n, \epsilon)$$

and

$$\sigma_{m,\epsilon}(A) = \boldsymbol{D}(\lambda_n, \epsilon), m \in \mathbb{Z}_+,$$

respectively.

Example 2. In the direct sum $H \coloneqq \bigoplus_{n=-\infty}^{\infty} \mathbb{C}^2$, consider the direct sum operator $A \coloneqq \bigoplus_{n=-\infty}^{\infty} A_n$, where $A_n = \begin{pmatrix} 0 & 0 \\ 1/n & 0 \end{pmatrix}$, $A_n: \mathbb{C}^2 \to \mathbb{C}^2, n \in \mathbb{Z} \setminus \{0\}$ and $A_0 = 0, A_0: \mathbb{C}^2 \to \mathbb{C}^2$. In this case, it is clear that

$$A_n^2 = 0, \|A_n\| = \frac{1}{n}, n \in \mathbb{Z} \setminus \{0\}$$

and

$$A \in C_\infty(H), A^2 = 0, \|A\| = \sup_{n \neq 0} \frac{1}{n} = 1.$$

Then, for $n \in \mathbb{Z} \setminus \{0\}$ and $\lambda \neq 0$ we get

$$\|R_\lambda(A_n)\| = sup\left\{\frac{1}{n}, \frac{1}{|\lambda|}\right\}$$

and

$$\|R_\lambda(A_0)\| = \frac{1}{|\lambda|}.$$

Consequently, by Theorem 4 we obtain

$$\sigma(A) = \{0\}.$$

Since $A^2 = 0$, then by Theorem 5 for each $\epsilon > 0$

$$\sigma_\epsilon(A) = \boldsymbol{D}\left(0, \sqrt{\epsilon^2 + \epsilon}\right).$$

On the other hand, for any $n \in \mathbb{Z}$ and $m \in \mathbb{Z}_+$, we have

$$(zI - A_n)^{2^m} = z^{2^m} - 2^m z^{2^m - 1} A_n.$$

Hence, for any $z \in \rho(A_n)$ and $\epsilon > 0$ we get

$$\left\|\left((zI - A_n)^{2^m}\right)^{-1}\right\| = |z|^{1-2^m} \|(zI - 2^m A_n)^{-1}\| > \frac{1}{\epsilon^{2^m}}.$$

For each $z \in \boldsymbol{C}(0, r)$ we have

$$\|(zI - 2^m A_n)^{-1}\| > \frac{1}{r^{1-2m}\epsilon^{2m}},\ n \in \mathbb{Z}, m \in \mathbb{Z}_+.$$

Thus, by Property 1.1 in [11] we obtain

$$\sigma_{r^{1-2m}\epsilon^{2m}}(2^m A_n) = 2^m \sigma_{\frac{r\epsilon^{2m}}{(2r^2)^m}}(A_n).$$

Consequently, we establish

$$\sigma_{m,\epsilon}(A_n) = 2^m \sigma(A) = 2^m \cup_{r>0} \sigma_{\frac{r\epsilon^{2m}}{(2r^2)^m}}(A_n).$$

Therefore, by Theorem 6 we have

$$\sigma_{m,\epsilon}(A) = 2^m \cup_{n=-\infty}^{\infty} \cup_{r>0} \sigma_{\frac{r\epsilon^{2m}}{(2r^2)^m}}(A_n).$$

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