Sigma J Eng & Nat Sci 36 (2), 2018, 373-388

Sigma Journal of Engineering and Natural Sciences
Sigma Mühendislik ve Fen Bilimleri Dergisi

Research Article

WEIGHTED VARIABLE EXPONENT SOBOLEV SPACES WITH ZERO BOUNDARY VALUES AND CAPACITY ESTIMATES

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Received: 19.06.2017 Revised: 12.02.2018 Accepted: 03.03.2018

ABSTRACT

In this paper, we define weighted variable exponent Sobolev space with zero boundary values and investigate some properties of this space with weighted variable Sobolev capacity. We obtain Poincaré inequality with respect to zero boundary values. We will introduce a capacity in sense to this defined space and, also, give several estimates.
Keywords: Weighted variable exponent Sobolev spaces, Sobolev capacity, energy integral.

1. INTRODUCTION

The history of potential theory begins in 17th century. Its development can be traced to such greats as Newton, Euler, Laplace, Lagrange, Fourier, Green, Gauss, Poisson, Dirichlet, Riemann, Weierstrass, Poincaré. We refer to the book by Kellogg [22] for references to some of the old works.

The study of variable exponent function spaces in higher dimensions was revealed in 1991 an article by Kovacik and Rakosnik [29]. They present some basic properties of the variable exponent Lebesgue space $L^{p(.)}\left(\mathbb{R}^n\right)$ and the Sobolev space $W^{k,p(.)}\left(\mathbb{R}^n\right)$ such as reflexivity and Hölder inequalities were obtained.

The boundedness of the maximal operator was an open problem in $L^{p(.)}\left(\mathbb{R}^n\right)$ for a long time. Diening [7] proved the first time this state over bounded domains if p(.) satisfies locally log-Hölder continuous condition, that is,

$$\left|p(x)-p(y)\right| \le \frac{C}{-\ln|x-y|}, \quad x, y \in \Omega, \ |x-y| \le \frac{1}{2}$$

where Ω is a bounded domain. We denote by $P^{\log}\left(\mathbb{R}^n\right)$ the class of variable exponents which satisfy the log-Hölder continuous condition. Diening later extended the result to unbounded domains by supposing, in addition, that the exponent p(.)=p is a constant function outside a large

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ball. After this study, many absorbing and crucial papers appeared in non-weighted and weighted variable exponent spaces, see [9], [13], [29] and [39]. Sobolev capacity for constant exponent spaces has found a great number of uses, see [12] and [34]. Moreover, the weighted Sobolev capacity was revealed by Kilpeläinen [23]. He investigated the role of capacity in the pointwise definition of functions in Sobolev spaces involving weights of Muckenhoupt's A_p-class. Harjulehto et al. [18] introduced variable Sobolev capacity in the spaces $W^{1,p(.)}(\mathbb{R}^n)$. Also, Aydın [3] generalized some results of the variable Sobolev capacity to the weighted variable exponent case.

The variational capacity has been used extensively in nonlinear potential theory on $\mathbb{R}^n$. Let $\Omega \subset \mathbb{R}^n$ is open and $K \subset \Omega$ is compact. Then the relative variational p-capacity is defined by

$$\mathrm{cap}_p(K,\Omega) = \inf_f \int_\Omega |\nabla f(x)|^p \, dx,$$

where the infimum is taken over smooth and zero boundary valued functions f in Ω such that $f \geq 1$ in K. The set of admissible functions f can be replaced by the continuous first order Sobolev functions with $f \geq 1$ in K. The p-capacity is a Choquet capacity relative to Ω. For more details and historical background, see [20]. Also, Harjulehto et al. [16] defined an another relative capacity. They studied properties of the capacity and compare it with the variable exponent Sobolev capacity.

The classical Dirichlet boundary value problem come out a partial differential equation: If $\Omega \subset \mathbb{R}^n$ and $h : \partial\Omega \to \mathbb{R}$ is a continuous function, then main problem is to find a continuous function $f : \overline{\Omega} \to \mathbb{R}$ such that the Laplace equation $-\Delta f = 0$ is satisfied on Ω and $f = h$ on $\partial\Omega$. Here, the function h gives the boundary values of f. One approach to solving the classical Dirichlet boundary value problem is to determine a minimizer for the energy operator within a certain function space. The energy operator, however, is dependent on the boundary value function. It is known that the Dirichlet energy integral does not always have a minimizer. It can be seen in [[17], Example 3.4].

Shanmugalingam studied the Dirichlet energy integral over metric spaces in [40]. She established the Dirichlet boundary value problem and investigated some properties of solutions (e.g. uniqueness, maximum principle property) to such problems.

In [1] and [6], the authors have explored some properties of the p(.)- Dirichlet energy integral $\int_\Omega |\nabla f(x)|^{p(x)} \, dx$ over a bounded domain $\Omega \subset \mathbb{R}^n$. They have discussed the existence and regularity of energy integral minimizers. As an alternative method the minimizers in one dimensional case have been studied by the authors in [17]. Moreover, Harjulehto et. al. [19] considered the Dirichlet energy integral, with boundary values given in the Sobolev sense, has a minimizer provided the variable exponent satisfies a certain jump condition.

Our purpose is to investigate some results of the variable Sobolev capacity in weighted case in sense to [3]. Using this capacity, we define weighted variable exponent Sobolev spaces with zero boundary values. We will consider $(p(.),\vartheta)$ - Poincaré inequality with respect to the space $W_{0,\vartheta}^{1,p(.)}(\Omega)$. Also, we will investigate the $p(.)$ - Dirichlet energy integral and generalize some results of Harjulehto et. al. [19] to the weighted variable exponent case. Moreover, we introduce a capacity in sense to $W_{0,\vartheta}^{1,p(.)}(\Omega)$ and give several estimates.

2. NOTATION AND PRELIMINARIES

In this paper, we work on $\mathbb{R}^n$ with Lebesgue measure dx. We denote by $C^\infty(\mathbb{R}^n)$ the space of all infinitely differentiable functions. Also, the elements of the space $C_0^\infty(\mathbb{R}^n)$ are the infinitely differentiable functions with compact support. The space $L^1_{loc}(\mathbb{R}^n)$ is to be space of all measurable functions f on $\mathbb{R}^n$ such that $f\chi_K \in L^1(\mathbb{R}^n)$ for any compact subset $K \subset \mathbb{R}^n$. We denote the family of all measurable functions $p(.):\mathbb{R}^n \to [1,\infty)$ (called the variable exponent on $\mathbb{R}^n$) by the symbol $P(\mathbb{R}^n)$

$$p^- = \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x), \qquad p^+ = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x)$$

For each $A \subset \mathbb{R}^n$ we set

$$p_A^- = \operatorname*{ess\,inf}_{x\in A} p(x), \qquad p_A^+ = \operatorname*{ess\,sup}_{x\in A} p(x).$$

The exponent p(.) is log-Hölder continuous in an open set Ω if and only if there is a constant $C > 0$ such that

$$|B|^{p^-_{B\cap\Omega} - p^+_{B\cap\Omega}} \le C \tag{2.1}$$

for every ball $B \cap \Omega \neq \varnothing$, see [7].

A measurable and locally integrable function $\vartheta:\mathbb{R}^n \to (0,\infty)$ is called a weight function. The weighted modular is defined by

$$\rho_{p(.),\vartheta}(f) = \int_{\mathbb{R}^n} |f(x)|^{p(x)}\,\vartheta(x)dx.$$

The weighted variable exponent Lebesgue spaces $L^{p(.)}_\vartheta(\mathbb{R}^n)$ consist of all real-valued measurable functions f on $\mathbb{R}^n$ endowed with the Luxemburg norm

$$\|f\|_{p(.),\vartheta} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \left|\frac{f(x)}{\lambda}\right|^{p(x)} \vartheta(x)dx \le 1\right\}.$$

When $\vartheta(x)=1$, the space $L^{p(.)}_\vartheta(\mathbb{R}^n)$ is the variable exponent Lebesgue space. The space $L^{p(.)}_\vartheta(\mathbb{R}^n)$ is a Banach space with respect to $\|.\|_{p(.),\vartheta}$. Also, some basic properties of this space were investigated in [2], [3], [28]. Moreover, it is known that the space $C_0^\infty(\mathbb{R}^n)$ is dense in $L^{p(.)}_\vartheta(\mathbb{R}^n)$, see [3], [7], [28], [30].

Let $\Omega \subset \mathbb{R}^n$ is bounded and ϑ is a weight function. It is known that a function $f \in C_0^\infty(\mathbb{R}^n)$ satisfy Poincaré inequality in $L^1_\vartheta(\Omega)$ if and only if there is a constant $c > 0$ such that the inequality

$$\int_\Omega |f(x)|\,\vartheta(x)dx \le c(\operatorname{diam}\Omega)\int_\Omega |\nabla f(x)|\,\vartheta(x)dx$$

holds [20].

In recent decades, variable exponent Lebesgue spaces $L^{p(.)}$ and the corresponding the variable exponent Sobolev spaces $W^{k,p(.)}$ have attracted more and more attention. Let $1 < p^- \le p(.) \le p^+ < \infty$ and $k \in \mathbb{N}$. The variable exponent Sobolev spaces $W^{k,p(.)}(\mathbb{R}^n)$ consist of all measurable functions $f \in L^{p(.)}(\mathbb{R}^n)$ such that the distributional derivatives $D^\alpha f$ are in $L^{p(.)}(\mathbb{R}^n)$ for all $0 \le |\alpha| \le k$ where $\alpha \in \mathbb{N}_0^n$ is a multiindex, $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$, and $D^\alpha = \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}...\partial_{x_n}^{\alpha_n}}$. The spaces $W^{k,p(.)}(\mathbb{R}^n)$ are a special class of so-called generalized Orlicz-Sobolev spaces with the norm

$$\|f\|_{k,p(.)} = \sum_{0\le|\alpha|\le k} \|D^\alpha f\|_{p(.)}.$$

Now, let $1 < p^- \le p(.) \le p^+ < \infty$, $k \in \mathbb{N}$ and $\vartheta^{-\frac{1}{p(.)-1}} \in L^1_{loc}(\mathbb{R}^n)$. We set the weighted variable exponent Sobolev spaces $W^{k,p(.)}_\vartheta(\mathbb{R}^n)$ by

$$W^{k,p(.)}_\vartheta(\mathbb{R}^n) = \left\{ f \in L^{p(.)}_\vartheta(\mathbb{R}^n) : D^\alpha f \in L^{p(.)}_\vartheta(\mathbb{R}^n), 0 \le |\alpha| \le k \right\}$$

equipped with the norm

$$\|f\|_{k,p(.),\vartheta} = \sum_{0\le|\alpha|\le k} \|D^\alpha f\|_{p(.),\vartheta}.$$

Since the embedding $L^{p(.)}_\vartheta(\mathbb{R}^n) \hookrightarrow L^1_{loc}(\mathbb{R}^n)$ holds, then the weighted variable exponent Sobolev spaces $W^{k,p(.)}_\vartheta(\mathbb{R}^n)$ is well-defined. Also, it is already known that $W^{k,p(.)}_\vartheta(\mathbb{R}^n)$ is a reflexive Banach space, see [3].

In particular, the space $W^{1,p(.)}_\vartheta(\mathbb{R}^n)$ is defined by

$$W^{1,p(.)}_\vartheta(\mathbb{R}^n) = \left\{ f \in L^{p(.)}_\vartheta(\mathbb{R}^n) : |\nabla f| \in L^{p(.)}_\vartheta(\mathbb{R}^n) \right\}.$$

The function $\rho_{1,p(.),\vartheta} : W^{1,p(.)}_\vartheta(\mathbb{R}^n) \to [0,\infty)$ is shown as $\rho_{1,p(.),\vartheta}(f) = \rho_{p(.),\vartheta}(f) + \rho_{1,p(.),\vartheta}(|\nabla f|)$. Also, the norm $\|f\|_{1,p(.),\vartheta} = \|f\|_{p(.),\vartheta} + \|\nabla f\|_{p(.),\vartheta}$ makes the space $W^{1,p(.)}_\vartheta(\mathbb{R}^n)$ a Banach space. The local weighted variable exponent Sobolev space $W^{1,p(.)}_{\vartheta,loc}(\mathbb{R}^n)$ is defined in the classical way. More information on the classic theory of variable exponent spaces can be found in [29].

If the exponent p(.) satisfies locally log-Hölder continuous condition, then a lot of regularities for variable exponent spaces holds. Because, the space $C_0^\infty(\mathbb{R}^n)$ is dense in $W^{1,p(.)}_\vartheta(\mathbb{R}^n)$ under the circumstances, see [3]. Therefore, it makes sense to define the space of zero boundary value Sobolev functions as the closure of $C_0^\infty(\mathbb{R}^n)$ in $W^{1,p(.)}_\vartheta(\mathbb{R}^n)$. But we will give an alternative definition to zero boundary value Sobolev functions space.

For $x \in \mathbb{R}^n$ and $r > 0$ we denote an open ball with center x and radius r by B(x,r). For $f \in L^1_{loc}(\mathbb{R}^n)$, the Hardy- Littlewood maximal operator Mf of f given by $Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy$ where the supremum is taken over all balls B(x,r).

Let $1 \le p(.) < \infty$. A weight ϑ satisfies Muckenhoupt's $A_p(\mathbb{R}^n) = A_p$ condition, briefly $\vartheta \in A_p$, if there are positive constants C_1 and C_2 such that, for all ball $B \subset \mathbb{R}^n$,

$$\left(\frac{1}{|B|}\int_B \vartheta(x)dx\right)\left(\frac{1}{|B|}\int_B \vartheta(x)^{-\frac{1}{p-1}}dx\right)^{p-1} \le C_1, \quad 1<p<\infty \text{ or}$$

$$\left(\frac{1}{|B|}\int_B \vartheta(x)dx\right)\left(\operatorname{ess\,sup}_B \frac{1}{\vartheta(x)}\right) \le C_2, \quad p=1.$$

The infimum over the constants C_1 and C_2 are called the A_p and A_1, respectively. Also it is known that $A_\infty = \bigcup_{1\le p<\infty} A_p$. Let $1 < p(.) < \infty$. Then it is known that $\vartheta \in A_p$ if and only if the Hardy-Littlewood maximal operator is bounded on $L^{p(.)}_\vartheta(\mathbb{R}^n)$, see [37].

In [8], the class $A_{p(.)}$ was defined to consist of weights ϑ such that

$$\|\vartheta\|_{A_{p(.)}} = \sup_{B\in\mathfrak{B}} |B|^{-p_B} \|\vartheta\|_{L^1(B)} \left\|\vartheta^{-1}\right\|_{L^{\frac{p'(.)}{p(.)}}(B)} < \infty$$

where $\mathfrak{B}$ denotes the family of all balls in $\mathbb{R}^n$, $p_B = \left(\frac{1}{|B|}\int_B \frac{1}{p(x)}dx\right)^{-1}$ and $p'(.)$ is the conjugate exponent of p(.).

Let $p(.), q(.) \in P^{\log}(\mathbb{R}^n)$, $1 < p^- \le p(.) \le p^+ < \infty$ and $1 < q^- \le q(.) \le q^+ < \infty$. If the inequality $q(.) \le p(.)$ is satisfied, then there exists a constant $C > 0$ depending on the characteristics of p(.) and q(.) such that $\|\vartheta\|_{A_{p(.)}} \le C\|\vartheta\|_{A_{q(.)}}$. Also, under these conditions, $M : L^{p(.)}_\vartheta(\mathbb{R}^n) \hookrightarrow L^{p(.)}_\vartheta(\mathbb{R}^n)$ if and only if $\vartheta \in A_{p(.)}$, see [8]. We denote

$$\mathfrak{P}(\mathbb{R}^n) = \left\{p(.) \in P^{\log}(\mathbb{R}^n) : 1 < p^- \le p(.) \le p^+ < \infty, \|Mf\|_{p(.),\vartheta} \le C\|f\|_{p(.),\vartheta}\right\},$$

that is, $\mathfrak{P}(\mathbb{R}^n)$ is the set of the maximal operator M is bounded on $L^{p(.)}_\vartheta(\mathbb{R}^n)$.

Given a subspace (Y, τ^*) of a topological space (X, τ), the closed subsets of the topological space (Y, τ^*) are called relatively closed in Y of briefly relatively closed. In other words the relatively closed subsets are the restriction to Y of the closed subsets of X. For more details about the relatively closed subsets can find in [[35], Section 6].

We say that a property holds $(p(.), \vartheta)$-quasieverywhere if it satisfies except in a set of capacity zero. Recall also a function f is $(p(.), \vartheta)$-quasicontinuous in $\mathbb{R}^n$ if for each $\varepsilon > 0$ there exists a set A with $C_{p(.),\vartheta}(A) < \varepsilon$ such that f restricted to $\mathbb{R}^n - A$ is continuous. If the capacity is an outer capacity, we can suppose that A is open. More detail can be found in [3].

Throughout this paper, we assume that $p(.) \in P^{\log}(\mathbb{R}^n)$ with $1 < p^- \le p(.) \le p^+ < \infty$ and $\vartheta^{-\frac{1}{p(.)-1}} \in L^1_{loc}(\mathbb{R}^n)$. We write that $a \approx b$ for two quantities if there exists positive constants c_1, c_2 such that $c_1 a \le b \le c_2 a$. Also, A⋐B means that A is a compact subset of B. We will denote $\mu_\vartheta(\Omega) = \int_\Omega \vartheta(x)dx$.

Also, we use the abbreviations; a.e., $(p(.),\vartheta)$ - q.e., $(p(.),\vartheta)$ - q.c. for almost everywhere, $(p(.),\vartheta)$ -quasieverywhere, $(p(.),\vartheta)$ - quasicontinuous, respectively.

3. THE SPACE $W^{1,p(.)}_{0,\vartheta}(\Omega)$

A capacity for subsets of $\mathbb{R}^n$ was introduced in [3]. To define this capacity we denote

$$S_{p(.),\vartheta}(A) = \left\{ f \in W^{1,p(.)}_{\vartheta}(\mathbb{R}^n) : f \ge 1 \text{ in an open set containing } A \right\}.$$

The Sobolev $(p(.),\vartheta)$ - capacity of A is defined by $C_{p(.),\vartheta}(E) = \inf_{f \in S_{p(.),\vartheta}(E)} \rho_{1,p(.),\vartheta}(f)$.

Thanks to meaning of the infimum, in case $S_{p(.),\vartheta}(A) = \varnothing$, we set $C_{p(.),\vartheta}(A) = \infty$. It is evident that the same number $C_{p(.),\vartheta}(A)$ is obtained if the infimum in the definition is taken over $f \in S_{p(.),\vartheta}(A)$ with $0 \le f \le 1$. The Sobolev $(p(.),\vartheta)$ - capacity has some basic properties such as outer measure, monotonicity, subadditivity, Choquet property etc. More details can be found in [3].

The proof of the following theorem can be easily shown with the same technique in [24]. Note that the second assertion of the theorem is a direct result of the first one, see [[25], Remark 3.3].

Theorem 1. Let f and g be $(p(.),\vartheta)$ -q.c. in $\mathbb{R}^n$. Assume that $U \subset \mathbb{R}^n$ is open. Then

(i) If $f = g$ a.e. in U, then $f = g$ $(p(.),\vartheta)$ -q.e. in U.

(ii) If $f \le g$ a.e. in U, then $f \le g$ $(p(.),\vartheta)$ -q.e. in U.

It is known that the space $W^{1,p(.)}_{\vartheta}(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ is not dense in general. But Zhikov and Surnachev proved this denseness under some conditions in [41]. Note that, the denseness of $W^{1,p(.)}_{\vartheta}(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ follows that the space $W^{1,p(.)}_{\vartheta}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ is dense in $W^{1,p(.)}_{\vartheta}(\mathbb{R}^n)$. From now, we will assume that the variable exponent p(.) is said to satisfy the density condition if the space $W^{1,p(.)}_{\vartheta}(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ is dense in $W^{1,p(.)}_{\vartheta}(\mathbb{R}^n)$.

Let $\Omega \subset \mathbb{R}^n$ be an open set. The space $W^{1,p(.)}_{0,\vartheta}(\Omega)$ is denoted as the set of all measurable functions f if there exists a $(p(.),\vartheta)$ - q.c. function $f^* \in W^{1,p(.)}_{\vartheta}(\mathbb{R}^n)$ such that $f = f^*$ a.e. in Ω and

$f^* = 0$ $(p(.),\vartheta)$-q.e. in $\mathbb{R}^n - \Omega$. In other words, $f \in W_{0,\vartheta}^{1,p(.)}(\Omega)$, if there exist a $(p(.),\vartheta)$- q.c. function $f^* \in W_\vartheta^{1,p(.)}(\mathbb{R}^n)$ such that the trace of f vanishes. Moreover the weighted variable exponent Sobolev spaces with zero boundary values equipped with the norm

$$\|f\|_{W_{0,\vartheta}^{1,p(.)}(\Omega)} = \|f^*\|_{W_\vartheta^{1,p(.)}(\mathbb{R}^n)}.$$

A $(p(.),\vartheta)$- q.c. function $f^* \in W_\vartheta^{1,p(.)}(\mathbb{R}^n)$ is called a canonical representative of the function $f \in W_{0,\vartheta}^{1,p(.)}(\Omega)$ if $f = f^*$ a.e. in Ω and $f^* = 0$ $(p(.),\vartheta)$-q.e. in $\mathbb{R}^n - \Omega$. From the definition of the space $W_{0,\vartheta}^{1,p(.)}(\Omega)$, it is clear that $W_{0,\vartheta}^{1,p(.)}(\mathbb{R}^n) = W_\vartheta^{1,p(.)}(\mathbb{R}^n)$. It can be shown that the space $W_{0,\vartheta}^{1,p(.)}(\Omega)$ is a reflexive Banach space.

We denote $H_{0,\vartheta}^{1,p(.)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in the space $W_\vartheta^{1,p(.)}(\Omega)$. More precisely, $f \in H_{0,\vartheta}^{1,p(.)}(\Omega)$ if and only if there exists a sequence $\{f_k\}_{k\in\mathbb{N}}$ of $C_0^\infty(\Omega)$ such that $\|f_k - f\|_{W_\vartheta^{1,p(.)}(\Omega)} \to 0$. Because of the fact that the space $W_\vartheta^{1,p(.)}(\Omega)$ is a Banach space and the inclusion $H_{0,\vartheta}^{1,p(.)}(\Omega) \subset W_\vartheta^{1,p(.)}(\Omega)$ holds, it is easy to see that the space $H_{0,\vartheta}^{1,p(.)}(\Omega)$ is a Banach space, as well.

Corollary 1. Let $\vartheta(x) \geq 1$ for $x \in \mathbb{R}^n$. If $1 < p^- \leq p(.) \leq p^+ < \infty$, then the inclusions $H_{0,\vartheta}^{1,p(.)}(\Omega) \subset W_{0,\vartheta}^{1,p(.)}(\Omega) \subset W_\vartheta^{1,p(.)}(\Omega)$ hold.

If we consider the definition of the space $W_\vartheta^{1,p(.)}(\Omega)$ instead of $W^{1,p(.)}(\Omega)$ in the proof of [[19], Theorem 3.3] and [[19], Theorem 3.4], then we obtain Theorem 2 and Theorem 3.

Theorem 2. Let $\vartheta(x) \geq 1$ for $x \in \mathbb{R}^n$. If the space $W_\vartheta^{1,p(.)}(\Omega) \cap C^\infty(\Omega)$ is dense in $W_\vartheta^{1,p(.)}(\Omega)$, then $H_{0,\vartheta}^{1,p(.)}(\Omega) = W_{0,\vartheta}^{1,p(.)}(\Omega)$.

Theorem 3. Let $1 < q^-, p^+ < \infty$ and $q(x) \leq p(x)$ for almost every $x \in \mathbb{R}^n$. Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded open set. Then $W_{0,\vartheta}^{1,p(.)}(\Omega) \hookrightarrow W_{0,\vartheta}^{1,q(.)}(\Omega)$.

The proofs of the following two theorems can be easily seen by considering the definition of the spaces $W_{0,\vartheta}^{1,p(.)}(\Omega)$ and $W_\vartheta^{1,p(.)}(\Omega)$ in [[10], Theorem 11.3.1] and [[10], Theorem 11.3.2], respectively.

Theorem 4. Let $\vartheta(x) \geq 1$ for $x \in \mathbb{R}^n$. Assume that $A \subset \Omega$ is a relatively closed subset. Then $W_{0,\vartheta}^{1,p(.)}(\Omega) = W_{0,\vartheta}^{1,p(.)}(\Omega - A)$ if and only if $C_{p(.),\vartheta}(A) = 0$.

Theorem 5. Assume that $A \subset \Omega$ is a relatively closed subset. If $C_{p(.),\vartheta}(A) = 0$, then $W_\vartheta^{1,p(.)}(\Omega) = W_\vartheta^{1,p(.)}(\Omega - A)$.

Theorem 6. Let $p(.) \in P(\mathbb{R}^n)$. Then $W_\vartheta^{1,p(.)}(\Omega) = W_{0,\vartheta}^{1,p(.)}(\Omega)$ if and only if $C_{p(.),\vartheta}(\mathbb{R}^n - \Omega) = 0$.

Proof. Assume that $W_{\vartheta}^{1,p(.)}(\Omega)=W_{0,\vartheta}^{1,p(.)}(\Omega)$. We define a function f as $f(x)=\max\{0,2r-|x|\}$ } for $0<r<\infty$. Then $f\in W_{\vartheta}^{1,p(.)}(\Omega)$. Indeed, suppose that $\max\{0,2r-|x|\}=2r-|x|$. In other case, the statement is clear. Since $\vartheta\in L_{loc}^{1}(\mathbb{R}^n)$, we have

$$\rho_{L_{\vartheta}^{p(.)}(\Omega)}(f)\leq\int_{\Omega}(|2r|+|x|)^{p(x)}\vartheta(x)dx\leq\int_{\Omega}(4r)^{p(x)}\vartheta(x)dx\leq\max\left((4r)^{p^-},(4r)^{p^+}\right)\int_{\Omega}\vartheta(x)dx<\infty.$$

It is easy to see that $\rho_{L_{\vartheta}^{p(.)}(\Omega)}(\nabla f)<\infty$. This follows that $f\in W_{\vartheta}^{1,p(.)}(\Omega)=W_{0,\vartheta}^{1,p(.)}(\Omega)$. Since the space $C_0^{\infty}(\Omega)$ is dense in $W_{\vartheta}^{1,p(.)}(\Omega)$, see [3], we can we can take the sequence (f_k) such that $f_k\to f$ in $W_{\vartheta}^{1,p(.)}(\Omega)$ and have compact supports in Ω. Hence $f-f_k$ are test functions for the capacity of $(\mathbb{R}^n-\Omega)\cap B(0,r)$ for $k\in\mathbb{N}$. Since $p^+<\infty$, we find that $\rho_{1,p(.),\vartheta}(f_k-f)\to 0$. If we take the infimum over $f-f_k\in S_{p(.),\vartheta}\left((\mathbb{R}^n-\Omega)\cap B(0,r)\right)$, then we get $C_{p(.),\vartheta}\left((\mathbb{R}^n-\Omega)\cap B(0,r)\right)=0$. Moreover, $\bigcup_{r=1}^{\infty}\left((\mathbb{R}^n-\Omega)\cap B(0,r)\right)=(\mathbb{R}^n-\Omega)\cap\bigcup_{r=1}^{\infty}B(0,r)=\mathbb{R}^n-\Omega$. Therefore we obtain

$$0\leq C_{p(.),\vartheta}\left((\mathbb{R}^n-\Omega)\right)\leq\sum_{r=1}^{\infty}C_{p(.),\vartheta}\left((\mathbb{R}^n-\Omega)\cap B(0,r)\right)=0.$$

To prove sufficient condition, suppose that $C_{p(.),\vartheta}(\mathbb{R}^n-\Omega)=0$. It is known that $W_{\vartheta}^{1,p(.)}(\mathbb{R}^n)=W_{0,\vartheta}^{1,p(.)}(\mathbb{R}^n)$. Moreover, it is easy to see that $\mathbb{R}^n-\Omega$ is relatively closed. Therefore, if we consider the Theorem 5 and Theorem 4, then we get

$$W_{\vartheta}^{1,p(.)}(\Omega)=W_{\vartheta}^{1,p(.)}\left(\mathbb{R}^n-(\mathbb{R}^n-\Omega)\right)=W_{\vartheta}^{1,p(.)}(\mathbb{R}^n)=W_{0,\vartheta}^{1,p(.)}(\mathbb{R}^n)=W_{0,\vartheta}^{1,p(.)}\left(\mathbb{R}^n-(\mathbb{R}^n-\Omega)\right)=W_{0,\vartheta}^{1,p(.)}(\Omega).$$

Now, we consider the Poincaré inequality in the space $W_{0,\vartheta}^{1,p(.)}(\Omega)$. Let $A\subset\mathbb{R}^n$. We define

$$p_A^-=\operatorname*{ess\,inf}_{x\in A\cap\Omega}p(x),\ p_A^+=\operatorname*{ess\,sup}_{x\in A\cap\Omega}p(x)$$

for $p(.)\in P(\mathbb{R}^n)$. If $p_{\Omega}^+<\infty$ and if there exists $r>0$ such that every $x\in\Omega$ either

$$p_{B(x,r)}^-\geq n\ \text{ or }\ p_{B(x,r)}^+\leq\frac{np_{B(x,r)}^-}{n-p_{B(x,r)}^-}$$

is valid, then the variable exponent p(.) is said to satisfies the jump condition in Ω with constant r. Moreover we put

$$p_{B(x,r)}^*=\begin{cases}\dfrac{np_{B(x,r)}^-}{n-p_{B(x,r)}^-}, & p_{B(x,r)}^-<n\\ p_{B(x,r)}^+, & p_{B(x,r)}^+\geq n\end{cases}$$

It is clear that if Ω is bounded and if p(.) is continuous in $\overline{\Omega}$, then p(.) satisfies the jump condition in Ω with some r>0, see [19].

Remark 1. Let $\Omega \subset \mathbb{R}^n$ be a bounded set. Then, the claim of Proposition 2.4 in [31] satisfies even if $p(.)=1$. This yields that the embedding $L_{\vartheta}^{p(.)}(\Omega) \hookrightarrow L_{\vartheta}^{1}(\Omega)$ holds.

Theorem 7. Suppose that $\Omega \subset \mathbb{R}^n$ be a bounded set. Let the exponent p(.) holds the jump condition in Ω with constant r>0. Then, for all $f \in W_{0,\vartheta}^{1,p(.)}(\Omega)$, the inequality

$$\|f\|_{L_{\vartheta}^{p(.)}(\Omega)} \leq C\|\nabla f\|_{L_{\vartheta}^{p(.)}(\Omega)}$$

is satisfied where the constant C depends on the exponent p(.), $|\Omega|$, diam(Ω), r and the dimension n.

Proof. Since Ω is a bounded set, $\overline{\Omega}$ is compact. Then we can find $x_1, x_2, ..., x_t$ such that $\Omega \subset \bigcup_{m=1}^{t} B(x_m, r)$. By using the fact that $f \in W_{0,\vartheta}^{1,p(.)}(\Omega)$, the function f^* can be taken as the canonical representative of f. If we consider [[31], Proposition 2.4], then we have

$$\|f\|_{L_{\vartheta}^{p(.)}(\Omega)} = \|f^*\|_{L_{\vartheta}^{p(.)}(\mathbb{R}^n)} \leq \left\|f^*\left[\chi_{B(x_1,r)} + ... + \chi_{B(x_m,r)}\right]\right\|_{L_{\vartheta}^{p(.)}(\mathbb{R}^n)} \leq \sum_{m=1}^{t}\|f^*\|_{L_{\vartheta}^{p(.)}(B(x_m,r))} \leq c\sum_{m=1}^{t}\|f^*\|_{L_{\vartheta}^{p_{B(x_m,r)}^*}(B(x_m,r))}$$

$$\leq c\sum_{m=1}^{t}\left(\left\|f^* - f_{B(x_m,r)}^*\right\|_{L_{\vartheta}^{p_{B(x_m,r)}^*}(B(x_m,r))} + \left|f_{B(x_m,r)}^*\right|\|1\|_{L_{\vartheta}^{p_{B(x_m,r)}^*}(B(x_m,r))}\right) \tag{3.1}$$

Here, the function $f_{B(x_m,r)}^*$ is average of f^* over the balls $B(x_m,r)$ and defined as $f_{B(x_m,r)}^* = \frac{1}{|B(x_m,r)|}\int_{B(x_m,r)} f^*(x)dx$, see [20]. It is clear that $p_{B(x_m,r)}^- \leq p(.)$. Moreover, if we use the Poincaré inequality over the balls [[20], Section 1] and the embedding $L_{\vartheta}^{p(.)}(B(x_m,r)) \hookrightarrow L_{\vartheta}^{p_{B(x_m,r)}^-}(B(x_m,r))$ [[31], Proposition 2.4], then we obtain

$$\left\|f^* - f_{B(x_m,r)}^*\right\|_{L_{\vartheta}^{p_{B(x_m,r)}^*}(B(x_m,r))} \leq cr\|\nabla f^*\|_{L_{\vartheta}^{p_{B(x_m,r)}^-}(B(x_m,r))} \leq cr\|\nabla f^*\|_{L_{\vartheta}^{p(.)}(B(x_m,r))} \leq cr\|\nabla f^*\|_{L_{\vartheta}^{p(.)}(\Omega)}$$

for all m=1,2,...,t. Moreover, if we use the Poincaré inequality in $L_{\vartheta}^{1}(\Omega)$ and Remark 1, then we get

$$\left|f_{B(x_m,r)}^*\right| \leq \frac{C}{r^n}\int_{\Omega}|f(x)|\vartheta(x)dx \leq \frac{C}{r^n}\text{diam}(\Omega)\int_{\Omega}|\nabla f(x)|\vartheta(x)dx \leq \frac{C}{r^n}\text{diam}(\Omega)c\|\nabla f\|_{L_{\vartheta}^{p(.)}(\Omega)}$$

for all m=1,2,...,t. Since $\vartheta \in L_{loc}^{1}(\mathbb{R}^n)$, we have

$$\rho_{p_{B(x_m,r)}^*,\vartheta}\left(\chi_{B(x_m,r)}\right) = \int_{B(x_m,r)}\vartheta(x)dx < \infty.$$

This yields that $\|1\|_{L_{\vartheta}^{p_{B(x_m,r)}^*}(B(x_m,r))}$ depends only on $p_{B(x_m,r)}^*$. Hence the claim follows from the inequality (3.1).

Corollary 2. By the previous theorem, $\|\nabla f\|_{L^{p(.)}_{\vartheta}(\Omega)}$ and $\|f\|_{W^{1,p(.)}_{\vartheta}(\Omega)}$ are the equivalents norms in $W^{1,p(.)}_{0,\vartheta}(\Omega)$. Hence, we can use the space $W^{1,p(.)}_{0,\vartheta}(\Omega)$ equipped with the norm $\|f\|_{W^{1,p(.)}_{0,\vartheta}(\Omega)} = \|\nabla f\|_{L^{p(.)}_{\vartheta}(\Omega)}$ for all $f \in W^{1,p(.)}_{0,\vartheta}(\Omega)$.

Now, we give an another capacity that has relationship with the Sobolev capacity. Let $A \subset \Omega$. We denote

$$\check{R}_{p(.),\vartheta}(A,\Omega) = \left\{ f \in W^{1,p(.)}_{0,\vartheta}(\Omega) : f \geq 1 \text{ in an open set containing } A \right\},$$

define

$$C_{p(.),\vartheta}(A,\Omega) = \inf_{f \in \check{R}_{p(.),\vartheta}(A,\Omega)} \int_{\Omega} |\nabla f(x)|^{p(x)} \vartheta(x)dx \tag{3.2}$$

Before the presenting relationship between defined new capacity above and Sobolev $(p(.),\vartheta)$ - capacity we will give an assertion.

Theorem 8. Let $B(x_0,r) \subset \Omega$ and $\vartheta \in A_{p(.)}$. For every $f \in W^{1,p(.)}_{0,\vartheta}(B(x_0,r))$ with $\rho_{L^{p(.)}_{\vartheta}(B(x_0,r))}(|\nabla f|) \leq 1$, there exist a constant C such that

$$\int_{B(x_0,r)} \left(\frac{|f(x)|}{r} \right)^{p(x)} \vartheta(x)dx \leq C \left(\int_{B(x_0,r)} |\nabla f(x)|^{p(x)} \vartheta(x)dx + \mu_{\vartheta}(B(x_0,r)) \right).$$

Proof. By Lemma 7.14 in [15], we obtain that $|f(x)| \leq C \int_{B(x_0,r)} \frac{|\nabla f(y)|}{|x-y|^{n-1}} dy$ for all $f \in W^{1,1}_{0,\vartheta}(B(x_0,r))$ and for almost all $x \in B(x_0,r)$. [[42], Lemma 2.8.3] shows us that the inequality

$$\int_{B(x_0,r)} \frac{|\nabla f(y)|}{|x-y|^{n-1}} dy \leq CrM(|\nabla f|)(x)$$

holds. Thus we get $\frac{|f(x)|}{r} \leq CM(|\nabla f|)(x)$. This follows that

$$\int_{B(x_0,r)} \left(\frac{|f(x)|}{r} \right)^{p(x)} \vartheta(x)dx \leq C \int_{B(x_0,r)} \left(M(|\nabla f|)(x) \right)^{p(x)} \vartheta(x)dx.$$

If we consider the weighted version of [[7], Lemma 3.3] with $\rho_{L^{p(.)}_{\vartheta}(B(x_0,r))}(|\nabla f|) \leq 1$, then we obtain

$$\int_{B(x_0,r)} \left(M(|\nabla f|)(x) \right)^{p(x)} \vartheta(x)dx \leq C^{p_B^-} \int_{B(x_0,r)} \left(\left(M(|\nabla f|)(x) \right)^{\frac{p(x)}{p_B^-}} + 1 \right)^{p_B^-} \vartheta(x)dx$$
$$\leq C^{p_B^-} 2^{p_B^- - 1} \left(\int_{B(x_0,r)} |\nabla f(x)|^{p(x)} \vartheta(x)dx + \mu_{\vartheta}(B(x_0,r)) \right)$$

where $M: L_\vartheta^{p(.)}(\mathbb{R}^n) \to L_\vartheta^{p(.)}(\mathbb{R}^n)$ is bounded due to $\vartheta \in A_{p(.)}$. This follows the claim.

Theorem 9. Let $B = B(x_0, r) \subset \mathbb{R}^n$ be a ball with $r \le 1$ and let $A \subset B$. Assume that ϑ is a weight function such that $\vartheta(x) \ge 1$ for all $x \in \mathbb{R}^n$. Then there exists a constant C such that

$$C_{p(.),\vartheta}(A) \le \left(Cr^{p(x_0)} + 1\right) C_{p(.),\vartheta}(A, 2B) + C\mu_\vartheta(2B) \tag{3.3}$$

and

$$C_{p(.),\vartheta}(A, 2B) \le \frac{C2^{p^+-1}}{r^{p(x_0)}} C_{p(.),\vartheta}(A). \tag{3.4}$$

Proof. Suppose that f is an admissible function for $C_{p(.),\vartheta}(A, 2B)$, that is, $f \in \check{R}_{p(.),\vartheta}(A, 2B)$. Then $f \in W_{0,\vartheta}^{1,p(.)}(2B)$ such that $f \ge 1$ in open set containing A. Therefore we have $f \in S_{p(.),\vartheta}(A)$. By (2.1) we get $r^{-p(x)} \approx r^{-p(x_0)}$ for every $x \in 2B$. If we consider the fact that $\vartheta(x) \ge 1$ for all $x \in \mathbb{R}^n$ and Theorem 8, then we have

$$|A| \le \int_A \vartheta(x)dx \le \int_A |f(x)|^{p(x)} \vartheta(x)dx \le C2^{p_{2B}^- -1} r^{p(x_0)} \left(\int_{2B} |\nabla f(x)|^{p(x)} \vartheta(x)dx + \mu_\vartheta(2B) \right).$$

This follows that

$$C_{p(.),\vartheta}(A) \le \left(C2^{p_{2B}^- -1} r^{p(x_0)} + 1\right) \int_{2B} |\nabla f(x)|^{p(x)} \vartheta(x)dx + C2^{p_{2B}^- -1} r^{p(x_0)} \mu_\vartheta(2B).$$

The inequality (3.3) is satisfied by taking the infimum over $f \in \check{R}_{p(.),\vartheta}(A, 2B)$ from the last inequality.

Now, let $f \in S_{p(.),\vartheta}(A)$. Also, suppose that $g \in C_0^\infty(2B)$ be a function such that $0 \le g \le 1$, $g = 1$ on B, and $|\nabla g| \le \frac{C}{r}$. Hence $fg \in \check{R}_{p(.),\vartheta}(A, 2B)$ and we get

$$\begin{aligned} C_{p(.),\vartheta}(A, 2B) &\le 2^{p^+-1} \left(\int_{2B} \frac{C}{r^{p(x_0)}} |f(x)|^{p(x)} \vartheta(x)dx + \int_{2B} |\nabla f(x)|^{p(x)} \vartheta(x)dx \right) \\ &\le \frac{C2^{p^+-1}}{r^{p(x_0)}} \left(\int_{2B} |f(x)|^{p(x)} \vartheta(x)dx + \int_{2B} |\nabla f(x)|^{p(x)} \vartheta(x)dx \right). \end{aligned}$$

The claim (3.4) follows by the infimum over all $f \in S_{p(.),\vartheta}(A)$.

Corollary 3. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $A \Subset \Omega$. Then $C_{p(.),\vartheta}(A) = 0$ if and only if $C_{p(.),\vartheta}(A, \Omega) = 0$.

4. DIRICHLET ENERGY INTEGRAL

Now, we will investigate Dirichlet energy integral as mentioned introduction. Assume that $\Omega \subset \mathbb{R}^n$ is an open set and let $h \in W_{\vartheta}^{1,p(.)}(\Omega)$. We define the energy operator corresponding to the boundary value function h on $W_{0,\vartheta}^{1,p(.)}(\Omega)$ as

$$E_{\Omega,h}^{p(.),\vartheta}(f) = \int_{\Omega} |\nabla f(x) + \nabla h(x)|^{p(x)} \vartheta(x)dx.$$

Our main goal of this section is to investigate a function that minimizes values of the energy operator $E_{\Omega,h}^{p(.),\vartheta}$ on the space $W_{0,\vartheta}^{1,p(.)}(\Omega)$.

The operator E is convex if for all $\alpha \in [0,1]$ and each pair $f, g \in X$ the inequality $E(\alpha f + (1-\alpha)g) \leq \alpha E(f) + (1-\alpha)E(g)$ is satisfied. Also, the operator E is said to be lower semicontinuous if $E(f) \leq \liminf_{m\to\infty} E(f_m)$ whenever $f_m \to f$ in X as $m \to \infty$. Finally, the operator E is coercive if $E(f_m) \to \infty$ whenever $\|f_m\|_X \to \infty$. The proof of the following theorem was given by [[27], Theorem 2.1].

Theorem 10. Suppose that X is a reflexive Banach space. If $E : X \to \mathbb{R}$ is a convex, lower semicontinuous and coercive operator, then there exists an element in X that minimizes E.

Now, we consider the existence of the minimizer for the energy operator $E_{\Omega,h}^{p(.),\vartheta}$ on the space $W_{0,\vartheta}^{1,p(.)}(\Omega)$.

Theorem 11. Let $\Omega \subset \mathbb{R}^n$ be a bounded set. If the exponent p(.) satisfies the jump condition in Ω, then there is a function $f \in W_{0,\vartheta}^{1,p(.)}(\Omega)$ such that

$$E_{\Omega,h}^{p(.),\vartheta}(f) = \inf_{g \in W_{0,\vartheta}^{1,p(.)}(\Omega)} E_{\Omega,h}^{p(.),\vartheta}(g). \tag{4.1}$$

Proof. Our motivation to proof is the previous theorem. It is known that the space $W_{0,\vartheta}^{1,p(.)}(\Omega)$ is a reflexive Banach space. Since the function $t \to t^{p(.)}$ is convex, we have

$$E_{\Omega,h}^{p(.),\vartheta}(\alpha f + (1-\alpha)g) = \int_{\Omega} |\alpha(\nabla f(x) + \nabla h(x)) + (1-\alpha)(\nabla g(x) + \nabla h(x))|^{p(x)} \vartheta(x)dx \leq \alpha E_{\Omega,h}^{p(.),\vartheta}(f) + (1-\alpha)E_{\Omega,h}^{p(.),\vartheta}(g)$$

for all $\alpha \in (0,1)$, and $f, g \in W_{0,\vartheta}^{1,p(.)}(\Omega)$. Hence the energy operator $E_{\Omega,h}^{p(.),\vartheta}$ is convex.

Now $(f_m)_{m\in\mathbb{N}}$ be a sequence of functions in the space $W_{0,\vartheta}^{1,p(.)}(\Omega)$ converging $f \in W_{0,\vartheta}^{1,p(.)}(\Omega)$. By the Corollary 2, we have

$$\|\nabla(f_m + h) - \nabla(f + h)\|_{L_{\vartheta}^{p(.)}(\Omega)} = \|f_m - f\|_{W_{0,\vartheta}^{1,p(.)}(\Omega)} \to 0$$

as $m \to \infty$. Since $p^+ < \infty$, we find that $\rho_{L_{\vartheta}^{p(.)}(\Omega)}(\nabla(f_m + h) - \nabla(f + h)) \to 0$ as $m \to \infty$, see [31]. This follows that

$$\rho_{L_{\vartheta}^{p(.)}(\Omega)}(\nabla(f_m + h)) \to \rho_{L_{\vartheta}^{p(.)}(\Omega)}(\nabla(f + h))$$

as $m\to\infty$, see [31]. Hence

$$\rho_{L_{\vartheta}^{p(.)}(\Omega)}\left(\nabla\left(f+h\right)\right)\le\liminf_{m\to\infty}\rho_{L_{\vartheta}^{p(.)}(\Omega)}\left(\nabla\left(f_m+h\right)\right)$$

that is the energy operator $E_{\Omega,h}^{p(.),\vartheta}$ is lower semicontinuous.

By the Theorem 7, we get that $\|\nabla f_m\|_{L_{\vartheta}^{p(.)}(\Omega)}\to\infty$ whenever $\|f_m\|_{W_{0,\vartheta}^{1,p(.)}(\Omega)}\to\infty$. If we consider the inequality

$$\|\nabla f_m\|_{L_{\vartheta}^{p(.)}(\Omega)}\le\|\nabla f_m+\nabla h\|_{L_{\vartheta}^{p(.)}(\Omega)}+\|\nabla h\|_{L_{\vartheta}^{p(.)}(\Omega)},$$

then we have $\|\nabla f_m+\nabla h\|_{L_{\vartheta}^{p(.)}(\Omega)}\to\infty$ as $m\to\infty$. Since $p^+<\infty$, we obtain that $\rho_{L_{\vartheta}^{p(.)}(\Omega)}\left(\nabla\left(f_m+h\right)\right)\to\infty$ as $m\to\infty$. This implies that the energy operator $E_{\Omega,h}^{p(.),\vartheta}$ is coercive. The proof is completed by Theorem 10.

We deals with quasilinear equations of the form

$$-\operatorname{div}\hat{A}\left(x,\nabla f\right)=0$$

where $\hat{A}:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ is a mapping satisfying the following assumptions for some constants $0<\alpha\le\beta<\infty$

(i) the mapping $x\to\hat{A}(x,\eta)$ is measurable for all $\eta\in\mathbb{R}^n$ and the mapping $\eta\to\hat{A}(x,\eta)$ is continuous for a.e. $x\in\mathbb{R}^n$.

(ii) $\hat{A}(x,\eta)\cdot\eta\ge\alpha\vartheta(x)|\eta|^{p(.)}$

(iii) $\left|\hat{A}(x,\eta)\right|\le\beta\vartheta(x)|\eta|^{p(.)-1}$

(iv) $\left(\hat{A}(x,\eta_1)-\hat{A}(x,\eta_2)\right)\cdot(\eta_1-\eta_2)>0$ whenever $\eta_1,\eta_2\in\mathbb{R}^n$, $\eta_1\ne\eta_2$

(v) $\hat{A}(x,\lambda\eta)=\lambda|\lambda|^{p(.)-2}\hat{A}(x,\eta)$ whenever $\lambda\in\mathbb{R}$, $\lambda\ne0$.

In particular, we give the growth condition $\hat{A}(x,\eta)\cdot\eta\approx\vartheta(x)|\eta|^{p(.)}$. Now, we define the weighted $p(.)$ - Laplace equation as

$$-\Delta_{p(.),\vartheta}=-\operatorname{div}\left(\vartheta(x)|\nabla f|^{p(.)-2}\nabla f\right)=0 \tag{4.2}$$

for every $f\in W_{0,\vartheta}^{1,p(.)}(\Omega)$.

Throughout this paper we assume that $\Omega\subset\mathbb{R}^n$ for $n\ge2$, is an open set. We say that a function $f\in W_{\vartheta,loc}^{1,p(.)}(\Omega)$ is a (weak) weighted solution of (4.2) in Ω, if

$$\int_{\Omega}|\nabla f(x)|^{p(x)-2}\nabla f(x)\cdot\nabla g(x)\vartheta(x)dx=0$$

whenever $g\in C_0^{\infty}(\Omega)$. Moreover, a function $f\in W_{\vartheta,loc}^{1,p(.)}(\Omega)$ is a (weak) weighted supersolution of (4.2) in Ω, if

$$\int_{\Omega}|\nabla f(x)|^{p(x)-2}\nabla f(x)\cdot\nabla g(x)\vartheta(x)dx\ge0 \tag{4.3}$$

whenever $g \in C_0^{\infty}(\Omega)$ is nonnegative. A function f is a weighted subsolution in Ω if -f is a $(p(.),\vartheta)$ - supersolution in Ω, and a weighted solution in Ω.

We recall the Dirichlet spaces as $D_{\vartheta}^{1,p(.)}(\Omega)=\left\{f \in W_{\vartheta,loc}^{1,p(.)}(\Omega): \nabla f \in L_{\vartheta}^{p(.)}(\Omega)\right\}$. Now we improve the definition of weighted solution and supersolution.

Theorem 12. If $f \in D_{\vartheta}^{1,p(.)}(\Omega)$ is a solution (respectively, a supersolution) of (4.2) in Ω, then

$$\int_{\Omega}|\nabla f(x)|^{p(x)-2}\nabla f(x)\cdot\nabla g(x)\vartheta(x)dx=0 \text{ (respectively, } \geq 0\text{)}$$

for all $g \in W_{\vartheta}^{1,p(.)}(\Omega)$ (respectively, for all nonnegative $g \in W_{\vartheta}^{1,p(.)}(\Omega)$).

Proof. Let the function $g \in W_{0,\vartheta}^{1,p(.)}(\Omega)$ be given. Hence we may take a sequence of functions $g_i \in C_0^{\infty}(\Omega)$ such that $g_i \to g$ in $W_{\vartheta}^{1,p(.)}(\Omega)$. By the variable case of Lemma 1.23 in [20], if g is nonnegative, then we may pick nonnegative functions g_i for each i. If we consider the assumption (iii) above, then we have

$$\begin{aligned}\left|\int_{\Omega}|\nabla f(x)|^{p(x)-2}\nabla f(x)\cdot\nabla g(x)\vartheta(x)dx-\int_{\Omega}|\nabla f(x)|^{p(x)-2}\nabla f(x)\cdot\nabla g_i(x)\vartheta(x)dx\right| &\leq \beta\int_{\Omega}|\nabla f(x)|^{p(x)-2}|\nabla g(x)-\nabla g_i(x)|\vartheta(x)dx\\ &\leq \beta C\|\nabla f\|_{L_{\vartheta}^{\frac{p(.)}{p(.)-1}}(\Omega)}\|\nabla g-\nabla g_i\|_{L_{\vartheta}^{p(.)}(\Omega)}\end{aligned}$$

by the variable exponent Hölder inequality. Since $p^{+}<\infty$, we get $\|\nabla g-\nabla g_i\|_{L_{\vartheta}^{p(.)}(\Omega)} \to 0$ as $i \to \infty$. This follows that

$$\int_{\Omega}|\nabla f(x)|^{p(x)-2}\nabla f(x)\cdot\nabla g(x)\vartheta(x)dx=\lim_{i\to\infty}\int_{\Omega}|\nabla f(x)|^{p(x)-2}\nabla f(x)\cdot\nabla g_i(x)\vartheta(x)dx \geq 0.$$

That is the desired result.

The proof of previous theorem give us an important fact that if f is any solution (respectively, supersolution) in Ω, then (4.3) is satisfied for all (respectively, nonnegative) $g \in W_{0,\vartheta}^{1,p(.)}(\Omega)$ with compact support.

Remark 2. In 2003, Fan and Zhang obtained a weak solution in $W_0^{1,p(.)}(\Omega)$ to the Dirichlet problem of $p(x)$ - Laplacian

$$\begin{cases}-\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right)=f(x,u), & x \in \Omega\\ u=0, & x \in \partial\Omega\end{cases}$$

where $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function which satisfies the growth condition, see [14]. Moreover, in recent years, $p(x)$ - Laplacian equations and variational problems with $p(x)$ - growth conditions have been studied by several authors, see [4], [5], [11], [14], [21], [26], [32], [33], [36], [38]. Hence ,weak solutions of weighted $p(x)$ - Laplacian equations

$$\begin{cases}-\operatorname{div}\left(\vartheta(x)|\nabla u|^{p(x)-2}\nabla u\right)=f(x,u), & x \in \Omega\\ u=0, & x \in \partial\Omega\end{cases}$$

can be found in $W_{0,\vartheta}^{1,p(.)}(\Omega)$ by using variational and topological methods under suitable conditions for f and ϑ functions.

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