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Research Article

ESTIMATION OF PARAMETER FOR INVERSE RAYLEIGH DISTRIBUTION UNDER TYPE-I HYBRID CENSORED SAMPLES

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ABSTRACT

In reliability analysis, censoring is often preferred due to cost and time restrictions. It is well-known that a hybrid censoring scheme is a mixture of conventional type-I and type-II censoring schemes. In this study, we deal with the problem of the point and interval estimation of parameter for inverse Rayleigh distribution based on the type-I hybrid censored samples. We consider the method of maximum likelihood to estimate the parameter of inverse Rayleigh distribution by using the expectation maximization algorithm. Furthermore, the approximate confidence interval based on the maximum likelihood estimator is obtained by using Fisher information. We also performed a Monte Carlo simulation study to evaluate the performance of the maximum likelihood estimator under hybrid censoring for different sizes of samples.
Keywords: Approximate confidence interval, expectation maximization algorithm, inverse Rayleigh distribution, maximum likelihood estimation, type-I hybrid censored samples.
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1. INTRODUCTION

Life test experiments are carried out under various restrictions like cost and time. Considering these restrictions the failure times of complete components tested are generally not observed. Such experiments lead to a censored samples. Type-I and type-II censoring scheme are the most widely used censoring schemes in the literature and in real world. It is extensively studied new censoring schemes by many authors recently. One of these censoring schemes is hybrid censoring (HC) scheme. Since a HC scheme is a mixture of Type-I and Type-II censoring schemes, it has a more flexible structure than the other censoring schemes. A HC sampling scheme can be summarized as follows:

Consider the lifetime test which includes n identical components. Life test is terminated at a pre-specified time T in the conventional Type-I censoring scheme. On the other hand it is completed when a pre-specified number of failures $r(r < n)$ has occurred in the conventional Type-II censoring scheme. In Type-I censoring, there may be a few failures or no failure until

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pre-specified time T. In Type-II censoring, experiment time can be very long since the time of r^{th} failure is unknown.

Consider the following life test experiment which includes n components. Suppose that the lifetime of components are independent and identically distributed (iid) random variables. Let ordered lifetimes of these components are represented by $X_1, X_2 ..., X_n$. Life test experiment continues during to pre-specified time T or during to a pre-specified number of failure r. The lifetime experiments are terminated in pre-specified $T_1^* = \min\left(X_{r:n}, T\right)$. The HC scheme was proposed by Epstein [1]. Epstein {1] examined the point and interval estimation issues for exponential distribution under a HC sample. Many authors have considered the problem of point and interval estimation for different distributions based on HC. Some of these papers are summarized as follows in Table 1.

Table 1. Some studies based on the HC

Base Distribution	Author
Weibull	Kundu [2]
Generalized Exponential	Kundu and Pradhan [3]
Log-Normal	Dube et al. [4]
Burr XII	Rastogi and Triphati [5]
Chen	Rastogi and Tripathi [6]
Birnbaum-Saunders	Balakrishnan and Zhu [7]
Exponential,Weibull, Log-Normal	Balakrishnan and Kundu [8]
Weighted Exponential	Kohansal et al. [9]
Burr-X	Rabie and Li [10]
Kumaraswamy	Sultana et al. [11]

The inverse Rayleigh distribution (IRD) has been used in many studies including lifetime experiments. It is stated that the distribution life of many experimental units is approximated by IRD. The probability density function (pdf) and cumulative distribution function (cdf) for the IRD is given by

$$f\ x\ = \left(\frac{2\theta}{x^3}\right)\exp\left(-\frac{\theta}{x^2}\right), \tag{1}$$

$$F\ x\ = \exp\left(-\frac{\theta}{x^2}\right), \tag{2}$$

respectively, where $\theta > 0$ is a scale parameter and $x > 0$. There are many studies about IRD. Some of these studies can be listed as follows. The estimation of IRD via the maximum likelihood method is given in [12]. Confidence Interval (CI) approacah and testing of hypotheses for IRD are also given in [12]. Gharraph [13] obtained some properties for IRD such as harmonic mean, geometric mean, mode and median. El-Helbawy and Abdel-Monem [14] studied the bayesian estimation of parameter for IRD.

The main aim of this study is to obtain maximum likelihood estimator the (MLE) and approximate CI based on MLE of the unknown parameter for IRD via expectation maximization (EM) algorithm under Type-I HC scheme. Thus, we provide a new extension of estimation of parameter for IRD by HC scheme. This study is organized as follows: The MLE of the unknown parameter of IRD based on the HC is obtained via the EM algorithm in Section 2. The approximate CI of MLE is given in Section 3. We perform a simulation study to exhibit the performance of this estimator in terms of bias and mean square error (MSE) in Section 4. The study is ended by giving the conclusions in Section 5.

2. THE MAXIMUM LIKELIHOOD ESTIMATION VIA EM ALGORITHM UNDER HYBRID CENSORING SCHEME

Suppose that, $(X_{1:n}, X_{2:n}, \ldots, X_{n:n})$ be the ordered lifetime of n independent components taken from IRD. Under the HC scheme, lifetime experiment is completed at a random time $T_1 = \min\ X_{r:n}, T$. The observed samples are given as follow for two conditions:

$$\begin{cases} I: & X_{(1:n)}, X_{(2:n)}, \ldots, X_{(r:n)}, \quad if\ X_{(r:n)} < T \\ II: & X_{(1:n)}, X_{(2:n)}, \ldots, X_{(r:n)}, \quad if\ m < r, X_{(m+1):n} > T, \end{cases} \tag{3}$$

under Eq. (3), the likelihood of θ can be given as follow

$$\begin{cases} I: & L(\theta) \propto \prod_{i=1}^{r} f(x_{i:n})[1 - F(x_{r:n})]^{(n-r)} \\ II: & L(\theta) \propto \prod_{i=1}^{m} f(x_{i:n})[1 - F(T)]^{(n-m)}. \end{cases} \tag{4}$$

Utilizing Eq. (3) and Eq. (4), the likelihood function for IRD can be obtained as follows:

$$L(\theta) \propto \prod_{i=1}^{d} \left(\frac{2\theta}{x_{i:n}^{3}} \right) exp(-\frac{\theta}{x_{i:n}^{2}})[1 - \exp\left(-\theta / c^2\right)]^{n-d}, \tag{5}$$

where d and c are as follows,

$$d = \begin{cases} r & for\ case\ I \\ m & for\ case\ II \end{cases} \quad c = \begin{cases} x_{r:n} & for\ case\ I \\ T & for\ case\ II \end{cases} \tag{6}$$

The log-likelihood function of Eq. (5) is given by

$$L(\theta) \propto d \log\left(2\theta\right) - 3\sum_{i=1}^{d} \log\left(x_{i:n}\right) - \theta \sum_{i=1}^{d} x_{i:n}^{-2} + \left(n-d\right) \log\left(1 - \exp\left(-\frac{\theta}{c^2}\right)\right) \tag{7}$$

By using equation Eq. (7), likelihood equation for θ parameter is found to be

$$\frac{\partial \ell(\theta)}{\partial \theta} = \frac{2}{2\theta} - \sum_{i=1}^{d} x_{i:n}^{-2} - (n-d) \frac{\exp\left(-\theta / c^2\right)}{c^2\left(1 - \exp\left(-\theta / c^2\right)\right)} = 0 \tag{8}$$

Since the MLE of θ can not be obtained in closed form, some numerical methods are needed to calculate this estimate. Dempster et al. [15] suggested the EM algorithm to obtain the MLE in case of censoring or incomplete observations. Assume that $\mathbf{Y} = (X_{1:n}, X_{2:n}, \ldots, X_{d:n})$ and $\mathbf{Z} = (Z_1, Z_2, \ldots, Z_{n-d})$ demonstrate observed data and censored or missing data, respectively. Then, $\mathbf{W} = (X, Z)$ represents complete data. The likelihood function for complete data can be given as follows:

$$\begin{aligned} L_c(\mathbf{W};\theta) &= \prod_{i=1}^{d} f\left(x_{i:n}\right) \prod_{i=1}^{n-d} f\left(z_i\right) \\ &= \prod_{i=1}^{d} 2\theta x_{i:n}^{-3} \exp\left(-\theta / x_{i:n}^{2}\right) \times \prod_{i=1}^{n-d} 2\theta z_{i:n}^{-3} \exp\left(-\theta / z_{i:n}^{2}\right). \end{aligned} \tag{9}$$

The log-likelihood function for complete data is written as follows.

$$\log\left(L_c(\mathbf{W};\theta)\right) = n\log(2\theta) - 3\sum_{i=1}^{d}\log\left(x_{i:n}\right) - \theta\sum_{i=1}^{d}\left(x_{i:n}^{-2}\right) -3\sum_{i=1}^{n-d}\log\left(z_{i:n}\right) - \theta\sum_{i=1}^{n-d}\left(z_{i:n}^{-2}\right) \quad (10)$$

The E-step of EM algorithm requires the calculation of conditional expected value, $E[LogL_c(\mathbf{W};\theta) \mid \mathbf{Y}]$, which is equal to the pseudo log-likelihood $L_s(\theta)$, where it is given by

$$\log\left(L_c(\mathbf{W};\theta)\right) = n\log(2\theta) - 3\sum_{i=1}^{d}\log\left(x_{i:n}\right) - \theta\sum_{i=1}^{d}\left(x_{i:n}^{-2}\right) -3\sum_{i=1}^{n-d}E\left(\log(Z_i) / Z_i > c\right) - \theta\sum_{i=1}^{n-d}E\left(Z_i^{-2} / Z_i > c\right) \quad (11)$$

Theorem: Let $Y_1 = y_1,...,Y_d = y_d < c$, the conditional pdf of Z_i for $i = 1,...,(n-d)$ is as follows [3,16].

$$f_{\mathbf{Z}/Y}(Z_i / Y_1 = y_1,...,Y_d = y_d < c) = \frac{f_{IR}(z_i;\theta)}{(1 - F_{IR}(c;\theta))}; \quad z_i > c \quad (12)$$

where c is defined in Eq. (5). By using Theorem 1, conditional expected values in Eq. (11) are obtained as follows,

$$E\left(\log(z_i)/z_i > c\right) = \frac{2\theta}{(1-F_{IR}(c;\theta))}\int_c^{\infty}\frac{\log(x)exp(-\frac{\theta}{x^2})}{x^3}dx \quad (13)$$

$$E\left(Z_i^{-2}/z_i > c\right) = \frac{2\theta}{(1-F_{IR}(c;\theta))}\int_c^{\infty}\frac{exp(-\frac{\theta}{x^2})}{x^5}dx \quad (14)$$

M-step includes maximization of $\log\ L_c(\mathbf{W};\theta)$ in Eq. (11). Let θ^k be the current estimation of parameter θ in k^{th} iteration. By substituting the Eqs. (12) and (13) into $\log\ L_c(\mathbf{W};\theta)$ in Eq.(14), the following equation is obtained

$$g(\theta) = n\log(2\theta) - 3\sum_{i=1}^{d}\log(x_{i:n}) - \theta\sum_{i=1}^{d}\left(x_{i:n}^{-2}\right) - 3(n-d)A\left(c;\theta^{(k)}\right) - \theta(n-d)B\left(c;\theta^{(k)}\right). \quad (15)$$

By using the fixed point algorithm, we first obtained a^{k+1} from Eq. (14).

$$h(\theta) = \theta$$

$h(\theta)$ function is given by

$$h(\theta) = \frac{n}{\sum_{i=1}^{d}\frac{1}{x^2} + (n-d)B\left(c;\theta^k\right)} \quad (16)$$

3. APPROXIMATE CONFIDENCE INTERVAL

In this section, we describe approximate CI using the observed Fisher information (FI) suggested by [17]. The FI of censored observation is indicated as follows,

$$I_{W/Y}(\theta) = -(n-d)E\left[\frac{\partial^2 \log f_z(\mathbf{Z}/Y,\boldsymbol{\theta})}{\partial \boldsymbol{\theta}^2}\right]. \tag{17}$$

Then, observed FI is obtained by;

$$I_{\mathbf{X}}(\theta) = I_{\mathbf{W}}(\theta) - I_{\mathbf{W}|Y}(\theta), \tag{18}$$

where $I_{\mathbf{W}}(\theta)$ are obtained as follows,

$$I_{\mathbf{W}}(\theta) = \frac{n}{\theta^2}, \tag{19}$$

and the $I_{\mathbf{W}/\mathbf{Y}}(\theta)$ can be written as;

$$I_{\mathbf{W}|Y}(\theta) = (n-d)\left\{\frac{1}{\theta^2} + \frac{\exp\left(-\theta/c^2\right)}{c^4\left(1-\exp\left(-\theta/c^2\right)\right)^2}\right\}. \tag{20}$$

The CI based on parameter always can not always be obtained in closed form for any distribution. However, it is possible to derive approximate CI using the large-sample approach to obtain approximate CI. The large sample approach is based on the following procedures. The asymptotic distribution $\sqrt{n}\left(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}\right)$ is given by,

$$\sqrt{n}\left(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}\right) \rightarrow N\left(0, I_Y^{-1}(\theta)\right), \tag{21}$$

where $I_X^{-1}(\boldsymbol{\theta})$ is the inverse of $I_X(\boldsymbol{\theta})$. The above approaches are used the approximate CIs of θ parameter. The $1-\eta$ 100% approximate CI of the θ parameter is obtained as in Eq. (22).

$$P\left(\hat{\theta} - z_{\eta/2}\sqrt{Var(\hat{\theta})} < a < \hat{\theta} + z_{\eta/2}\sqrt{Var(\hat{\theta})}\right) = 1-\eta. \tag{22}$$

4. SIMULATION STUDY

We provide a simulation study to examine the MSE, bias, length, coverage probability (CP) of MLE of the parameter of IRD under Type-I HC in this section. The biases, MSEs and approximate CI are simulated by means of 10.000 replications with $(\theta = 0.5, \theta = 2)$ of IRD for different censoring schemes. The simulation results are given for different n, T_1 and r values in Table 2 and 3.

Table 2. MLE, bias, MSE, mean lenght and CP for IRD under Type-I HC for $\theta = 0.5$ and $T = 15$.

n,r	MLE	Bias	MSE	Length	CP
30,15	0.5059	0.0059	0.0023	0.1832	0.9520
30,25	0.5061	0.0061	0.0022	0.1812	0.9536
50,25	0.5040	0.0040	0.0014	0.1412	0.9500
50,45	0.5041	0.0041	0.0012	0.1397	0.9502
100,20	0.5023	0.0023	0.0008	0.1072	0.9556
100,50	0.5021	0.0021	0.0007	0.0994	0.9506
100,70	0.5019	0.0019	0.0006	0.0986	0.9504
100,90	0.5017	0.0017	0.0005	0.0982	0.9512
200,50	0.5014	0.0014	0.0004	0.0737	0.9532
200,75	0.5013	0.0013	0.0003	0.0712	0.9526
200,100	0.5013	0.0013	0.0003	0.0702	0.9508
200,150	0.5008	0.0008	0.0003	0.0695	0.9512
200,175	0.5006	0.0006	0.0002	0.0690	0.9501

Table 3. MLE, bias, MSE, mean length and CP for IRD under Type-I HC for $\theta = 2$ and $T = 15$.

n,r	MLE	Bias	MSE	Length	CP
30,15	2.0265	0.0265	0.0366	0.7337	0.9518
30,25	2.0271	0.0271	0.0359	0.7257	0.9524
50,25	2.0159	0.0159	0.0211	0.5650	0.9528
50,45	2.0158	0.0158	0.0206	0.5588	0.9538
100,20	2.0084	0.0084	0.0122	0.4288	0.9530
100,50	2.0096	0.0096	0.0105	0.3980	0.9504
100,70	2.0095	0.0095	0.0103	0.3945	0.9504
100,90	2.0086	0.0086	0.0102	0.3939	0.9485
200,50	2.0031	0.0031	0.0056	0.2945	0.9502
200,75	2.0030	0.0030	0.0053	0.2846	0.9512
200,100	2.0033	0.0033	0.0050	0.2805	0.9508
200,150	2.0029	0.0020	0.0040	0.2778	0.9510
200,175	2.0025	0.0025	0.0040	0.2772	0.9522

From Tables 2-3, it is clearly seen that as the number of observed sample in the same sample sizes increases, MSE and bias decreases and length is narrowing as expected. Also, MSE and bias approach zero as proportion of censoring decreases. Further, it is observed that approximate CIs generally about 0.95. When the results in Tables 2-3 are compared, it can be said that the MSE, bias and length are smaller for $\theta = 0.5$. Therefore, it is can also be said that more consistent estimates have been obtained for $\theta = 0.5$.

5. CONCLUSION

In this paper, we consider to provide a new extend in point and interval estimation estimation for IRD under type-I HC schemes. The motivation of this study is lack of the study on point and interval estimation based on type-I HC for IRD in literature. In this regard, we estimated the parameter of IRD using maximum likelihood method under type-I HC schemes. We also used EM algoritm to obtain MLE and approximate CI of θ for IRD. It is carried out a comprehensive

Monte Carlo simulation study to evaluate the performances of MLE in terms of MSE and bias for different sample sizes and censoring schemes. According to simulation results, it is concluded that biases and MSEs decreases as n and r increases when T is constant. As a result, this study provides to see the performance of the maximum likelihood method for IRD based on HC. A new study based on HC has been added to the literature.

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