Sigma J Eng & Nat Sci 36 (2), 2018, 351-359

Sigma Journal of Engineering and Natural Sciences
Sigma Mühendislik ve Fen Bilimleri Dergisi

Research Article

INTERVAL OSCILLATION CRITERIA FOR SECOND-ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS

Süleyman ÖĞREKÇİ*

Amasya University, Faculty of Arts & Science, AMASYA; ORCID:0000-0003-1205-6848

Received: 17.08.2017 Accepted: 09.01.2018

ABSTRACT

This paper concerns the oscillation problem of a general class of second-order differential equations. New interval oscillation criteria for a class of second- order functional nonlinear differential equations with damping and forcing terms have been established by using the classical Riccati technique and averaging function of Philos type. Obtained results extend some of previous works and particularly answer a comment published previously. Illustrative examples also stated.
Keywords: Differential equation, functional term, oscillati.
MSC Number: 34C10, 34C15, 34K11.

1. INTRODUCTION

This paper is concerned with the problem of oscillation of the second order forced nonlinear functional differential equations with nonlinear damping terms of the form

$$\left[r(t)k_1\big(x(t),x'(t)\big)\right]' + p(t)\left[k_2\big(x(t),x'(t)\big)x'(t) + k_3\big(x(t),x'(t)\big)\right] + F\Big(t, x(t), x\big(\tau(t)\big), x'(t), x'\big(\tau(t)\big)\Big) = e(t), \tag{1.1}$$

on the half line $[t_0,\infty)$, $t_0 \geq 0$. In what follows we assume with respect to (1.1) that $r \in C^1\big([t_0,\infty),(0,\infty)\big)$, $p,e \in C([t_0,\infty),\mathbb{R})$, $k_1 \in C^1(\mathbb{R}^2,\mathbb{R})$, $k_2,k_3 \in C(\mathbb{R}^2,\mathbb{R})$, $F \in C([t_0,\infty)\times\mathbb{R}^4,\mathbb{R})$, $\tau \in C\big([t_0,\infty),(0,\infty)\big)$ with $\lim_{t\to\infty}\tau(t) = \infty$.

We restrict our attention to solutions of Eq. (1.1) which exists on $[t_0,\infty)$. As usual, such a solution, $x(t)$, is said to be oscillatory if it has arbitrarily zeros for all $t_0 \geq 0$, otherwise, it is called nonoscillatory. Eq. (1.1) is called oscillatory if all solutions are oscillatory.

There is a great number of papers devoted to particular cases of Eq. (1.1) in the absence of functional and forcing terms such as

* Corresponding Author: e-mail: suleyman.ogrekci@amasya.edu.tr, tel: (358) 242 16 13

$$x''(t)+p(t)x'(t)+q(t)f\big(x(t)\big)=0,$$
$$\big(r(t)x'(t)\big)'+p(t)x'(t)+q(t)f\big(x(t)\big)=0,$$
$$\Big(r(t)\psi\big(x(t)\big)x'(t)\Big)'+p(t)x'(t)+q(t)f\big(x(t)\big)=0,$$
$$\Big(r(t)k_1\big(x(t),x'(t)\big)\Big)'+p(t)k_2\big(x(t),x'(t)\big)x'(t)+q(t)f\big(x(t)\big)=0. \tag{1.2}$$

Numerous oscillation cri teria have been obtained for these equations and their generalizations (see, for example [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12] and references therein).

In 2006, Zhao and Meng [13] obtained some oscillation results for the Eq. (1.2) by using well-known Riccati Technique and kernel functions of Philos type. But in 2007, Çakmak and Tiryaki [14] showed that the proofs given in [13] are inaccurate when $x(t)<0$ and suggested an appropriate replacement for the conditions assumed in proofs.

In 2008, Huang and Meng [15], taking into considerations of Çakmak and Tiryaki, obtained some new oscillation results for the Eq. (1.2). Then, in 2011, Shang and Qin [16] showed that the proofs given in [15] still need revisement since the conditions used in their paper are inappropriate. Shang and Qin showed that the examples given in [15] do not oscillate even they satisfy the conditions of their theorems and guaranteed to be oscillatory. Furthermore, same conditions used in the proofs given in some recent papers such as [17, 18].

Motivated by this fact, in this paper, we first investigate the oscillatory behavior of the second order nonlinear functional differential equation (1.1), which contains the Eq. (1.2) as a special case, by using similar techniques with the proofs given in [17, 18] but revising the inappropriate conditions mentioned above. Furthermore, we use a more general and applicable functional $A(\cdot,n)$ instead of the integral operator, which contains improper integrals, used in [17].

2. MAIN RESULTS

First we introduce a functional that will be used in proofs of some results.

Let

$$D(s_i,t_i)=\{u\in C^1[s_i,t_i]: u(t)\neq 0 \text{ for } t\in(s_i,t_i), u(s_i)=u(t_i)=0\}, \tag{2.1}$$

for $i=1,2$. We define the functional $A_{s_i}^{t_i}(\cdot;n)$ for $H\in D(s_i,t_i)$ and $n\geq 0$ such as;

$$A_{s_i}^{t_i}(h;n)=\int_{s_i}^{t_i}|H(t)|^n h(t)dt, \quad s_i\leq t\leq t_i, i=1,2, \tag{2.2}$$

where $h\in C([t_0,\infty),[0,\infty))$. It is easily seen that the linear functional $A_{s_i}^{t_i}(\cdot;n)$ satisfies

1. $A_{s_i}^{t_i}(h;n)=A_{s_i}^{t_i}(|H|^k h;n-k)$,for $i=1,2$ and $k\in\mathbb{R}$;
2. $A_{s_i}^{t_i}(h';n)\geq -A_{s_i}^{t_i}(n|H'h|;n-1)$, for $i=1,2$.

In proofs of some of our results, we will also use another class of averaging functions $G(t,s)\in C(D_1,R)$ which satisfy

1. $G(t,t)=0$, $G(t,s)>0$ for $t>s$,
2. G has partial derivatives $\partial G/\partial t$ and $\partial G/\partial s$ on D_1 such that
$\frac{\partial G}{\partial t}=g_1(t,s)\sqrt{G(t,s)}, \qquad \frac{\partial G}{\partial s}=-g_2(t,s)\sqrt{G(t,s)}$

where $D_1=\{(t,s): t_0\leq s\leq t<\infty\}$ and $g_1,g_2\in L_{loc}(D_1,\mathbb{R}^+)$.

Next, we state two useful lemmas that will be used as important tools in some of our proofs.

Lemma 1 *[19] If A and B are non-negative constants and $m,n\in\mathbb{R}$ such that $\frac{1}{m}+\frac{1}{n}=1$, then* $\frac{1}{m}A+\frac{1}{n}B\geq A^{1/m}B^{1/n}$.

Lemma 2 *[20] Assume t hat $\tau \in C([t_0,\infty),\mathbb{R}^+)$, $\tau(t) < t$ for $t \geq t_0$ and $lim_{t\to\infty}\tau(t) = \infty$. Suppose also that $x(t) \in C^2([T,\infty),\mathbb{R})$ for some $T > 0$, $x(t) > 0$ and $x''(t) \leq 0$ for $t \geq T > 0$. Then, for each $k \in (0,1)$, there exists a constant $T_k \geq T$ such that*

$$\frac{x(\tau(t))}{x(t)} \geq \left(\frac{\tau(t)}{t}\right)^{\frac{1}{k}} \quad \text{for} t > \tau(t) \geq T_k. \tag{2.3}$$

We shall make use of the following conditions in our results:

1. For any $T \geq t_0$, there exists $T \leq s_1 < t_1 \leq s_2 < t_2$ such that
   $e(t) \leq 0 \text{for} t \in s_1, t_1]$,
   $e(t) \geq 0 \text{for} t \in s_2, t_2]$,
2. there exist a function $q_1(t) > 0$ and a constant $\gamma \geq 1$ such that $F(t,x,u,v,w)/x \geq q(t)|x|^{\gamma-1}$ holds for $t \in s_1, t_1] \cup s_2, t_2]$ and $x \neq 0, u, v, w \in \mathbb{R}$,
3. $vk_1(u,v) \geq \beta|k_1(u,v)|^{(\alpha+1)/\alpha}|u|^{(\alpha-1)/\alpha}$, for some $\alpha > 0$, $\beta > 0$ and for all $u \in \mathbb{R}$, $v \neq 0$.
4. $uvk_2(u,v) \geq 0$ and $uk_3(u,v) \geq 0$ for all $(u,v) \in \mathbb{R}^2$,
5. $\tau(t) \leq t$ for $t \in t_0, \infty)$,
6. $sgnF(t,x,u,v,w) = sgnx$ for each $t \geq t_0$, and $x, u, v, w \in \mathbb{R}$,
7. there exist a function $q_2(t) > 0$ and a constant $\eta \geq 1$ such that $F(t,x,u,v,w)/u \geq q_2(t)|u|^{\eta-1}$ holds for $t \in s_1, t_1] \cup s_2, t_2]$ and $x, u \neq 0, v, w \in \mathbb{R}$.

We are now able to state our results.

Theorem 1 *Suppose the conditions $(C_1 - C_4)$ hold and $p(t) \geq 0$ for $t \in s_1, t_1] \cup s_2, t_2]$. If there exists $H \in D(s_i, t_i)$ and a nonnegative constant n such that the inequality*

$$A_{s_i}^{t_i}(Q_1; n+\alpha+1) > A_{s_i}^{t_i}(\delta_1 r|H'|^{\alpha+1}; n), \tag{2.4}$$

holds for $i = 1,2$ where $\delta_1 = \left(\frac{\alpha}{\beta}\right)^{\alpha}\left(\frac{n+\alpha+1}{\alpha+1}\right)^{\alpha+1}$, $Q_1(t) = \gamma(\gamma-1)^{(1-\gamma)/\gamma}[q_1(t)]^{1/\gamma}|e(t)|^{(\gamma-1)\gamma}$ with the convention $0^0 = 1$, the functional A and the set D are defined with (2.2), (2.1) respectively. Then the Eq. (1.1) is oscillatory.

Proof. On the contrary, suppose that Eq. (1.1) has a nonoscillatory solution $x(t)$. Then $x(t)$ eventually must have one sign, i.e. $x(t) \neq 0$ on $[T_0, \infty)$ for some large $T_0 \geq t_0$. Define

$$w_1(t) = \frac{r(t)k_1(x(t),x'(t))}{x(t)} \tag{2.5}$$

for $t \in s_1, t_1] \cup s_2, t_2]$. Then differentiating (2.5) and using Eq. (1.1) we obtain

$$w_1'(t) = \frac{e(t)}{x(t)} - \frac{p(t)k_2(x(t),x'(t))x'(t)}{x(t)} - \frac{p(t)k_3(x(t),x'(t))}{x(t)}$$
$$-\frac{r(t)k_1(x(t),x'(t))x'(t)}{x^2(t)} - F\left(t, x(t), x'(t), x(\tau(t)), x'(\tau(t))\right)$$

for $t \in s_1, t_1] \cup s_2, t_2]$. By assuming $(C_1 - C_4)$ we obtain for $t \in s_1, t_1] \cup s_2, t_2]$

$$q(t)|x(t)|^{\gamma-1} - \frac{e(t)}{x(t)} \leq -w_1'(t) - \beta r^{-1/\alpha}(t)|w_1(t)|^{(\alpha+1)/\alpha}. \tag{2.6}$$

Which means the inequality

$$q(t)|x(t)|^{\gamma-1} + \left|\frac{e(t)}{x(t)}\right| \leq -w_1'(t) - \beta r^{-1/\alpha}(t)|w_1(t)|^{(\alpha+1)/\alpha}$$

holds for $t \in [s_1, t_1]$ and $t \in [s_2, t_2]$.

For $\gamma > 1$, by setting $m = \gamma$, $n = \gamma/(\gamma-1)$, $A = \gamma q_1(t)|x(t)|^{\gamma-1}$, $B = \left(\frac{\gamma}{\gamma-1}\right)|\frac{e(t)}{x(t)}|$ and using Lemma 1, we obtain

$$q_1(t)|x(t)|^{\gamma-1} + |\frac{e(t)}{x(t)}| \geq Q_1(t).$$

Hence, on the intervals $[s_1, t_1]$ and $[s_2, t_2]$, $w_1(t)$ satisfies

$$Q_1(t) \leq -w_1'(t) - \beta r^{-1/\alpha}(t)|w_1(t)|^{(\alpha+1)/\alpha}. \tag{2.7}$$

Note that the inequality holds for $\gamma = 1$ also with the convention $0^0 = 1$.

Now multiplying $|H(t)|^{n+\alpha+1}$ throughout Eq.(2.7) and integrating from s_i to t_i, we obtain

$$A_{s_i}^{t_i}(Q; n+\alpha+1) \leq A_{s_i}^{t_i}((n+\alpha+1)|H|^\alpha|H'||w| - a|H|^{\alpha+1}|w|^{(\alpha+1)/\alpha}; n), \tag{2.8}$$

where $a = \beta r^{-1/\alpha}$ and $D(s_i, t_i)$ is given by hypotheses. Setting

$$m_1(v) := (n+\alpha+1)|H|^\alpha|H'|v - a|H|^{\alpha+1}v^{(\alpha+1)/\alpha}, \qquad v > 0,$$

we have $m_1'(v^*) = 0$ and $m_1''(v^*) < 0$, where $v^* = \left(\frac{\alpha(n+\alpha+1)}{\alpha+1}\frac{1}{a}|\frac{H'}{H}|\right)^\alpha$, which implies that $F(v)$ obtains its maximum at v^*. So we have

$$m_1(v) \leq m_1(v^*) = \left(\frac{\alpha}{\beta}\right)^\alpha \left(\frac{n+\alpha+1}{\alpha+1}\right)^{\alpha+1} |H'|^{\alpha+1}. \tag{2.9}$$

Then we get, by using (2.9) in (2.8), we obtain

$$A_{s_i}^{t_i}(Q; n+\alpha+1) \leq A_{s_i}^{t_i}(\delta_1 r|H'|^{\alpha+1}; n), \tag{2.10}$$

which contradicts to (2.4). Thus the proof is complete.

Theorem 2 *Suppose the conditions* $(C_1 - C_4)$ *hold and* $p(t) \geq 0$ *for* $t \in s_1, t_1] \cup s_2, t_2]$. *If there exist some* $\varepsilon_i \in (s_i, t_i)$, $i = 1,2$, $G(t,s)$ *satisfying (iii)-(iv) and a positive function* $\rho \in C^1([t_0, \infty), \mathbb{R}^+)$ *such that*

$$\frac{1}{G^{\alpha+1}(\varepsilon_i, s_i)} \int_{s_i}^{\varepsilon_i} [G^{\alpha+1}(\tau, s_i)Q_1(\tau)\rho(\tau) - \delta_2 G_1^{\alpha+1}(\tau, s_i)r(\tau)\rho(\tau)]d\tau$$
$$+ \frac{1}{G^{\alpha+1}(t_i, \varepsilon_i)} \int_{\varepsilon_i}^{t_i} [G^{\alpha+1}(t_i, \tau)Q_1(\tau)\rho(\tau) - \delta_2 G_2^{\alpha+1}(t_i, \tau)r(\tau)\rho(\tau)]d\tau \tag{2.11}$$
$$> 0$$

for $i = 1,2$ where

$$\delta_2 = \frac{\alpha^\alpha}{\beta^\alpha(\alpha+1)^{\alpha+1}},$$
$$G_1(t,s) = \left|(\alpha+1)g_1(t,s)\sqrt{G(t,s)} + G(t,s)\frac{\rho'(s)}{\rho(s)}\right|,$$
$$G_2(t,s) = \left|(\alpha+1)g_2(t,s)\sqrt{G(t,s)} - G(t,s)\frac{\rho'(s)}{\rho(s)}\right|.$$

Then Eq. (1.1) is oscillatory.

Proof. On the contrary, suppose that Eq. (1.1) has a nonoscillatory solution $x(t)$. Then $x(t) \neq 0$ on $[T, \infty)$ for some sufficiently large $T \geq t_0$. Define

$$w_2(t) = \rho(t)\frac{r(t)k_1(x(t), x'(t))}{x(t)}, t \in s_1, t_1] \cup s_2, t_2]. \tag{2.12}$$

Differentiating (2.12) , using conditions $(C_1 - C_4)$ and Eq. (1.1) we obtain

$$\rho(t)\left(q_1(t)|x(t)|^{\gamma-1} + \left|\frac{e(t)}{x(t)}\right|\right) \leq -w_2'(t) + \frac{\rho'(t)}{\rho(t)}w_2(t) - \beta r^{-1/\alpha}(t)\rho^{-1/\alpha}(t)|w_2(t)|^{(\alpha+1)/\alpha}$$

for $t \in s_1, t_1]$ or $t \in s_2, t_2]$. As in the proof of previous result, using Lemma 1, we obtain

$$\rho(t)Q_1(t) \leq -w_2'(t) + \frac{\rho'(t)}{\rho(t)}w_2(t) - \beta r^{-1/\alpha}(t)\rho^{-1/\alpha}(t)|w_2(t)|^{(\alpha+1)/\alpha} \tag{2.13}$$

for $t \in s_1, t_1]$ or $t \in s_2, t_2]$ and for $\gamma \geq 1$.

Now multiplying (2.13) with $G^{\alpha+1}(t,s)$ and integrating (with t replaced by s) over $[\varepsilon_i, t)$ for $t \in \varepsilon_i, t_i)$ and $i = 1,2$ we have

$\int_{\varepsilon_i}^{t} G^{\alpha+1}(t,s)Q_1(s)\rho(s)ds \le G^{\alpha+1}(t,\varepsilon_i)w_2(\varepsilon_i) +$
$\int_{\varepsilon_i}^{t} G^{\alpha}(t,s)G_2(t,s)|w_2(s)|ds -$
$\int_{\varepsilon_i}^{t} \beta r^{-1/\alpha}(t)\rho^{-1/\alpha}(t)G^{\alpha+1}(t,s)|w_2(t)|^{(\alpha+1)/\alpha}ds.$ (2.14)

For a given t and s, set

$$m_2(v) = G^{\alpha}G_2 v - \beta r^{-1/\alpha}\rho^{-1/\alpha}G^{\alpha+1}v^{(\alpha+1)/\alpha}, \quad v > 0.$$

m_2 yields its maximum at the point $v^* = \left(\frac{\alpha}{\alpha+1}\frac{G_2}{\beta G r^{-1/\alpha}\rho^{-1/\alpha}}\right)^{\alpha}$ and

$$m_2(v) \le m_{2_{\max}} = m_2(v^*) = \delta_2 G_2 r\rho. \tag{2.15}$$

Then, by using (2.15) and letting $t \to t_i^-$ in (2.14), we get

$$\int_{\varepsilon_i}^{t_i} G^{\alpha+1}(t_i,s)Q_1(s)\rho(s)ds \le G^{\alpha+1}(t_i,\varepsilon_i)w_2(\varepsilon_i) + \delta_2\int_{\varepsilon_i}^{t_i} G_2^{\alpha+1}(t_i,s)r(s)\rho(s)ds. \tag{2.16}$$

On the other hand, multiplying (2.13) with $G^{\alpha+1}(s,t)$, then integrating (with t replaced by s) over $[t,\varepsilon_i)$ for $t \in t_i,\varepsilon_i)$, $i = 1,2$ and using similar calculations with the proof of (2.16) we get

$$\int_{t}^{\varepsilon_i} G^{\alpha+1}(s,s_i)Q_1(s)\rho(s)ds \le -G^{\alpha+1}(\varepsilon_i,s_i)w_2(\varepsilon_i) + \delta_2\int_{t}^{\varepsilon_i} G_1^{\alpha+1}(s,s_i)r(s)\rho(s)ds. \tag{2.17}$$

Letting $t \to s_i^+$ in (2.17), it follows that

$$\int_{s_i}^{\varepsilon_i} G^{\alpha+1}(s,s_i)Q_1(s)\rho(s)ds \le -G^{\alpha+1}(\varepsilon_i,s_i)w_2(\varepsilon_i) + \delta_2\int_{s_i}^{\varepsilon_i} G_1^{\alpha+1}(s,s_i)r(s)\rho(s)ds. \tag{2.18}$$

Finally, dividing (2.16) and (2.18) by $G^{\alpha+1}(t_i,\varepsilon_i)$ and $G^{\alpha+1}(\varepsilon_i,s_i)$ respectively, and then adding them, we have the desired contradiction with (2.11). Thus the proof is complete.

Corollary 1 *Suppose the conditions $(C_1 - C_4)$ hold and $p(t) \ge 0$ for $t \in s_1,t_1] \cup s_2,t_2]$. If there exist some $\varepsilon_i \in (s_i,t_i)$, $i = 1,2$, $G(t,s)$ satisfying (iii)-(iv) and a positive function $\rho \in C^1([t_0,\infty),\mathbb{R}^+)$ such that*

$$\int_{s_i}^{\varepsilon_i} [G^{\alpha+1}(\tau,s_i)Q_1(\tau)\rho(\tau) - \delta_2 G_1^{\alpha+1}(\tau,s_i)r(\tau)\rho(\tau)]d\tau > 0 \tag{2.19}$$

and

$$\int_{\varepsilon_i}^{t_i} [G^{\alpha+1}(t_i,\tau)Q_1(\tau)\rho(\tau) - \delta_2 G_2^{\alpha+1}(t_i,\tau)r(\tau)\rho(\tau)]d\tau > 0 \tag{2.20}$$

for $i = 1,2$. Then the Eq. (1.1) is oscillatory.

Now, we consider the following special case of Eq. (1.1), namely

$$\left(\psi(x)k(x')\right)' + F\left(t,x(t),x\left(\tau(t)\right),x'(t),x'\left(\tau(t)\right)\right) = e(t) \tag{2.21}$$

where $\psi: \mathbb{R} \to \mathbb{R}^+$ is a differentiable function with $0 < \psi(x) \le L$ and $\psi_x(x) \ge 0$ for $x \in \mathbb{R}$, $L \in \mathbb{R}$, the function $k: \mathbb{R} \to \mathbb{R}$ is differentiable with $vk(v) > 0$ and $k_v(v) \ge 0$, the functions F, τ and e are defined as before.

Theorem 3 *Suppose the conditions (C_1), (C_3) for $k_1(u,v) = \psi(u)k(v)$, $(C_5 - C_7)$ hold and $p(t) \ge 0$ for $t \in s_1,t_1] \cup s_2,t_2]$. If there exists $H \in D(s_i,t_i)$ and a nonnegative constant n such that the inequality*

$$A_{s_i}^{t_i}(Q_2; n+\alpha+1) > A_{s_i}^{t_i}(\delta_1|H'|^{\alpha+1}; n), \tag{2.22}$$

holds for $i = 1,2$, $Q_2(t) = \eta(\eta-1)^{(1-\eta)/\eta}[q_2(t)(\tau(t)/t)^{\eta/k}]^{1/\eta}|e(t)|^{(\eta-1)\eta}$ with the convention $0^0 = 1$ and $k \in (0,1)$, the functional A and the set D are defined with (2.2), (2.1) respectively. Then the Eq. (2.21) is oscillatory.

Proof. On the contrary, suppose that Eq. (1.1) has a nonoscillatory solution $x(t)$. Then $x(t)$ eventually must have one sign, i.e. $x(t) \neq 0$ on $[T_0, \infty)$ for some large $T_0 \geq t_0$. Firstly, we suppose that $x(t) > 0$ on $[T_0, \infty)$ Define

$w_3(t) = \frac{\psi(x(t))k(x'(t))}{x(t)}$ (2.23)

for $t \in s_1, t_1] \cup s_2, t_2]$. Then differentiating (2.23), using Eq. (1.1) and the assumptions of theorem, we obtain

$$w_3'(t) \leq -q_2(t)\left|x(\tau(t))\right|^{\eta-1} \frac{x(\tau(t))}{x(t)} - \beta |w_3(t)|^{(\alpha+1)/\alpha} + \frac{e(t)}{x(t)}. \tag{2.24}$$

By the assumptions, we can choose $s_1, t_1 > T_0$ such that $e(t) \leq 0$ on the interval $[s_1, t_1]$. On this interval we have

$$\left[\psi(x(t))k(x'(t))\right]' \leq 0$$

i.e.,

$$\psi_x(x(t))k(x'(t))x'(t) + \psi(x(t))k_{x'}(x'(t))x''(t) \leq 0$$

which implies that $x''(t) \leq 0$ for $t \in [s_1, t_1]$. Hence, by Lemma.2, we have

$$x(\tau(t)) \geq \left(\frac{\tau(t)}{t}\right)^{\frac{1}{k}} x(t) \tag{2.25}$$

for any $k \in (0,1)$ and $t \in [s_1, t_1]$. Using (2.25) in (2.24) we obtain

$$q_3(t)|x(t)|^{\eta-1} + \left|\frac{e(t)}{x(t)}\right| \leq -w_3'(t) - \beta |w_3(t)|^{(\alpha+1)\alpha} \tag{2.26}$$

for $q_3(t) = \left(\frac{\tau(t)}{t}\right)^{\eta/k} q_2(t)$, and $t \in [s_1, t_1]$. Note that if $x(t)$ is eventually negative, thanks to condition (C_6), by choosing $s_2, t_2 > T_0$ such that $e(t) \geq 0$ on the interval $[s_2, t_2]$ we can also obtain the inequality (2.26) for $t \in [s_2, t_2]$. Thus, the rest of proof is similar with the proof of Theorem.1, hence omitted.

Theorem 4 *Suppose the conditions* (C_1), (C_3) *for* $k_1(u,v) = \psi(u)k(v)$, $(C_5 - C_7)$ *hold and* $p(t) \geq 0$ *for* $t \in s_1, t_1] \cup s_2, t_2]$. *If there exist some* $\varepsilon_i \in (s_i, t_i)$, $i = 1,2$, $G(t,s)$ *satisfying (iii)-(iv) and a positive function* $\rho \in C^1([t_0, \infty), \mathbb{R}^+)$ *such that*

$$\frac{1}{G^{\alpha+1}(\varepsilon_i, s_i)} \int_{s_i}^{\varepsilon_i} [G^{\alpha+1}(\tau, s_i)Q_2(\tau)\rho(\tau) - \delta_2 G_1^{\alpha+1}(\tau, s_i)\rho(\tau)]d\tau$$
$$+ \frac{1}{G^{\alpha+1}(t_i, \varepsilon_i)} \int_{\varepsilon_i}^{t_i} [G^{\alpha+1}(t_i, \tau)Q_2(\tau)\rho(\tau) - \delta_2 G_2^{\alpha+1}(t_i, \tau)\rho(\tau)]d\tau \tag{2.27}$$
$$> 0$$

for $i = 1,2$. Then the Eq. (2.21) is oscillatory.

Proof. On the contrary, suppose that Eq. (1.1) has a nonoscillatory solution $x(t)$. Then $x(t) \neq 0$ on $[T, \infty)$ for some sufficiently large $T \geq t_0$. Define

$$w_4(t) = \rho(t) \frac{\psi(x(t))k(x'(t))}{x(t)}, t \in s_1, t_1] \cup s_2, t_2]. \tag{2.28}$$

As in the proof of previous theorem, differentiating (2.28), using the Eq. (2.21) and Lemma.2, we obtain the inequality

$$\rho(t)\left(q_3(t)|x(t)|^{\eta-1} + \left|\frac{e(t)}{x(t)}\right|\right) \leq -w_4'(t) + \frac{\rho'(t)}{\rho(t)} w_4(t) - \beta \rho^{-1/\alpha}(t)|w_4(t)|^{(\alpha+1)/\alpha} \tag{2.29}$$

where q_3 is as defined before, $\eta \geq 1$ and $t \in [s_1, t_1]$ or $t \in [s_2, t_2]$. Using Lemma.1, one can easily obtain that the inequality

$$\rho(t)Q_2(t) \leq -w_4'(t) + \frac{\rho'(t)}{\rho(t)} w_4(t) - \beta\rho^{-1/\alpha}(t)|w_4(t)|^{(\alpha+1)/\alpha} \tag{2.30}$$

for $t \in [s_1, t_1]$ or $t \in [s_2, t_2]$. The rest of the proof is similar with the proof of Theorem.2, hence omitted.

Corollary 2 *Suppose the conditions (C_1), (C_3) for $k_1(u,v) = \psi(u)k(v)$, $(C_5 - C_7)$ hold and $p(t) \geq 0$ for $t \in s_1, t_1] \cup s_2, t_2]$. If there exist some $\varepsilon_i \in (s_i, t_i)$, $i = 1,2$, $G(t,s)$ satisfying (iii)-(iv) and a positive function $\rho \in C^1([t_0, \infty), \mathbb{R}^+)$ such that*

$$\int_{s_i}^{\varepsilon_i} [G^{\alpha+1}(\tau, s_i)Q_2(\tau)\rho(\tau) - \delta_2 G_1^{\alpha+1}(\tau, s_i)\rho(\tau)]d\tau > 0$$

and

$$\int_{\varepsilon_i}^{t_i} [G^{\alpha+1}(t_i, \tau)Q_2(\tau)\rho(\tau) - \delta_2 G_2^{\alpha+1}(t_i, \tau)\rho(\tau)]d\tau > 0$$

for $i = 1,2$. Then the Eq. (2.21) is oscillatory.

Finally, now we apply Theorem.3 and Tehorem.4 to following second order nonlinear differential equation with delayed argument

$$\Big(\psi\big(x(t)\big)k\big(x'(t)\big)\Big)' + q(t)\big|x\big(\tau(t)\big)\big|^{\eta-1}x\big(\tau(t)\big) = e(t) \tag{2.31}$$

where $q \in C([t_0, \infty), \mathbb{R})$, ψ, k, e and τ are defined as before.

Corollary 3 *Let $q(t) \geq 0$ for $t \in [s_1, t_1] \cup [s_2, t_2]$ and the conditions of Theorem.3 satisfies except (C_6) and (C_7). Then equation (2.31) is oscillatory.*

Corollary 4 *Let $q(t) \geq 0$ for $t \in [s_1, t_1] \cup [s_2, t_2]$ and the conditions of Theorem.4 satisfies except (C_6) and (C_7). Then equation (2.31) is oscillatory.*

Corollary 5 *Let $q(t) \geq 0$ for $t \in [s_1, t_1] \cup [s_2, t_2]$ and the conditions of Corollary.2 satisfies except (C_6) and (C_7). Then equation (2.31) is oscillatory.*

Example 1 *Consider the second-order nonlinear differential equation*

$$\left(t^{3\lambda+1}\frac{x'(t)}{1+\big(x'(t)\big)^2}\right) + p(t)\left[\frac{x(t)\big(x'(t)\big)^2}{1+\big(x'(t)\big)^2} + \frac{x(t)\big(x'(t)\big)^2}{1+\big(x(t)\big)^2}\right]$$

$$+N_1 t^{3\lambda}x(t)\left[1 + \textstyle\sum_{k=1}^{m} b_k\left(\Big(x\big(\tau(t)\big)\Big)^{2k} + \Big(x'\big(\tau(t)\big)\Big)\right)\right] = \sin t \tag{2.32}$$

where p is any nonnegative function, $\lambda > 0$, $m > 1$, $N_1 > 0$, $b_k \geq 0$ and $t \geq t_0 > 1$. Note that the functions

$$k_1(u,v) = \frac{v}{1+v^2}, k_2(u,v) = \frac{uv}{1+v^2}, k_3(u,v) = \frac{uv^2}{1+u^2}$$

satisfies the conditions $(C_3 - C_4)$ with $\alpha = \beta = 1$ and the function $F(t,x,u,v,w)$ satisfies the condition (C_2) with $\eta = 1$ and $q_1(t) = N_1 t^{3\lambda}$. So we obtain $Q(t) = q_1(t) = N_1 t^{3\lambda}$.

Now, choosing $n = 1$, $s_1 = k\pi$, $t_1 = s_2 = (k+1)\pi$, $t_2 = (k+2)\pi$, and $H(t) = t^{-\lambda}\sin^2 t$, we obtain

$$A_{s_i}^{t_i}\big(N_1 t^{3\lambda}; 3\big) = \int_{k\pi}^{(k+1)\pi} \sin^6 t dt = \tfrac{5N_1}{16}. \tag{2.33}$$

On the other hand, with elemantery calculations, one can have

$$A_{s_i}^{t_i}(\delta_1 r|H'|^2;1) \le \frac{9}{4}\int_{k\pi}^{(k+1)\pi}\left(\frac{\lambda^2}{t}+4t-4\lambda\sin^5 t\cos t\right)dt$$
$$=\frac{27}{2}\pi^2+\frac{9}{4}\lambda^2\ln 2. \tag{2.34}$$

From Theorem.1, by combining (2.33) and (2.34), equation (2.32) is oscillatory if $N_1 > \frac{9}{10\pi}(6\pi^2+\lambda^2\ln 2)$.

Example 2 *Consider the second-order nonlinear differential equation*

$$x''(t)+x(t)\left[1+\sum_{k=1}^{m} b_k\left(\left(x(\tau(t))\right)^{2k}+\left(x'(\tau(t))\right)\right)\right]=e(t) \tag{2.35}$$

where $m > 1$, $b_k \ge 0$ and $t \ge t_0 > 1$. Note that the conditions $(C_2 - C_4)$ of Teorem.2 are satisfied with $\alpha=\beta=\gamma=1$, $Q_1(t)=q_1(t)=1$.

Now, choosing $s_1=\pi$, $t_1=s_2=3\pi$, $t_2=5\pi$, $\varepsilon_1=2\pi$, $\varepsilon_2=4\pi$ and the functions

$$\rho(t)=1, G(t,s)=(t-s)$$

we have

$$g_1(t,s)=g_2(t,s)=\frac{1}{\sqrt{t-s}}$$
$$G_1(t,s)=G_2(t,s)=\alpha+1.$$

By elemantery calculations, one can see that the inequality (2.11) holds. Thus, by Theorem.2, the equation (2.35) is oscillatory.

REFERENCES

[1] J.S.W. Wong, On Kamenev-type oscillation for second order differential equations with damping, J. Math. Anal. Appl. 248 (2001) 244–257.

[2] X. Yang, Oscillation criteria for nonlinear differential equations with damping, Applied Mathematics and Computation 136 (2003) 549–557.

[3] Y.V. Rogovchenko, Oscillation theorems for second order equations with damping, Nonlinear Anal. 41 (2000) 1005–1028.

[4] A. Tiryaki and A. Zafer, Oscillation criteria for second order nonlinear differential equation with damping, Turkish J. Math. 24 (2000) 185–196.

[5] O.G. Mustafa and S.P. Rogovchenko, Oscillation of nonlinear second order differential equations with damping term, J. Math. Anal. Appl. 298 (2004) 604–620.

[6] S.P. Rogovchenko and Yu.V. Rogovchenko, Oscillation of second order differential equations with damping, Dynam. Contin. Discrete Impuls. Syst. Ser. A 10 (2003) 447–461.

[7] B. Ayanlar and A. Tiryaki, Oscillation theorems for nonlinear second order differential equation with damping, Acta Math. Hungar. 89 (2000) 1-13.

[8] S.P. Rogovchenko and Yu.V. Rogovchenko, Oscillation theorems for differential equations with a nonlinear damping term, J. Math. Anal. Appl.279 (2003) 139–152.

[9] A. Tiryaki and A. Zafer, Interval oscillation of a general class of second-order nonlinear differential equations with nonlinear damping, Nonlinear Anal. 60 (2005) 49–63.

[10] M. Bayram, A. Seçer, H. Adıgüzel, Oscillatory behavior of solutions of differential equations with fractional order, Appliead Mathematics & Information Sciences, 11 (3) 683-691.

[11] M. Bayram, A. Seçer, H. Adıgüzel, On the oscillation of fractional order nonlinear differential equations, Sakarya University Journal of Science, 21 (6) 1-20.

[12] M. Bayram, H. Adıgüzel, A. Seçer, Oscillation criteria for nonlinear fractional differential equation with damping term, Open Physics, 14 (1) 119-128.

[13] X. Zhao and F. Meng , Oscillation of second-order nonlinear ODE with damping, Applied Mathematics and Computation 182 (2006) 1861–1871.

[14] D. Çakmak and A. Tiryaki, Comment on the paper "Oscillation of second-ordernonlinear ODE with damping", Applied Mathematics and Computation 191 (2007) 298.

[15] Y. Huang and F. Meng, Oscillation of second-order nonlinear ODE with damping, Applied Mathematics and Computation 199 (2008) 644–652.

[16] N.Shang and H. Qin, Comments on the paper: "Oscillation of second-order nonlinear ODEwith damping" [Applied Mathematics and Computation 199 (2008) 644–652], Applied Mathematics and Computation 218 (2011) 2979–2980.

[17] F. Meng and Y. Huang, Interval oscillation criteria for a forced second-order nonlinear differential equations with damping, Applied Mathematics and Computation 218 (2011) 1857-1861.

[18] Y. Huang and F. Meng, Oscillation criteria for forced second-order nonlinear differential equations with damping, Journal of Computational and Applied Mathematics 224 (2009) 339-345.

[19] G.H. Hardy, J.E. Littlewood, G. Polya, Inequalities, second ed., Cambridge University Press, Cambridge, 1988.

[20] A. Tiryaki and Y Başçı, Interval oscillation criteria for second-order quasi-linear functional differential equations, Dynamics of Continuous, Discrete and Impulsive Systems Serias A: Mathematical Analysis 16 (2009) 233-252.