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Research Article

AN EFFICIENT LOCAL TRANSFORM METHOD FOR INITIAL VALUE PROBLEMS

Huseyin TUNC[1], Murat SARI*[2]

[1] *Department of Mathematics, Yildiz Technical University, ISTANBUL;* ORCID: 0000-0001-6450-5380
[2] *Department of Mathematics, Yildiz Technical University, ISTANBUL;* ORCID: 0000-0003-0508-2917

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ABSTRACT

This paper has proposed a local differential transform method in analysing various types of linear and nonlinear initial value problems (IVP) representing physical models encountered in a broad range of science. Local and global error analyses of the proposed scheme are presented to demonstrate the capacity and priorities of the local differential transform method (LDTM). The produced results show that even using coarser meshes the present scheme produce quite a little error in finite time. The present solution technique in solving the IVPs is compared with the Runge-Kutta method. It is proved that the LDTM produces more accurate results than the Runge-Kutta methods studied in the literature. By considering various types of initial value problems, the stabilities of the LDTM and the RK4 are examined with various time intervals in a comparative way.
Keywords: Local differential transform method, initial value problem, system of differential equations, error analysis, Runge-Kutta method.
MSC2010: 65L05, 65L06, 34K28.

1. INTRODUCTION

Initial value problems of differential equations arise in many fields of science including physics, chemistry, engineering, mathematical biology etc. Depending on the problems, the exact solutions of these problems may or may not be derived mathematically. The derivation of these exact solution may have computational complexities especially in large systems. These drawbacks of analytical approaches can be overcome by considering accurate and economic numerical methods. For solving IVPs there are various methods in the literature such as Euler method, θ- method, Crank-Nicolson method, Taylor method, Runge-Kutta methods and so on [1]. Among these methods, the Runge-Kutta and the Crank-Nicolson methods are commonly used. Because the Runge-Kutta methods generally provide higher order accuracy and the Crank-Nicolson method is a second order unconditionally stable method. Although the Taylor method provides higher order accuracy, the method has computational complexity because of using exact derivatives of the functions.

The concept of the DTM was first proposed by Zhou [2] and the method was used to solve initial value problems occurred in electric circuit analysis. This method was exerted to the system

* Corresponding Author: e-mail: sarim@yildiz.edu.tr, tel: (212) 383 43 60

of differential equations [3], fluid flow problems [4,5], magnetohydrodynamics boundary-layer equations [6], nonline ar partial differential equations [7,8] and Duffing oscillator equation with damping effects [9]. Many of the considered references are preferred to use the DTM in global sense, i.e. analytic approximation series are considered about the initial time $t = 0$ without considering any time discretization. Generally, this semi-analytic approach gives good agreement with exact solution only about the initial position. To avoid this drawback, the modified differential transform method (MDTM) was developed and applied to various scientific problems [6,9]. The MDTM is based on using Laplace transformation and Pade approximation after obtaining approximate solution with differential transform method in global sense. The main goal of the MDTM is to decrease numerical error of the global DTM, which is the natural result of the truncated Taylor series expansion when the time moves away from the zero.

The DTM can also be used to produce discrete or continuous approximate solutions of differential equations in local sense, which first introduced by Jang et. al. [10]. The main advantage of the local transform method (LDTM) is that the order of the method and accuracy of the results can be increased with and without changing time increment, also provides high accuracy not only at the neighbourhood of the initial position but also the entire domain. The proposed method has important priorities such as less computational time, high order accuracy, adaptivity and flexibility.

In this study, it is proved the order of the LDTM is stable with increasing time. The priori error estimate is established for the LDTM and the priorities of the LDTM over the DTM in global sense are shown by analysing the global error bounds of the considered methods. Thus, the LDTM produces more accurate and economic results than both the MDTM and the DTM in the whole-time domain as will be demonstrated. By considering linear and nonlinear IVPs, using the produced results, the LDTM method is compared with the DTM, the fourth-order Runge-Kutta (RK4) method and the exact solution. As demonstrated, the LDTM method provides more excellent results for various IVPs.

2. DIFFERENTIAL TRANSFORMATION

The definitions of the DTM are reorganized from reference [11] by considering local sense as follows:

Definition 1. Let $x(t)$ is analytic in the domain T, the function $\varphi(t,k)$ can be defined as follows

$$\frac{d^k x(t)}{dt^k} = \varphi(t,k) \text{ for all } t \in T, \tag{1}$$

where k belongs to non-negative integer. The differential transform of function $x(t)$ at any time $t = t_n$ in the domain is locally defined as follows:

$$X(k) = \frac{\varphi(t_n,k)}{k!} = \frac{1}{k!}\left[\frac{d^k x(t)}{dt^k}\right]\Bigg|_{t=t_n}. \tag{2}$$

Definition 2. If $x(t)$ is analytic in the domain T, then $x(t)$ can be denoted by Taylor series at $t = t_n$ as follows:

$$x(t) = \sum_{k=0}^{\infty}(t-t_n)^k X(k) = D^{-1}X(k), \tag{3}$$

where D^{-1} denotes the inverse differential transform operator. By truncating series (3), $x(t)$ can be represented as follows

$$x(t) = \sum_{k=0}^{N}(t-t_n)^k X(k) + O((t-t_n)^{N+1}). \tag{4}$$

It is noticeable that the operator D satisfies linearity and the differential transform of the derivative function can be written in terms of the differential transform of the function itself. This property is main point to solve differential equations by using the DTM. In reference [11], the

DTM tables of some well-known func tions can be found in detail. Using the definition of the DTM operator, the required properties can be generated in Table 1.

3. THE LOCALIZED DIFFERENTIAL TRANSFORM METHOD

We assume the following first-order ODE system to give the implementation procedure of the LDTM,

$$\frac{dx(t)}{dt} = A\boldsymbol{x}(t) + B(t) \text{ for } 0 \leq t \leq T, \tag{5}$$

with the following initial condition

$$\boldsymbol{x}(0) = C, \tag{6}$$

where $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$, C and $B(t)$ are the column vectors of $n \times 1$, A is $n \times n$ matrix. The interval $[0, T]$ is partitioned into N subdomains which have equally spaced grid points defined as $0 = t_0 < t_1 \dots < t_N = T$ such that $t_{i+1} = t_i + h$ where $h = \frac{T}{N}$. Use of the differential transform of equation (5) leads to the following iteration,

Table 1. Algebraic properties of transformations of well-known functions

Function	*Transformed Function*
$x(t) = y(t) \pm z(t)$	$X(k) = Y(k) \pm Z(k)$
$x(t) = \alpha y(t)$	$X(k) = \alpha Y(k)$
$x(t) = \dfrac{dy(t)}{dt}$	$X(k) = (k+1)Y(k+1)$
$x(t) = \dfrac{d^m x(t)}{dt^m}$	$X(k) = (k+1)(k+2)\dots(k+m)X(k+m)$
$x(t) = t^m$	$X(k) = \begin{cases} 1 & if\ k = m \\ C(m,k)t_k^{(k-m)} & if\ k \neq m \end{cases}$
$x(t) = \exp(\gamma t)$	$X(k) = \exp(\gamma t_k)\dfrac{\gamma^k}{k!}$
$x(t) = y(t)z(t)$	$X(k) = \sum_{l=0}^{k} Y(l)Z(k-l)$

$$X_n(k+1) = \frac{1}{k+1}\left(AX_n(k) + F(k)\right), \tag{7}$$

where $X(k)$ and $F(k)$ are the transformed function of $\boldsymbol{x}(t)$ and $B(t)$**,** respectively**.** For the first subdomain, the function $\boldsymbol{x}(t)$ can be approximated by $\boldsymbol{x}_0(t)$ such that

$$\boldsymbol{x}_0(t) = X_0(0) + X_0(1)t + X_0(2)t^2 + \dots + X_0(K)t^K, \tag{8}$$

where K is the order of differential transformation, $X_0(0) = \boldsymbol{x}_0 = C$ and $X_0(k)$ can be obtained from iteration (7). Approximate value of the dependent variable $\boldsymbol{x}(\boldsymbol{t})$ at $t = t_1$ can be evaluated as follows,

$$\boldsymbol{x}(t_1) \cong \boldsymbol{x}_0(t_1) = \boldsymbol{x}_1 = \textstyle\sum_{k=0}^{K} X_0(k)h^k. \tag{9}$$

The critical idea for constructing the LDTM is that the obtained approximate solution is taken to be initial value of the next iteration, i.e. $\boldsymbol{x}_1(t_1) = X_1(0) = \boldsymbol{x}_0(t_1)$. Therefore, the approximate solution $\boldsymbol{x}_1(t)$ for the second subdomain can be expressed as follows,

$$\boldsymbol{x}_1(t) = X_1(0) + X_1(1)(t-t_1) + X_1(2)(t-t_1)^2 + \dots + X_1(K)(t-t_1)^K, \tag{10}$$

and the approximate solution $\boldsymbol{x}_1(t)$ will then be

$$\boldsymbol{x}(t_2) \cong \boldsymbol{x}_1(t_2) = \boldsymbol{x}_2 = \textstyle\sum_{k=0}^{K} X_1(k)h^k. \tag{11}$$

Therefore, the approximate solution at grid point t_{i+1} can be stated as follows

$$\boldsymbol{x}(t_{n+1}) \cong \boldsymbol{x}_n(t_{n+1}) = \boldsymbol{x}_{n+1} = \sum_{k=0}^{K} X_n(k)h^k. \tag{12}$$

where the subscript is taken to be $n = 0,1,2,\dots,N-1$.

4. ERROR ANALYSIS

Consider the following initial value problem,

$$\boldsymbol{x}'(t) = G(\boldsymbol{x}(t),t),\ t > 0,\quad \boldsymbol{x}(0) = C, \tag{13}$$

where $C \in R^m$ and $G: R \times R^m \to R^m$. Considering the procedure mentioned in the previous section, the LDTM solution of equation (13) can be stated as follows,

$$X_n(k+1) = \frac{1}{k+1}[F(k,t_n)],\ k = 0,1,2,\dots,K-1,\ n = 0,1,2,\dots,N-1, \tag{14}$$

$$\boldsymbol{x}(t_{n+1}) \cong \boldsymbol{x}_n(t_{n+1}) = \boldsymbol{x}_{n+1} = \sum_{k=0}^{K} X_n(k)h^k. \tag{15}$$

where X_n and F are the transformed vector valued functions. The exact solution of equation (13) at point $t = t_{n+1}$ can be expressed in the Taylor expansion form as follows,

$$\boldsymbol{x}(t_{n+1}) = \sum_{k=0}^{\infty} X_n(k)h^k = \sum_{k=0}^{K} X_n(k)h^k + h\,\rho_n \tag{16}$$

where $n = 0,1,2,\dots,N-1$ and ρ_n is local truncation error. The local truncation error can be written by residual formula of Taylor series as follows

$$\rho_n = X_{\boldsymbol{n}}^*(K+1)h^K = \frac{h^K}{K+1}[F(K,t_c)] = \frac{h^K}{(K+1)!}\left[\frac{d^K G(x(t),t)}{dt^K}\right]\Big|_{t=t_c},\quad t_c \in (t_n, t_{n+1}). \tag{17}$$

Hence, the present method approximates locally to the exact solution with order K. To have convergent numerical scheme, we need also to have bounded global error of the scheme. For further analysis about global discretization error, assume that the problem is linear, $G(\boldsymbol{x}(t),t) = A\boldsymbol{x}(\boldsymbol{t}) + B(t)$ and let $\varepsilon_n = \boldsymbol{x}(t_n) - \boldsymbol{x}_n$ for $n = 0,1,\dots,N$. The present method for this equation yields the following discretization,

$$\boldsymbol{x}_{n+1} = \sum_{k=0}^{K} X_n(k)h^k. \tag{18}$$

With the use of recursive relation (7), the general term $X_n(k)$ can be stated as

$$X_n(k) = \frac{1}{k!}A^k X_n(0) + \sum_{p=0}^{k-1}\frac{p!}{k!}F(p) \tag{19}$$

where F is the transformed form of the function $B(t)$. Subtraction of (18) from (16) and using the general term (19) yield,

$$\varepsilon_{n+1} = \sum_{k=0}^{K}\frac{1}{k!}A^k h^k \varepsilon_n + h\rho_n. \tag{20}$$

Using the equality $\sum_{k=0}^{K}\frac{1}{k!}A^k h^k = e^{hA}$, equation (20) can be expressed as follows

$$\varepsilon_{n+1} = e^{hA}\varepsilon_n + \delta_n \tag{21}$$

where $\delta_n = h\rho_n$. Note that the method is said to be consistent of order p if $||\delta_n|| = O(h^{p+1})$. To relate global discretization error with the initial error ε_0 and local discretization error, recursive relation (21) becomes,

$$\varepsilon_n = e^{nhA}\varepsilon_0 + \sum_{p=0}^{n-1} e^{(n-p-1)hA}\,\delta_p. \tag{22}$$

Stability of the current scheme is also depending on the bound of the term e^{nhA} for all $nh \le T$. So, considering the following stability criterion

$$e^{nhA} \le S,\ \text{for all } n \ge 0 \text{ and } nh \le T, \tag{23}$$

the error norm bound becomes

$$\|\epsilon_{\boldsymbol{n}}\| \leq S\left|\left|\varepsilon_0\right|\right| + S\sum_{p=0}^{n-1}||\delta_p||. \tag{24}$$

Then, use of the definition of the local discretization error $||\delta_p|| \leq Rh^{K+1}$ for all p and $R = \underbrace{max}_{t\in[0,T]}\left(\left|\frac{dx^K(\boldsymbol{x}(t),t)}{dt^K}\right|\right)$, and also assuming $\boldsymbol{x}(0) = \boldsymbol{x}_0$ lead to the following error norm inequality

$$\left|\left|\boldsymbol{x}(t_n) - \boldsymbol{x}_n\right|\right| \leq R^*h^K$$

where $R^* = SRt_n$ for all $n = 0,1,\dots,N$. Thus, whenever the exact solution is smooth and stability criterion (23) is satisfied, then the currently presented local differential transform method converges to the exact solution with order K.

5. NUMERICAL EXPERIMENTS

This section is devoted to numerical illustration of the proposed method through various test problems by producing quantitative and qualitative results. Accuracy of the obtained results are figured out by using error norms and pointwise solutions. The produced results are compared with the literature, exact solutions, the RK4, various versions of the DTM. To evaluate error norms of the present results, we prefer to use the following norm definition

$$E_i = \left|x_i^{exact} - x_i^{numerical}\right|.$$

Problem 1 [11].
Consider the following second-order IVP

$$x''(t) - 2x'(t) + 2x(t) = \exp(2t)\sin(t), \quad 0 \leq t \leq a, \tag{25}$$

with the initial conditions

$$x(0) = -0.4, \tag{26}$$

$$x'(0) = -0.6. \tag{27}$$

IVP problem (25)-(27) can be transformed to the following system of equations,

$$u_1'(t) = u_2(t), \tag{28}$$

$$u_2'(t) = -2u_1(t) + 2u_2(t) + \exp(2t)\sin(t), \tag{29}$$

with the initial conditions

$$u_1(0) = -0.4, \tag{30}$$

$$u_2(0) = -0.6. \tag{31}$$

The differential equation system can be expressed in the matrix notation

$$U' = AU + B, \tag{32}$$

where $U = (u_1, u_2)$ and $U(0) = (-0.4, -0.6)$. With the use of the LDTM as stated in (5)-(12), the approximate solutions of equation (32) are produced.

In Figs. 1-2, the LDTM and the RK4 solutions of Problem 1 are illustrated by comparing absolute errors of the methods. As seen in the figures, the LDTM is of higher accuracy than the RK4 for short and long-time values. In Table 2, the LDTM results are compared with the results produced by both the DTM [11] and the RK4 using various values of the parameter K, i.e. LTDM10 means that local differential transform method of order 10. The currently produced results have far less absolute errors than both the global DTM results and the widely used RK4 method.

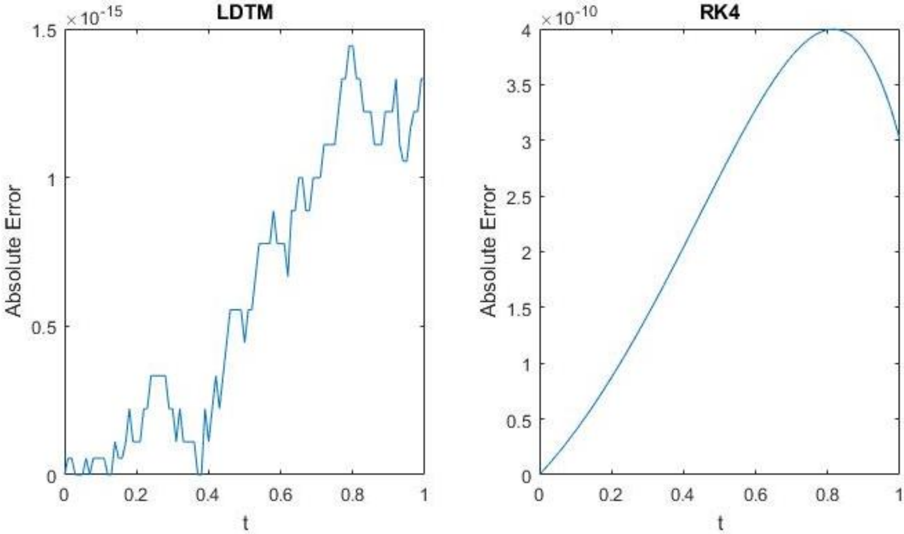

Figure 1. Comparison of the LDTM and the RK4 solutions of Problem 1 produced with $dt = 0.1$, $K = 10$ and $a = 1$.

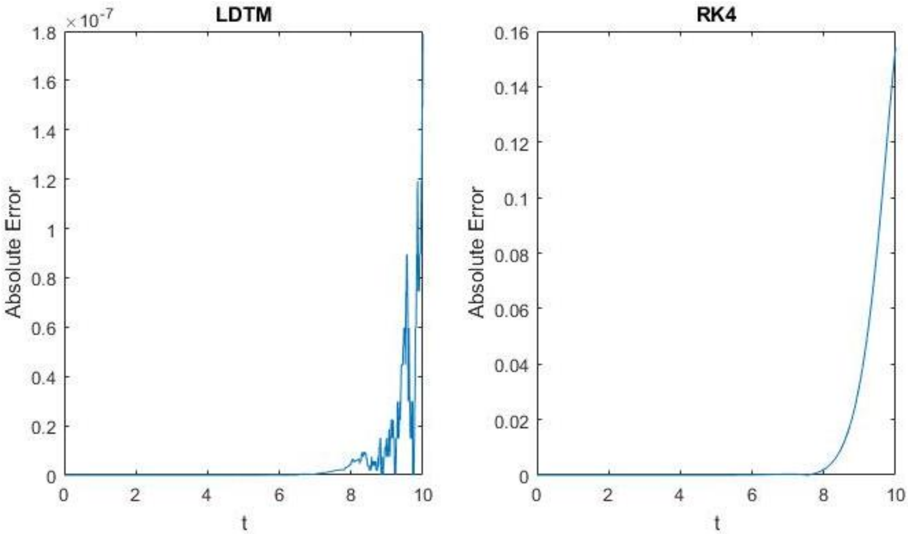

Figure 2. Comparison of the LDTM and the RK4 solutions of Problem 1 produced with $dt = 0.1$, $K = 10$ and $a = 10$.

Table 2. Comparison of absolute errors produced by the LDTM, RK4 method and DTM [11] for $h = 0.1$.

t	RK4	DTM4 [11]	LDTM4	DTM5 [11]	LDTM5	DTM10 [11]	LDTM10
0.1	3.73E-05	5.90E-06	2.03E-06	1.90E-06	7.93E-08	2.10E-06	5.00E-16
0.2	8.39E-05	1.67E-05	5.09E-06	3.40E-06	1.90E-07	3.70E-06	1.55E-15
0.3	1.39E-04	3.99E-05	9.44E-06	7.40E-06	3.38E-07	7.00E-06	3.00E-15
0.4	2.03E-04	5.31E-05	1.54E-05	4.70E-06	5.28E-07	4.40E-06	5.44E-15
0.5	2.72E-04	8.63E-05	2.32E-05	9.50E-06	7.65E-07	9.20E-06	8.55E-15
0.6	3.42E-04	1.01E-04	3.33E-05	4.30E-06	1.05E-06	4.60E-06	1.25E-14
0.7	4.07E-04	1.47E-04	4.59E-05	7.30E-06	1.39E-06	6.90E-06	1.77E-14
0.8	4.58E-04	1.85E-04	6.15E-05	4.00E-06	1.78E-06	3.70E-06	2.41E-14
0.9	4.80E-04	2.47E-04	8.03E-05	2.03E-05	2.20E-06	2.01E-06	3.21E-14
1.0	4.54E-04	3.06E-04	1.02E-04	3.46E-05	2.66E-06	3.44E-06	4.24E-14

Problem 2.
Consider the following nonlinear initial value problem

$$x'(t) = \frac{x(t)^2 + \exp(t)}{5}, \quad 0 \le t \le a, \quad x(0) = 1. \tag{33}$$

In Fig. 3, the LDTM solutions and the RK4 solutions of nonlinear differential equation (28) are compared. The results have been produced for the parameters $dt = 0.1$ and $K = 10$. As illustrated, the results of the LDTM have been seen to be far more accurate than the results of the RK4 method. Even if the time interval is extended to [0,2.5], as demonstrated in Fig. 4, the LDTM produces acceptable results. But in this case, the RK4 solutions diverge to infinity after the time $t \cong 2$. As demonstrated in Fig. 5, decreasing of the time increment as $dt = 0.01$ leads to convergent solutions of the RK4 method, but the accuracy of the RK4 is less than the accuracy of the present technique.

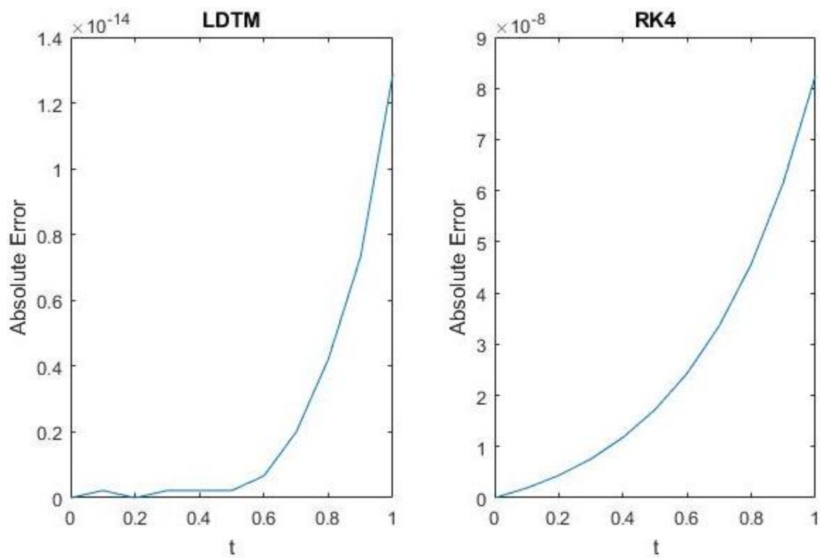

Figure 3. Comparison of the LDTM and the RK4 solutions of Problem 2 for $dt = 0.1$, $K = 10$ and $a = 1$.

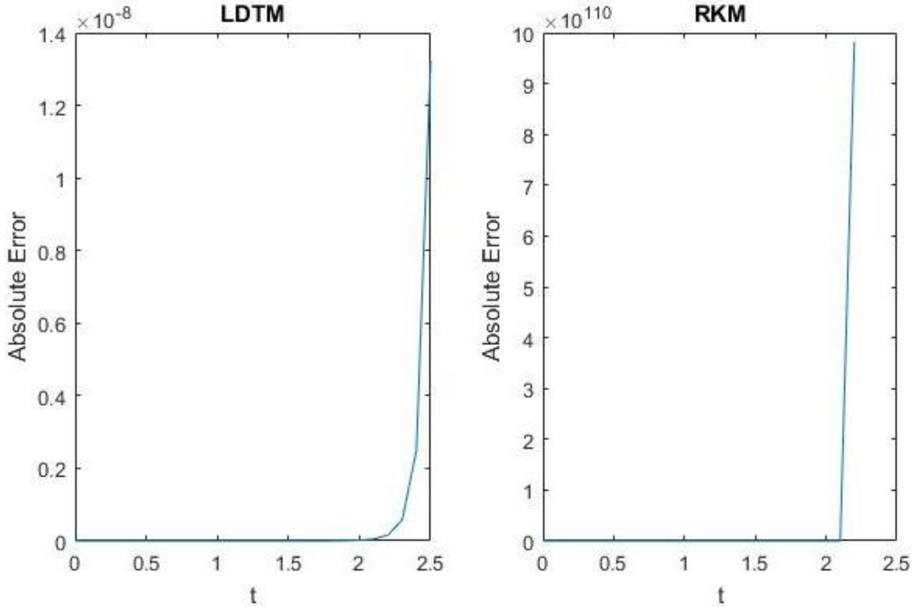

Figure 4. Comparison of the LDTM and the RK4 solutions of Problem 2 for $dt = 0.1$, $K = 10$ and $a = 2.5$.

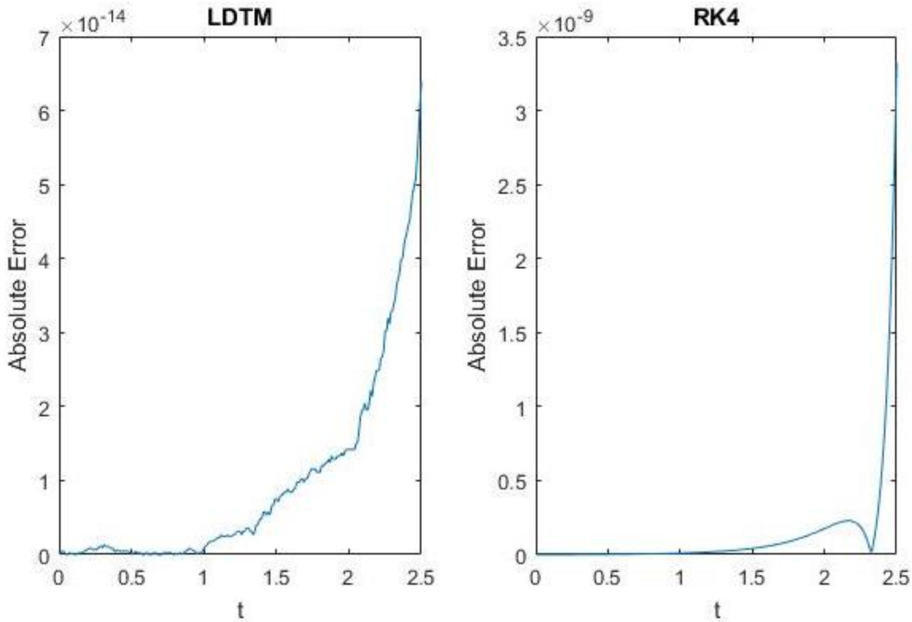

Figure 5. Comparison of the LDTM and the RK4 solutions of Problem 2 for $dt = 0.01$, $K = 10$ and $a = 2.5$.

Problem 3 [11].
Let us consider the following third order initial value problem

$$x'''(t) + 2x''(t) - x'(t) - 2x(t) = \exp(t), \quad 0 \le t \le a, \tag{34}$$

with the conditions $x(0) = 1$, $x'(0) = 2$ and $x''(0) = 0$. Using $u_1(t) = x(t)$, $u_2(t) = x'(t)$ and $u_3(t) = x''(t)$, equation (34) can be transformed to the following system of differential equations,

$$u_1'(t) = u_2(t), \tag{35}$$

$$u_2'(t) = u_3(t), \tag{36}$$

$$u_3'(t) = 2u_1(t) + u_2(t) - 2u_3(t) + \exp(t), \tag{37}$$

with the initial conditions $u_1(0) = 1$, $u_2(0) = 2$ and $u_3(0) = 0$.

In Figs. 6-8, the approximate solutions of the third order non-homogeneous differential equation (34) are obtained by the present LDTM and the RK4 method using different time intervals. The parameters are taken to be $dt = 0.1$, $dt = 0.2$ and $K = 10$.

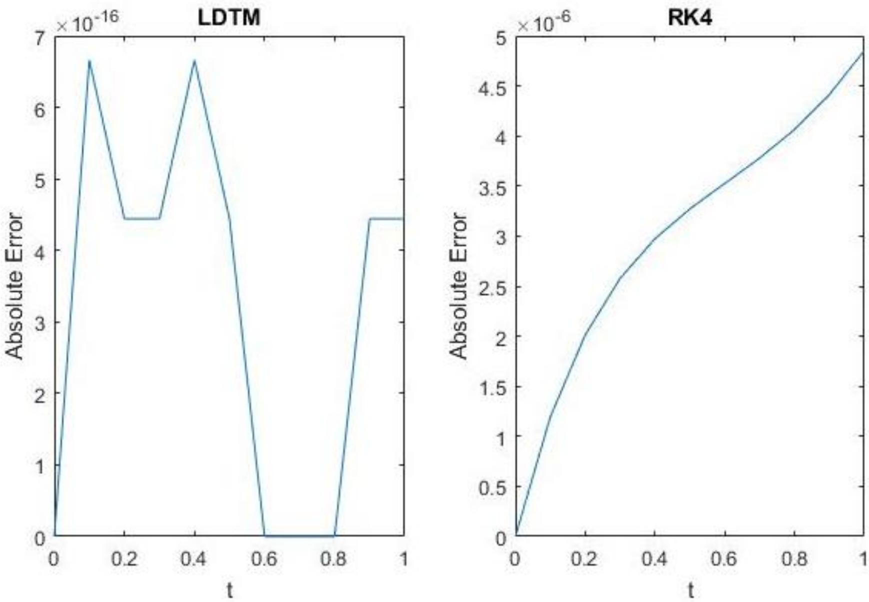

Figure 6. Comparison of the LDTM and the RK4 solutions of Problem 3 for $dt = 0.1$, $K = 10$ and $a = 1$.

Problem 4 [3].
Consider the following stiff system of differential equations

$$x_1'(t) = -20x_1(t) - 0.25x_2(t) - 19.75x_3(t), \tag{38}$$

$$x_2'(t) = 20x_1(t) - 20.25x_2(t) - 0.25x_3(t), \quad 0 \le t \le a, \tag{39}$$

$$x_3'(t) = 20x_1(t) - 19.75x_2(t) - 0.25x_3(t) \tag{40}$$

with the initial conditions $x_1(0) = 1$, $x_2(0) = 0$ and $x_3(0) = -1$.

In Figs. 8-9, approximate solutions of the system of differential equations (38)-(40) produced by the LDTM and the RK4 are compared for different time intervals. The parameters are taken to be $dt = 0.01$ and $K = 10$. For various time intervals, the present numerical approach has more accuracy than the RK4 method as illustrated in Figs. 8-9.

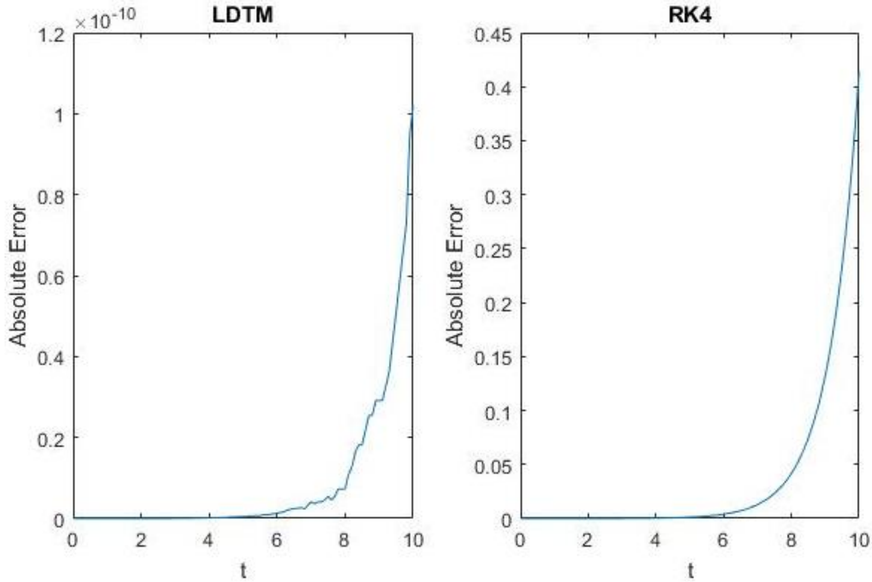

Figure 7. Comparison of LDTM and RK4 solutions of Problem 3 for $dt = 0.1$, $K = 10$ and $a = 10$.

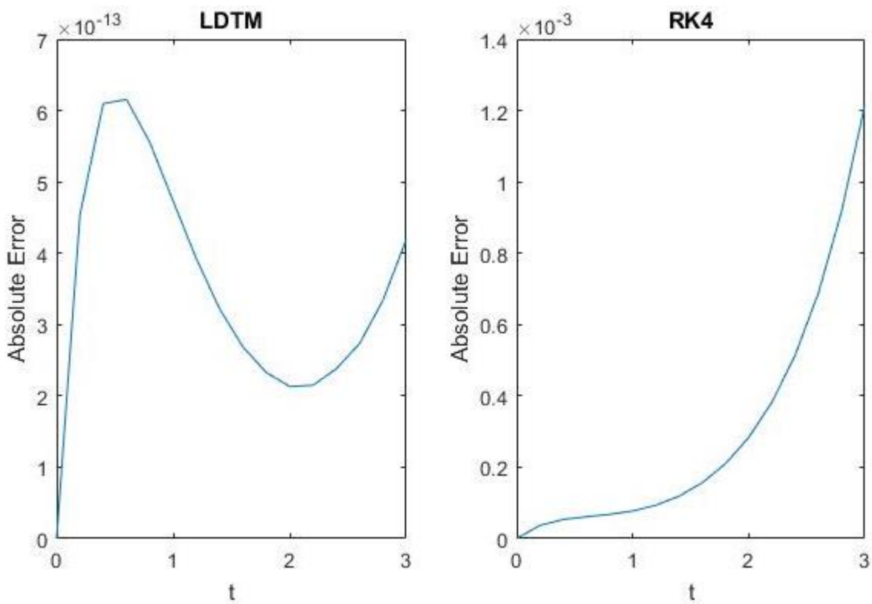

Figure 8. Comparison of the LDTM and the RK4 solutions of Problem 3 for $dt = 0.2$, $K = 10$ and $a = 3$.

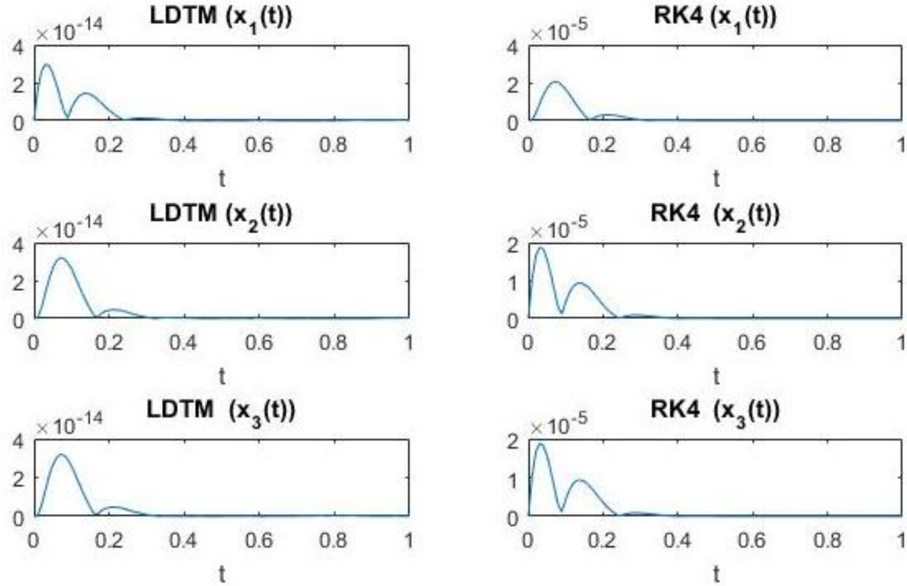

Figure 8. Comparison of the LDTM and the RK4 solutions of Problem 3 for $dt = 0.01$, $K = 10$ and $a = 1$.

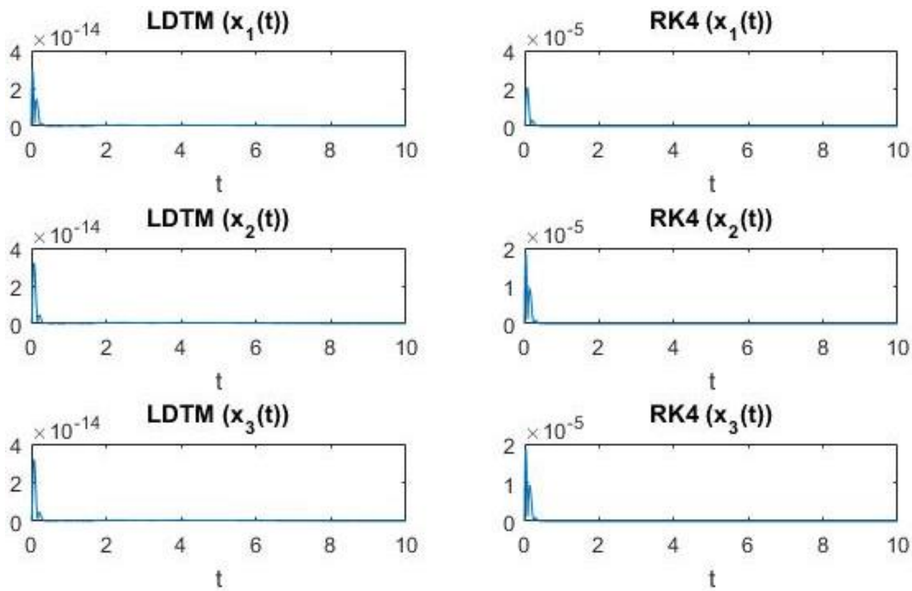

Figure 9. Comparison of the LDTM and the RK4 solutions of Problem 3 for $dt = 0.01$, $K = 10$ and $a = 10$.

6. CONCLUSIONS AND RECOMMENDATIONS

This paper has been organized for the construction and implementation of a local differential transform method (LTDM) for various types of initial value problems. The global error analysis of the LTDM has been studied to find properly out the advantages of the presented method over the semi-analytic global DTM. By considering various types of test problems, the present approach and the widely-used time integration method RK4 have been compared in terms of the produced errors. As both qualitatively and quantitatively illustrated in experiments, the present method has been seen to be in very good agreement with the exact solution even for large time values. Note that the proposed method has been seen to be far more accurate than the RK4 method even when the long-time interval is accepted.

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