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Research Article

IMPROVED HERMITE-HADAMARD TYPE INEQUALITIES FOR CONVEX FUNCTIONS VIA KATUGAMPOLA FRACTIONAL INTEGRALS

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ABSTRACT

In this paper, we gave adjustments for some results in the paper [1], and proved three new Katugampola fractional Hermite-Hadamard type inequalities for convex functions by using the left and the right fractional integrals independently. One of our Katugampola fractional Hermite-Hadamard type inequalities is better than given by Chen and Katugampola. Also, we gave two new Katugampola fractional identities for differentiable functions. By using these identities, we obtained some new trapezoidal type inequalities for convex functions. Our results generalize earlier results.
Keywords: Hermite-Hadamard inequalities, Katugampola fractional integrals, convex functions.

1. INTRODUCTION

Let $f: I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

is well known in the literature as Hermite-Hadamard's inequality. (See [3,4]).

The inequality (1.1) attract widely attention for many researchers, so in recent decades, many generalizations and extensions of inequalities (1.1) for various classes of functions have been studied.

One of the most generalization of the inequalities (1.1) is fractional type, for instance Riemann-Liouville, Hadamard's, conformable, Katugampola fractional integrals etc.

In this work, we focused on Katugampola fractional integrals and Riemann-Liouville fractional integrals which are the special case of Katugampola fractional integrals. Some generalizations of Hermite-Hadamard type inequalities in Riemann-Liouville fractional integrals are widely studied in the papers [8, 9, 10, 14] and references therein. In addition, the Katugampola fractional generalizations of Hermite-Hadamard type inequalities have been presented in [1, 13].

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Let see the definitions and some results about Riemann-Liouville and Katugampola fractional integrals.

Definition 1. *Let $a, b \in \mathbb{R}$ with $a < b$ and $f \in L[a, b]$. The left and right Riemann-Liouville fractional integrals $J_{a+}^{\alpha}f$ and $J_{b-}^{\alpha}f$ of order $\alpha > 0$ are defined by*

$$J_{a+}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

$$J_{b-}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int_0^{\infty} e^{-t}t^{\alpha-1}dt$, (*see* [7, page 69]).

In [14], Sarıkaya et al. presented Hermite-Hadamard type inequalities for convex functions in fractional integral forms as follow.

Theorem 1. *Let $f: [a, b] \to \mathbb{R}$ be a positive function with $0 \le a < b$ and $f \in L[a, b]$. If f is a convex function on $[a, b]$, then the following inequalities hold:*

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[J_{b-}^{\alpha}f(a) + J_{a+}^{\alpha}f(b)] \le \frac{f(a)+f(b)}{2} \tag{1.2}$$

with $\alpha > 0$.

Remark 1. *In Theorem 1, the assumption of the positivity of the function f is not necessary. At the same time, $a, b \in \mathbb{R}$ could be any numbers such that $a < b$.*

In [8,9,10], Kunt et al. proved the following three Riemann-Liouville fractional Hermite-Hadamard type inequalities for convex functions as follows:

Theorem 2. *Let $f: [a, b] \to \mathbb{R}$ be a a convex function for real numbers $a < b$. If $f \in L[a, b]$, then the following inequalities for the lefht Riemann-Liouville fractional integral hold:*

$$f\left(\frac{\alpha a+b}{\alpha+1}\right) \le \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}} J_{a+}^{\alpha}f(b) \le \frac{\alpha f(a)+f(b)}{\alpha+1} \tag{1.3}$$

where $\alpha > 0$.

Theorem 3. *Let $f: [a, b] \to \mathbb{R}$ be a a convex function for real numbers $a < b$. If $f \in L[a, b]$, then the following inequalities for the right Riemann-Liouville fractional integral hold:*

$$f\left(\frac{a+\alpha b}{\alpha+1}\right) \le \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}} J_{b-}^{\alpha}f(a) \le \frac{f(a)+\alpha f(b)}{\alpha+1} \tag{1.4}$$

where $\alpha > 0$.

Theorem 4. *Let $f: [a, b] \to \mathbb{R}$ be a a convex function for real numbers $a < b$. If $f \in L[a, b]$, then the following inequalities for the Riemann-Liouville fractional integral hold:*

$$\frac{f\left(\frac{\alpha a+b}{\alpha+1}\right)+f\left(\frac{a+\alpha b}{\alpha+1}\right)}{2} \le \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} [J_{b-}^{\alpha}f(a) + J_{a+}^{\alpha}f(b)] \le \frac{f(a)+f(b)}{2} \tag{1.5}$$

where $\alpha > 0$.

The following definitions of Katugampola fractional integrals could be found in [1,6]. Consider the space $X_c^p(a, b)$ $(c \in \mathbb{R},\ 1 \le p \le \infty)$ of those complex-valued Lebesgue measurable functions f on $[a, b]$ for which $||f||_{X_c^p} < \infty$ where the norm is defined by

$$||f||_{X_c^p} = \left(\int_a^b |t^c f(t)|^p \frac{dt}{t}\right)^{\frac{1}{p}} < \infty, \quad (1 \le p < \infty,\ c \in \mathbb{R})$$

and for the case $p = \infty$

$$||f||_{X_c^p} = ess\ sup_{a \le t \le b}|f(t)| \qquad (c \in \mathbb{R}).$$

Definition 2. *L et $[a,b] \subset \mathbb{R}$ be a finite interval. Then the left and right side Katugampola fractional integrals of order $\alpha > 0$ of $f \in X_c^p(a,b)$ are defined by*

$$^{\rho}I_{a+}^{\alpha}f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^x \frac{t^{\rho-1}}{(x^\rho - t^\rho)^{1-\alpha}} f(t)dt,$$

and

$$^{\rho}I_{b-}^{\alpha}f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_x^b \frac{t^{\rho-1}}{(t^\rho - x^\rho)^{1-\alpha}} f(t)dt,$$

with $a < x < b$ and $\rho > 0$, respectively.

(See [5], *for the definition of the set $f \in X_c^p(a,b)$)*

It is easily seen that if one that $\rho \to 1$ in the Definition 2, one has the Definition 1.

The following properties of convex functions and lemma are used for forward results.

Definition 3. [12, page 12] *A function f defined on I has a support at $x_0 \in I$ if there exists an affine functions $A(x) = f(x_0) + m(x - x_0)$ such that $A(x) \le f(x)$ for all $x \in I$. The graph of the support function A is called a line of support for f at x_0.*

Theorem 5. [12, page 12] *A function $f: (a,b) \to \mathbb{R}$ is a convex function if and only if there is at least one line of support for f at each $x_0 \in (a,b)$.*

In this paper, our aim is to obtain new Katugampola fractional Hermite-Hadamard type inequalities by using only the right or the left fractional integrals separately for convex functions.

2. SOME CORRECTIONS FOR THE PAPER BY CHEN AND KATUGAMPOLA

In this section, we want to give adjustments for some results in the paper [1].

For the Theorem 2.1 in [1], f must be a convex function on $[a^\rho, b^\rho]$ instead of $[a,b]$, and the assumption of the positivity of the function f is not necessary. At the same time, $a, b \in \mathbb{R}$ could be any numbers. So, the correct theorem should be expressed as follows:

Theorem 6. *Let $f: [a^\rho, b^\rho] \to \mathbb{R}$ be a convex function on $[a^\rho, b^\rho]$ with $f \in X_c^p(a^\rho, b^\rho)$, then the following Hermite-Hadamard type inequality for the Katugampola fractional integrals hold:*

$$f\left(\frac{a^\rho + b^\rho}{2}\right) \le \frac{\rho^\alpha \Gamma(\alpha+1)}{2(b^\rho - a^\rho)^\alpha}\left[\,^{\rho}J_{a+}^{\alpha}f(b^\rho) + \,^{\rho}J_{b-}^{\alpha}f(a^\rho)\right] \le \frac{f(a^\rho) + f(b^\rho)}{2} \tag{2.1}$$

where $-\infty < a < b < \infty$, $\alpha > 0$, $\rho > 0$ and the fractional integrals are considered for the function $f(x^\rho)$ and evaluated at a and b, respectively.

Remark 2. *In Theorem 6*

(1) *If one takes $\rho \to 1$, one has the inequality* (1.2),

(2) *If one takes $\rho \to 1$, and after that if one takes $\alpha = 1$, one has the inequality* (1.1).

For the Theorem 2.2 in [1], the correct expression should be expressed as follows:

Theorem 7. *Let $f{:}I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° (the interior of the interval I), $a^\rho, b^\rho \in I^\circ$ with $a^\rho < b^\rho$. If f' is differentiable on I°, then the following inequality holds:*

$$\left|\frac{f(a^\rho) + f(b^\rho)}{2} - \frac{\rho^\alpha \Gamma(\alpha+1)}{2(b^\rho - a^\rho)^\alpha}\left[\,^{\rho}J_{a+}^{\alpha}f(b^\rho) + \,^{\rho}J_{b-}^{\alpha}f(a^\rho)\right]\right|$$
$$\le \frac{(b^\rho - a^\rho)^2}{\alpha\rho(\alpha+1)(\alpha+2)}\left(\alpha + \frac{1}{2^\alpha}\right) sup_{\xi \in [a^\rho, b^\rho]}|f''(\xi)|.$$

For the Theorem 2.3 in [1], the correct expressions should be expressed as follows:

Theorem 8. *Let $f{:}I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $a^\rho, b^\rho \in I^\circ$ with $a^\rho < b^\rho$. If $|f'|$ is a convex on $[a^\rho, b^\rho]$, following inequality holds:*

$$\left|\frac{f(a^\rho)+f(b^\rho)}{2}-\frac{\rho^\alpha\Gamma(\alpha+1)}{2(b^\rho-a^\rho)^\alpha}\left[\ {}^\rho J^\alpha_{a+}f(b^\rho)+\ {}^\rho J^\alpha_{b-}f(a^\rho)\right]\right| \leq \frac{(b^\rho-a^\rho)^2}{\alpha\rho(\alpha+1)}\left[|f'(a^\rho)|+|f'(b^\rho)|\right].$$

For the Lemma 2.4 in [1], the correct expression should be expressed as follows:

Lemma 1. *Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable function on I°, $a^\rho,b^\rho\in I^\circ$ with $a^\rho<b^\rho$. If the fractional integrals exist and $f'\in L[a^\rho,b^\rho]$, then the following equality for the Katugampola fractional integrals hols:*

$$\frac{f(a^\rho)+f(b^\rho)}{2}-\frac{\rho^\alpha\Gamma(\alpha+1)}{2(b^\rho-a^\rho)^\alpha}\left[\ {}^\rho J^\alpha_{a+}f(b^\rho)+\ {}^\rho J^\alpha_{b-}f(a^\rho)\right] = \frac{b^\rho-a^\rho}{2}\int_0^1\left[(1-t^\rho)^\alpha-t^{\rho\alpha}\right]t^{\rho-1}f'(t^\rho a^\rho+(1-t^\rho)b^\rho)dt$$

where $\alpha>0$ and $\rho>0$.

For the Theorem 2.5 in [1], the correct theorem should be expressed as follows:

Theorem 9. *Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable function on I°, $a^\rho,b^\rho\in I^\circ$ with $a^\rho<b^\rho$. If $|f'|$ is convex on $[a^\rho,b^\rho]$, then the following inequality for the Katugampola fractional integrals hold:*

$$\left|\frac{f(a^\rho)+f(b^\rho)}{2}-\frac{\rho^\alpha\Gamma(\alpha+1)}{2(b^\rho-a^\rho)^\alpha}\left[\ {}^\rho J^\alpha_{a+}f(b^\rho)+\ {}^\rho J^\alpha_{b-}f(a^\rho)\right]\right| \leq \frac{(b^\rho-a^\rho)^2}{2\rho(\alpha+1)}\left(1-\frac{1}{2^\alpha}\right)\left[|f'(a^\rho)|+|f'(b^\rho)|\right].$$

with $\alpha>0$ and $\rho>0$.

3. KATUGAMPOLA FRACTIONAL HERMITE HADAMARD TYPE INEQUALITIES FOR CONVEX FUNCTIONS

Theorem 10. *Let $f:[a^\rho,b^\rho]\to\mathbb{R}$ be a convex function on $[a^\rho,b^\rho]$ with $f\in X^p_c(a^\rho,b^\rho)$, then the following Hermite-Hadamard type inequality for the left Katugampola fractional integral holds:*

$$f\left(\frac{\alpha a^\rho+b^\rho}{\alpha+1}\right)\leq\frac{\rho^\alpha\Gamma(\alpha+1)}{(b^\rho-a^\rho)^\alpha}\ {}^\rho I^\alpha_{a+}f(b^\rho)\leq\frac{\alpha f(a^\rho)+f(b^\rho)}{\alpha+1} \tag{3.1}$$

where $-\infty<a^\rho<b^\rho<\infty$, $\alpha>0$ and *$\rho>0$.*

Proof. Let $\alpha>0$.Since f is convex on $[a^\rho,b^\rho]$, using Theorem 5, there is at least one line of support

$$A(x^\rho)\leq f\left(\frac{\alpha a^\rho+b^\rho}{\alpha+1}\right)+m\left(x^\rho-\frac{\alpha a^\rho+b^\rho}{\alpha+1}\right)\leq f(x^\rho) \tag{3.2}$$

for all $x^\rho\in[a^\rho,b^\rho]$ and $m\in\left[f'_-\left(\frac{\alpha a^\rho+b^\rho}{\alpha+1}\right),f'_+\left(\frac{\alpha a^\rho+b^\rho}{\alpha+1}\right)\right]$. From (3.2) and convexity of f, we have

$$A(t^\rho a^\rho+(1-t^\rho)b^\rho)=f\left(\frac{\alpha a^\rho+b^\rho}{\alpha+1}\right)+m\left(t^\rho a^\rho+(1-t^\rho)b^\rho-\frac{\alpha a^\rho+b^\rho}{\alpha+1}\right)\leq f(t^\rho a^\rho+(1-t^\rho)b^\rho)\leq t^\rho f(a^\rho)+(1-t^\rho)f(b^\rho) \tag{3.3}$$

for all $t\in[0,1]$. Multiplying all sides of (3.3) with $t^{\alpha\rho-1}$ and integrating over (0,1) respect to t, we have

$$\int_0^1 t^{\alpha\rho-1}\left[f\left(\frac{\alpha a^\rho+b^\rho}{\alpha+1}\right)+m\left(t^\rho a^\rho+(1-t^\rho)b^\rho-\frac{\alpha a^\rho+b^\rho}{\alpha+1}\right)\right]dt=f\left(\frac{\alpha a^\rho+b^\rho}{\alpha+1}\right)\int_0^1 t^{\alpha\rho-1}dt+ m\left[\int_0^1 t^{\alpha\rho-1}(t^\rho a^\rho+(1-t^\rho)b^\rho)\,dt-\frac{\alpha a^\rho+b^\rho}{\alpha+1}\int_0^1 t^{\alpha\rho-1}dt\right]=\frac{1}{\alpha\rho}f\left(\frac{\alpha a^\rho+b^\rho}{\alpha+1}\right)+m\left[\frac{\alpha a^\rho+b^\rho}{\alpha\rho(\alpha+1)}- \frac{\alpha a^\rho+b^\rho}{\alpha\rho(\alpha+1)}\right]=\frac{1}{\alpha\rho}f\left(\frac{\alpha a^\rho+b^\rho}{\alpha+1}\right)\le\int_0^1 t^{\alpha\rho-1}f(t^\rho a^\rho+(1-t^\rho)b^\rho)\,dt= \int_a^b\left(\frac{b^\rho-t^\rho}{b^\rho-a^\rho}\right)^{\alpha-1}f(t^\rho)\frac{t^{\rho-1}}{a^\rho-b^\rho}dt=\frac{\rho^{\alpha-1}\Gamma(\alpha)}{(b^\rho-a^\rho)^\alpha}\;{}^\rho I_{a+}^\alpha f(b^\rho)\le f(a^\rho)\int_0^1 t^{\alpha\rho-1+\rho}dt+f(b^\rho)\int_0^1(t^{\alpha\rho-1}-t^{\alpha\rho-1+\rho})dt=\frac{1}{\alpha\rho}\frac{\alpha f(a^\rho)+f(b^\rho)}{\alpha+1}.$$

It means that

$$f\left(\frac{\alpha a^\rho+b^\rho}{\alpha+1}\right)\le\frac{\rho^\alpha\Gamma(\alpha+1)}{(b^\rho-a^\rho)^\alpha}\;{}^\rho I_{a+}^\alpha f(b^\rho)\le\frac{\alpha f(a^\rho)+f(b^\rho)}{\alpha+1}. \tag{3.4}$$

This completes the proof. ■

Remark 3. *In Theorem 10*

(1) *If one takes* $\rho\to 1$, *one has the inequality* (1.3),

(2) *If one takes* $\rho\to 1$, *and after that if one takes* $\alpha=1$, *one has the inequality* (1.1).

Theorem 11. *Let* $f:[a^\rho,b^\rho]\to\mathbb{R}$ *be a convex function on* $[a^\rho,b^\rho]$ *with* $f\in X_c^P(a^\rho,b^\rho)$, *then the following Hermite-Hadamard type inequality for the right Katugampola fractional integral holds:*

$$f\left(\frac{a^\rho+\alpha b^\rho}{\alpha+1}\right)\le\frac{\rho^\alpha\Gamma(\alpha+1)}{(b^\rho-a^\rho)^\alpha}\;{}^\rho I_{b-}^\alpha f(a^\rho)\le\frac{f(a^\rho)+\alpha f(b^\rho)}{\alpha+1} \tag{3.5}$$

where $-\infty<a^\rho<b^\rho<\infty$, $\alpha>0$ and $\rho>0$.

Proof. Let $\alpha>0$.Since f is convex on $[a^\rho,b^\rho]$, using Theorem 5, there is at least one line of support

$$A(x^\rho)\le f\left(\frac{a^\rho+\alpha b^\rho}{\alpha+1}\right)+m\left(x^\rho-\frac{a^\rho+\alpha b^\rho}{\alpha+1}\right)\le f(x^\rho) \tag{3.6}$$

for all $x^\rho\in[a^\rho,b^\rho]$ and $m\in\left[f'_-\left(\frac{a^\rho+\alpha b^\rho}{\alpha+1}\right),f'_+\left(\frac{a^\rho+b\alpha^\rho}{\alpha+1}\right)\right]$. From (3.6) and convexity of f, we have

$$A(t^\rho b^\rho+(1-t^\rho)a^\rho)=f\left(\frac{a^\rho+\alpha b^\rho}{\alpha+1}\right)+m\left(t^\rho b^\rho+(1-t^\rho)a^\rho-\frac{a^\rho+\alpha b^\rho}{\alpha+1}\right)\le f(t^\rho b^\rho+(1-t^\rho)a^\rho)\le t^\rho f(b^\rho)+(1-t^\rho)f(a^\rho) \tag{3.7}$$

for all $t\in[0,1]$. Multiplying all sides of (3.7) with $t^{\alpha\rho-1}$ and integrating over (0,1) respect to t, we have

$$\int_0^1 t^{\alpha\rho-1}\left[f\left(\frac{a^\rho+\alpha b^\rho}{\alpha+1}\right)+m\left(t^\rho b^\rho+(1-t^\rho)a^\rho-\frac{a^\rho+\alpha b^\rho}{\alpha+1}\right)\right]dt$$
$$=f\left(\frac{a^\rho+\alpha b^\rho}{\alpha+1}\right)\int_0^1 t^{\alpha\rho-1}dt$$
$$+m\left[\int_0^1 t^{\alpha\rho-1}(t^\rho b^\rho+(1-t^\rho)a^\rho)\,dt-\frac{a^\rho+\alpha b^\rho}{\alpha+1}\int_0^1 t^{\alpha\rho-1}dt\right]$$
$$=\frac{1}{\alpha\rho}f\left(\frac{a^\rho+\alpha b^\rho}{\alpha+1}\right)+m\left[\frac{a^\rho+\alpha b^\rho}{\alpha\rho(\alpha+1)}-\frac{a^\rho+\alpha b^\rho}{\alpha\rho(\alpha+1)}\right]=\frac{1}{\alpha\rho}f\left(\frac{a^\rho+\alpha b^\rho}{\alpha+1}\right)$$
$$\le\int_0^1 t^{\alpha\rho-1}f(t^\rho b^\rho+(1-t^\rho)a^\rho)\,dt=\int_a^b\left(\frac{a^\rho-t^\rho}{a^\rho-b^\rho}\right)^{\alpha-1}f(t^\rho)\frac{t^{\rho-1}}{b^\rho-a^\rho}dt$$
$$=\frac{\rho^{\alpha-1}\Gamma(\alpha)}{(b^\rho-a^\rho)^\alpha}\;{}^\rho I_{b-}^\alpha f(b^\rho)$$
$$\le f(b^\rho)\int_0^1 t^{\alpha\rho-1+\rho}dt+f(a^\rho)\int_0^1(t^{\alpha\rho-1}-t^{\alpha\rho-1+\rho})dt$$
$$=\frac{1}{\alpha\rho}\frac{f(a^\rho)+\alpha f(b^\rho)}{\alpha+1}.$$

It means that

$$f\left(\frac{a^\rho+b\alpha^\rho}{\alpha+1}\right)\le\frac{\rho^\alpha\Gamma(\alpha+1)}{(b^\rho-a^\rho)^\alpha}\;{}^\rho I_{b-}^\alpha f(a^\rho)\le\frac{f(a^\rho)+\alpha f(b^\rho)}{\alpha+1}. \tag{3.8}$$

This completes the proof. ■

Remark 4. *In Theorem 11,*

(1) *If one takes* $\rho\to1$, *one has the inequality* (1.4),
(2) *If one takes* $\rho\to1$, *and after that if one takes* $\alpha=1$, *one has the inequality* (1.1).

Theorem 12. *Let* $f:[a^\rho,b^\rho]\to\mathbb{R}$ *be a convex function on* $[a^\rho,b^\rho]$ *with* $f\in X_c^p(a^\rho,b^\rho)$, *then the following Hermite-Hadamard type inequality for the Katugampola fractional integral holds:*

$$\frac{f\left(\frac{\alpha a^\rho+b^\rho}{\alpha+1}\right)+f\left(\frac{a^\rho+\alpha b^\rho}{\alpha+1}\right)}{2}\le\frac{\rho^\alpha\Gamma(\alpha+1)}{2(b^\rho-a^\rho)^\alpha}\left[{}^\rho I_{b-}^\alpha f(a^\rho)+{}^\rho I_{b-}^\alpha f(b^\rho)\right]\le\frac{f(a^\rho)+\alpha f(b^\rho)}{2} \tag{3.9}$$

where $-\infty<a^\rho<b^\rho<\infty$, $\alpha>0$ and $\rho>0$.

Proof. Adding the inequalities (3.1) and (3.5) side by side, then the multiplying the resulting inequalities by $\frac{1}{2}$, we have the inequalities (3.9). ■

Remark 5. *In Theorem 12,*

(1) *If one takes* $\rho\to1$, *one has the inequality* (1.5),
(2) *If one takes* $\rho\to1$, *and after that if one takes* $\alpha=1$, *one has the inequality* (1.1).

Corollary 1. *The left hand side of (3.9) is better than the left hand side of (2.1).*
Proof. Since f is convex on $[a^\rho,b^\rho]$, it is clear from

$$f\left(\frac{a^\rho+b^\rho}{2}\right)=f\left(\frac{\frac{\alpha a^\rho+b^\rho}{\alpha+1}+\frac{a^\rho+\alpha b^\rho}{\alpha+1}}{2}\right)\le\frac{f\left(\frac{\alpha a^\rho+b^\rho}{\alpha+1}\right)+f\left(\frac{a^\rho+\alpha b^\rho}{\alpha+1}\right)}{2}$$

■

4. LEMMAS

In this section we will prove two new identities used in forward results.

Lemma 2. *Let* $f: I \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable function* $I^{\circ}, a^{\rho}, b^{\rho} \in I^{\circ}$ *with* $a^{\rho} < b^{\rho}$*. If the fractional integrals exist and* $f' \in L[a^{\rho}, b^{\rho}]$*, then the following equality for the left Katugampola fractional integral holds:*

$$\frac{\alpha f(a^{\rho})+f(b^{\rho})}{\alpha+1} - \frac{\rho^{\alpha}\Gamma(\alpha+1)}{(b^{\rho}-a^{\rho})^{\alpha}} \; {}^{\rho}I^{\alpha}_{a+}f(b^{\rho}) = \frac{\rho(b^{\rho}-a^{\rho})}{\alpha+1}\int_0^1 (1-(\alpha+1)t^{\rho\alpha})t^{\rho-1}f'(t^{\rho}a^{\rho} + (1-t^{\rho})b^{\rho})dt \tag{4.1}$$

where $\alpha > 0$ *and* $\rho > 0$.

Proof. It could be proven directly by applying the partial integration to the right hand side of the equation (4.1) as follows:

$$\frac{\rho(b^\rho - a^\rho)}{\alpha+1}\int_0^1 (1-(\alpha+1)t^{\rho\alpha})t^{\rho-1}f(t^\rho a^\rho + (1-t^\rho)b^\rho)dt$$

$$= \frac{\rho(b^\rho - a^\rho)}{\alpha+1}\left[\int_0^1 t^{\rho-1}f(t^\rho a^\rho + (1-t^\rho)b^\rho)dt\right.$$

$$\left. -(\alpha+1)\int_0^1 t^{\alpha\rho}t^{\rho-1}f(t^\rho a^\rho + (1-t^\rho)b^\rho)dt\right]$$

$$= \frac{\rho(b^\rho - a^\rho)}{\alpha+1}\left[\left.\frac{f(t^\rho a^\rho + (1-t^\rho)b^\rho)}{\rho(a^\rho - b^\rho)}\right|_0^1\right.$$

$$-(\alpha+1)\left[\left.\frac{f(t^\rho a^\rho + (1-t^\rho)b^\rho)}{\rho(a^\rho - b^\rho)}\ t^{\alpha\rho}\right|_0^1\right.$$

$$\left.\left. -\frac{\alpha}{a^\rho - b^\rho}\int_0^1 t^{\alpha\rho-1}f(t^\rho a^\rho + (1-t^\rho)b^\rho)dt\right]\right]$$

$$= \frac{\rho(b^\rho - a^\rho)}{\alpha+1}\left[\frac{f(a^\rho)-f(b^\rho)}{\rho(a^\rho - b^\rho)} - \frac{(\alpha+1)f(a^\rho)}{\rho(a^\rho - b^\rho)}\right.$$

$$\left. +\frac{(\alpha+1)\alpha}{a^\rho - b^\rho}\int_0^1 t^{\alpha\rho-1}f(t^\rho a^\rho + (1-t^\rho)b^\rho)dt\right]$$

$$= \frac{\rho(b^\rho - a^\rho)}{\alpha+1}\left[\frac{\alpha f(a^\rho)+f(b^\rho)}{\rho(b^\rho - a^\rho)}\right.$$

$$\left. -\frac{(\alpha+1)\alpha}{b^\rho - a^\rho}\int_0^1 t^{\alpha\rho-1}f(t^\rho a^\rho + (1-t^\rho)b^\rho)dt\right]$$

$$= \frac{\alpha f(a^\rho)+f(b^\rho)}{\alpha+1} - \alpha\rho\int_0^1 t^{\alpha\rho-1}f(t^\rho a^\rho + (1-t^\rho)b^\rho)dt$$

$$= \frac{\alpha f(a^\rho)+f(b^\rho)}{\alpha+1} - \alpha\rho\int_a^b \left(\frac{b^\rho - t^\rho}{b^\rho - a^\rho}\right)^{\alpha-1} f(t^\rho)\frac{t^{\rho-1}}{a^\rho - b^\rho}dt$$

$$= \frac{\alpha f(a^\rho)+f(b^\rho)}{\alpha+1} - \frac{\rho^\alpha\Gamma(\alpha+1)}{(b^\rho - a^\rho)^\alpha}\ {}^\rho I_{a+}^\alpha f(b^\rho).$$

This completes the proof. ■

Remark 6. *In Lemma 2,*

(1) *If one takes* $\rho \to 1$, *one has the inequality proved in* [8, Lemma 3].
(2) *If one takes* $\rho \to 1$, *and after that if one takes* $\alpha = 1$, *one has the inequality* [2, Lemma 2.1].

Lemma 3. *Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function $I^{\circ}, a^{\rho}, b^{\rho} \in I^{\circ}$ with $a^{\rho} < b^{\rho}$. If the fractional integrals exist and $f' \in L[a^{\rho}, b^{\rho}]$, then the following equality for the right Katugampola fractional integral holds:*

$$\frac{f(a^{\rho})+\alpha f(b^{\rho})}{\alpha+1} - \frac{\rho^{\alpha}\Gamma(\alpha+1)}{(b^{\rho}-a^{\rho})^{\alpha}} \quad {}^{\rho}I_{b-}^{\alpha} f(a^{\rho}) = \frac{\rho(b^{\rho}-a^{\rho})}{\alpha+1}\int_0^1 [(\alpha+1)(1-t^{\rho})^{\alpha}-1]t^{\rho-1}f'(t^{\rho}a^{\rho} + (1-t^{\rho})b^{\rho})dt \tag{4.1}$$

where $\alpha > 0$ and $\rho > 0$.

Proof. It could be proven directly by applying the partial integration to the right hand sie of the equation (4.2) as follows:

$$\frac{\rho(b^{\rho}-a^{\rho})}{\alpha+1}\int_0^1 [(\alpha+1)(1-t^{\rho})^{\alpha}-1]t^{\rho-1}f'(t^{\rho}a^{\rho} + (1-t^{\rho})b^{\rho})dt$$

$$= \frac{\rho(b^{\rho}-a^{\rho})}{\alpha+1}\int_0^1 [(\alpha+1)(1-t^{\rho})^{\alpha}t^{\rho-1}f'(t^{\rho}a^{\rho} + (1-t^{\rho})b^{\rho}) - t^{\rho-1}f'(t^{\rho}a^{\rho} + (1-t^{\rho})b^{\rho})]dt$$

$$= \frac{\rho(b^{\rho}-a^{\rho})}{\alpha+1}\left[(\alpha+1)\left[\frac{(1-t^{\rho})f(t^{\rho}a^{\rho} + (1-t^{\rho})b^{\rho})}{\rho(a^{\rho}-b^{\rho})}\bigg|_0^1 + \frac{\alpha}{(a^{\rho}-b^{\rho})}\int_0^1 (1-t^{\rho})^{\alpha-1}f(t^{\rho}a^{\rho} + (1-t^{\rho})b^{\rho})dt\right] - \int_0^1 t^{\rho-1}f'(t^{\rho}a^{\rho} + (1-t^{\rho})b^{\rho})dt\right]$$

$$= \frac{\rho(b^{\rho}-a^{\rho})}{\alpha+1}\left[\left[\frac{(\alpha+1)f(b^{\rho})}{\rho(a^{\rho}-b^{\rho})} - \frac{(\alpha+1)\alpha}{a^{\rho}-b^{\rho}}\int_0^1 (1-t^{\rho})^{\alpha-1}f(t^{\rho}a^{\rho} + (1-t^{\rho})b^{\rho})dt\right] - \frac{f(t^{\rho}a^{\rho} + (1-t^{\rho})b^{\rho})}{\rho(a^{\rho}-b^{\rho})}\bigg|_0^1\right]$$

$$= \frac{\rho(b^{\rho}-a^{\rho})}{\alpha+1}\left[\left[\frac{(\alpha+1)f(b^{\rho})}{\rho(a^{\rho}-b^{\rho})} - \frac{(\alpha+1)\alpha}{a^{\rho}-b^{\rho}}\int_0^1 (1-t^{\rho})^{\alpha-1}f(t^{\rho}a^{\rho} + (1-t^{\rho})b^{\rho})dt\right] + \frac{f(a^{\rho})-f(b^{\rho})}{\rho(b^{\rho}-a^{\rho})}\right]$$

$$= \frac{f(a^{\rho})+\alpha f(b^{\rho})}{\alpha+1} - \alpha\rho\int_0^1 t^{\alpha\rho-1}f(t^{\rho}a^{\rho} + (1-t^{\rho})b^{\rho})dt$$

$$= \frac{f(a^{\rho})+\alpha f(b^{\rho})}{\alpha+1} - \alpha\rho\int_a^b \left(\frac{t^{\rho}-a^{\rho}}{b^{\rho}-a^{\rho}}\right)^{\alpha-1} f(t^{\rho})\frac{t^{\rho-1}}{a^{\rho}-b^{\rho}}dt$$

$$= \frac{f(a^{\rho})+\alpha f(b^{\rho})}{\alpha+1} - \frac{\rho^{\alpha}\Gamma(\alpha+1)}{(b^{\rho}-a^{\rho})^{\alpha}} \quad {}^{\rho}I_{b-}^{\alpha} f(a^{\rho}).$$

This completes the proof. ■

Remark 7. *In Lemma 3,*

(1) *If one takes* $\rho \to 1$, *one has the inequality proved in* [9, Lemma 3].
(2) *If one takes* $\rho \to 1$, *and after that if one takes* $\alpha = 1$, *one has the inequality* [2, Lemma 2.1].

5. SOME NEW KATUGAMPOLA FRACTIONAL TRAPEZOID TYPE INEQUALITIES FOR CONVEX FUNCTIONS

In this section, we will prove some new Katugampola fractional trapezoid type inequalities for convex functions by using Lemma 2 and Lemma 3.

Theorem 13. *Let* $f: I \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable function on* $I^{\circ}, a^{\rho}, b^{\rho} \in I^{\circ}$ *with* $a^{\rho} < b^{\rho}$. *If* $f' \in L[a^{\rho}, b^{\rho}]$ *and* $|f'|^q$ *is convex on* $[a^{\rho}, b^{\rho}]$ *for* $q \geq 1$, *then the following inequality for the left Katugampola fractional integral holds:*

$$\left|\frac{\alpha f(a^{\rho})+f(b^{\rho})}{\alpha+1} - \frac{\rho^{\alpha}\Gamma(\alpha+1)}{(b^{\rho}-a^{\rho})^{\alpha}} \; {}^{\rho}I_{a+}^{\alpha} f(b^{\rho})\right| \leq \frac{\rho(b^{\rho}-a^{\rho})}{\alpha+1} Z_1(\alpha,\rho)^{1-\frac{1}{q}} \left(|f'(a^{\rho})|^q Z_2(\alpha,\rho) + |f'(b^{\rho})|^q Z_3(\alpha,\rho)\right)^{\frac{1}{q}} \tag{5.1}$$

where

$$Z_1(\alpha,\rho) = \frac{2\alpha}{\rho(\alpha+1)}\left(\frac{1}{\alpha+1}\right)^{\frac{1}{\alpha}},$$
$$Z_2(\alpha,\rho) = \frac{\alpha\left(2+(\alpha+1)^{\frac{2}{\alpha}}\right)}{2\rho(\alpha+2)(\alpha+1)^{\frac{2}{\alpha}}},$$
$$Z_3(\alpha,\rho) = Z_1(\alpha,\rho) - Z_2(\alpha,\rho),$$

with $\alpha > 0$ *and* $\rho > 0$.

Proof. By using Lemma 2, power mean inequality and the convexity of $|f'|^q$, we have

$$\left|\frac{\alpha f(a^{\rho})+f(b^{\rho})}{\alpha+1} - \frac{\rho^{\alpha}\Gamma(\alpha+1)}{(b^{\rho}-a^{\rho})^{\alpha}} \; {}^{\rho}I_{a+}^{\alpha} f(b^{\rho})\right| \leq \frac{\rho(b^{\rho}-a^{\rho})}{\alpha+1} \int_0^1 |1-(\alpha+1)t^{\rho\alpha}| t^{\rho-1} |f'(t^{\rho}a^{\rho} + (1-t^{\rho})b^{\rho})| dt \leq$$
$$\frac{\rho(b^{\rho}-a^{\rho})}{\alpha+1}\left(\int_0^1 |(1-(\alpha+1)t^{\rho\alpha})t^{\rho-1}| dt\right)^{1-\frac{1}{q}} \times \left(\int_0^1 |1-(\alpha+1)t^{\rho\alpha}| t^{\rho-1} |f'(t^{\rho}a^{\rho} + (1-t^{\rho})b^{\rho})| dt\right)^{\frac{1}{q}} \leq$$
$$\frac{\rho(b^{\rho}-a^{\rho})}{\alpha+1}\left(\int_0^1 |(1-(\alpha+1)t^{\rho\alpha})t^{\rho-1}| dt\right)^{1-\frac{1}{q}} \times \left(\int_0^1 |1-(\alpha+1)t^{\rho\alpha}| t^{\rho-1} [t^{\rho}|f'(a^{\rho})|^q + (1-t^{\rho})|f'(b^{\rho})|^q] dt\right)^{\frac{1}{q}} \leq \frac{\rho(b^{\rho}-a^{\rho})}{\alpha+1}\left(\int_0^1 |(1-(\alpha+1)t^{\rho\alpha})t^{\rho-1}| dt\right)^{1-\frac{1}{q}} \times \left(|f'(a^{\rho})|^q \int_0^1 |(1-(\alpha+1)t^{\rho\alpha})t^{2\rho-1}| dt + |f'(b^{\rho})|^q \int_0^1 |(1-(\alpha+1)t^{\rho\alpha})t^{\rho-1}| (1-t^{\rho})\, dt\right)^{\frac{1}{q}} \tag{5.2}$$

Calculating the appearing integrals in (5.2), we have

$\int_0^1 |(1-(\alpha+1)t^{\rho\alpha})|\, t^{\rho-1}dt = \int_0^{\sqrt[\alpha\rho]{\frac{1}{\alpha+1}}} (1-(\alpha+1)t^{\rho\alpha})t^{\rho-1}dt + \int_{\sqrt[\alpha\rho]{\frac{1}{\alpha+1}}}^1 \big((\alpha+1)t^{\rho\alpha} - 1\big)t^{\rho-1}dt = \int_0^{\sqrt[\alpha\rho]{\frac{1}{\alpha+1}}} t^{\rho-1}dt - (\alpha+1)\int_0^{\sqrt[\alpha\rho]{\frac{1}{\alpha+1}}} t^{\rho(\alpha+1)-1}dt + (\alpha+1)\int_{\sqrt[\alpha\rho]{\frac{1}{\alpha+1}}}^1 t^{\rho(\alpha+1)-1}dt - \int_{\sqrt[\alpha\rho]{\frac{1}{\alpha+1}}}^1 t^{\rho-1}dt = \left(\frac{t^\rho}{\rho} - \frac{t^{\rho(\alpha+1)}}{\rho}\right)\Big|_0^{\sqrt[\alpha\rho]{\frac{1}{\alpha+1}}} + \left(\frac{t^{\rho(\alpha+1)}}{\rho} - \frac{t^\rho}{\rho}\right)\Big|_{\sqrt[\alpha\rho]{\frac{1}{\alpha+1}}}^1 = \frac{2\alpha}{\rho(\alpha+1)}\left(\frac{1}{\alpha+1}\right)^{\frac{1}{\alpha}} = Z_1(\alpha,\rho),$ (5.3)

and

$\int_0^1 |(1-(\alpha+1)t^{\rho\alpha})|\, t^{2\rho-1}dt = \int_0^{\sqrt[\alpha\rho]{\frac{1}{\alpha+1}}} (1-(\alpha+1)t^{\rho\alpha})t^{2\rho-1}dt + \int_{\sqrt[\alpha\rho]{\frac{1}{\alpha+1}}}^1 \big((\alpha+1)t^{\rho\alpha} - 1\big)t^{2\rho-1}dt = \int_0^{\sqrt[\alpha\rho]{\frac{1}{\alpha+1}}} t^{2\rho-1}dt - (\alpha+1)\int_0^{\sqrt[\alpha\rho]{\frac{1}{\alpha+1}}} t^{\rho(\alpha+2)-1}dt + (\alpha+1)\int_{\sqrt[\alpha\rho]{\frac{1}{\alpha+1}}}^1 t^{\rho(\alpha+2)-1}dt - \int_{\sqrt[\alpha\rho]{\frac{1}{\alpha+1}}}^1 t^{2\rho-1}dt = \left(\frac{t^{2\rho}}{2\rho} - \frac{(\alpha+1)t^{\rho(\alpha+2)}}{\rho(\alpha+2)}\right)\Big|_0^{\sqrt[\alpha\rho]{\frac{1}{\alpha+1}}} + \left(\frac{(\alpha+1)t^{\rho(\alpha+2)}}{\rho(\alpha+2)} - \frac{t^{2\rho}}{2\rho}\right)\Big|_{\sqrt[\alpha\rho]{\frac{1}{\alpha+1}}}^1 = \frac{1}{\rho}\left(\left(\frac{1}{\alpha+1}\right)^{\frac{2}{\alpha}} - \frac{2}{\alpha+1}\left(\frac{1}{\alpha+1}\right)^{\frac{2}{\alpha}} + \frac{\alpha}{2(\alpha+1)}\right) = Z_2(\alpha,\rho),$ (5.4)

and

$\int_0^1 |(1-(\alpha+1)t^{\rho\alpha})|\, t^{\rho-1}(1-t^\rho)dt = \int_0^1 |(1-(\alpha+1)t^{\rho\alpha})t^{\rho-1}|\, dt - \int_0^1 |(1-(\alpha+1)t^{\rho\alpha})t^{2\rho-1}|\, dt = Z_1(\alpha,\rho) - Z_2(\alpha,\rho) = Z_3(\alpha,\rho).$ (5.5)

If we use (5.3) − (5.5) in (5.2) , we have (5.1). This completes the proof. ■

Remark 8. *In Theorem 13,*

(1) *If one takes* $\rho \to 1$, *one has the inequality proved in* [8, Theorem 5].
(2) *If one takes* $\rho \to 1$, *and after that if one takes* $\alpha = 1$, *one has the inequality* [11, Theorem 1].

Theorem 14. *Let* $f: I \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable function* $I^\circ, a^\rho, b^\rho \in I^\circ$ *with* $a^\rho < b^\rho$. *If* $f' \in L[a^\rho, b^\rho]$ *and* $|f'|^q$ *is convex on* $[a^\rho, b^\rho]$ *for* $q > 1$, *and* $\frac{1}{q} + \frac{1}{p} = 1$, *then the following inequality for the left Katugampola fractional integral holds:*

$$\left|\frac{\alpha f(a^\rho)+f(b^\rho)}{\alpha+1} - \frac{\rho^\alpha \Gamma(\alpha+1)}{(b^\rho-a^\rho)^\alpha}\ {}^\rho I_{a+}^\alpha f(b^\rho)\right| \le \frac{\rho(b^\rho-a^\rho)}{\alpha+1}\left(Z_4(\alpha,\rho,p) + Z_5(\alpha,\rho,p)\right)^{\frac{1}{q}} \times \left(\frac{|f'(a^\rho)|^q(q(\rho-1)+1) + |f'(b^\rho)|^q \rho}{(q(\rho-1)+1)(\rho(q+1)-q+1)}\right)^{\frac{1}{q}} \quad (5.6)$$

where

$$Z_4(\alpha,\rho,p) = \int_0^{\sqrt[\alpha\rho]{\frac{1}{\alpha+1}}} (1-(\alpha+1)t^{\rho\alpha})^p dt,$$

$$Z_5(\alpha,\rho,p) = \int_{\sqrt[\alpha\rho]{\frac{1}{\alpha+1}}}^1 \big((\alpha+1)t^{\rho\alpha} - 1\big)^p dt\,,$$

with $\alpha > 0$ and $\rho > 0$.

Proof. By using Lemma 2, Hölder inequality and the convexity of $|f'|^q$, we have

$$\left|\frac{\alpha f(a^\rho)+f(b^\rho)}{\alpha+1} - \frac{\rho^\alpha\Gamma(\alpha+1)}{(b^\rho-a^\rho)^\alpha}\ {}^\rho I^\alpha_{a+} f(b^\rho)\right| \le \frac{\rho(b^\rho-a^\rho)}{\alpha+1}\int_0^1 |1-(\alpha+1)t^{\rho\alpha}| t^{\rho-1}|f'(t^\rho a^\rho + (1-t^\rho)b^\rho)|dt \le \frac{\rho(b^\rho-a^\rho)}{\alpha+1}\left(\int_0^1 |1-(\alpha+1)t^{\rho\alpha}|^p dt\right)^{\frac{1}{p}} \times \left(\int_0^1 t^{q\rho-q}|f'(t^\rho a^\rho+(1-t^\rho)b^\rho)|^q dt\right)^{\frac{1}{q}} \le \frac{\rho(b^\rho-a^\rho)}{\alpha+1}\left(\int_0^1 |1-(\alpha+1)t^{\rho\alpha}|^p dt\right)^{\frac{1}{p}} \times \left(\int_0^1 [t^{\rho(q+1)-q}|f'(a^\rho)|^q + t^{q\rho-q}(1-t^\rho)|f'(b^\rho)|^q]dt\right)^{\frac{1}{q}} \le \frac{\rho(b^\rho-a^\rho)}{\alpha+1}\left(\int_0^1 |1-(\alpha+1)t^{\rho\alpha}|^p dt\right)^{\frac{1}{p}} \times \left(|f'(a^\rho)|^q\int_0^1 t^{\rho(q+1)-q}dt + |f'(b^\rho)|^q\int_0^1 t^{q\rho-q}(1-t^\rho)\,dt\right)^{\frac{1}{q}} \le$$

$$\frac{\rho(b^\rho-a^\rho)}{\alpha+1}\left(\int_0^1 |1-(\alpha+1)t^{\rho\alpha}|^p dt\right)^{\frac{1}{p}} \times \left(\frac{|f'(a^\rho)|^q(q(\rho-1)+1)+|f'(b^\rho)|^q\rho}{(q(\rho-1)+1)(\rho(q+1)-q+1)}\right)^{\frac{1}{q}} \le \frac{\rho(b^\rho-a^\rho)}{\alpha+1} \times \left(\int_0^{\sqrt[\alpha\rho]{\frac{1}{\alpha+1}}}(1-(\alpha+1)t^{\rho\alpha})^p dt + \int_{\sqrt[\alpha\rho]{\frac{1}{\alpha+1}}}^1 \left((\alpha+1)t^{\rho\alpha}-1\right)^p dt\right)^{\frac{1}{p}} \times \left(\frac{|f'(a^\rho)|^q(q(\rho-1)+1)+|f'(b^\rho)|^q\rho}{(q(\rho-1)+1)(\rho(q+1)-q+1)}\right)^{\frac{1}{q}} \tag{5.7}$$

This completes the proof. ∎

Remark 9. *In Theorem 14,*

(1) *If one takes $\rho \to 1$, one has the inequality proved in* [8, Theorem 6].
(2) *If one takes $\rho \to 1$, and after that if one takes $\alpha = 1$, one has the inequality* [2, Theorem 2.3].

Theorem 15. *Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function I°, $a^\rho, b^\rho \in I^\circ$ with $a^\rho < b^\rho$. If $f' \in L[a^\rho, b^\rho]$ and $|f'|^q$ is convex on $[a^\rho, b^\rho]$ for $q \ge 1$, then the following inequality for the right Katugampola fractional integral holds:*

$$\left|\frac{f(a^\rho)+\alpha f(b^\rho)}{\alpha+1} - \frac{\rho^\alpha\Gamma(\alpha+1)}{(b^\rho-a^\rho)^\alpha}\ {}^\rho I^\alpha_{b-} f(a^\rho)\right| \le \frac{\rho(b^\rho-a^\rho)}{\alpha+1} Z_6(\alpha,\rho)^{1-\frac{1}{q}}(|f'(b^\rho)|^q Z_7(\alpha,\rho) + |f'(a^\rho)|^q Z_8(\alpha,\rho))^{\frac{1}{q}} \tag{5.8}$$

where

$$Z_6(\alpha,\rho) = \frac{\alpha}{\alpha\rho+1}\left[2\rho\left(\frac{1}{\alpha+1}\right)^{\frac{1}{\alpha\rho}} - \rho + 1\right],$$

$$Z_7(\alpha,\rho) = \frac{\alpha}{(\rho+1)(\rho(\alpha+1)+1)}\left[2\rho\left(\frac{1}{\alpha+1}\right)^{\frac{\rho+1}{\alpha\rho}} + 1\right],$$

$Z_8(\alpha,\rho) = Z_6(\alpha,\rho) - Z_7(\alpha,\rho)$,
with $\alpha > 0$ and $\rho > 0$.

Proof. Similarly the proof of the Theorem 13, by using Lemma 3, power mean inequality and convexity of $|f'|^q$, we have (5.8). ∎

Remark 10. *In Theorem 15,*

(1) *If one takes $\rho \to 1$, one has the inequality proved in* [9, Theorem 5].
(2) *If one takes $\rho \to 1$, and after that if one takes $\alpha = 1$, one has the inequality* [11, Theorem 1].

Theorem 16. *Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function $I^{\circ}, a^{\rho}, b^{\rho} \in I^{\circ}$ with $a^{\rho} < b^{\rho}$. If $f' \in L[a^{\rho}, b^{\rho}]$ and $|f'|^q$ is convex on $[a^{\rho}, b^{\rho}]$ for $q > 1$, and $\frac{1}{q} + \frac{1}{p} = 1$, then the following inequality for the right Katugampola fractional integral holds:*

$$\left| \frac{\alpha f(a^{\rho}) + f(b^{\rho})}{\alpha+1} - \frac{\rho^{\alpha}\Gamma(\alpha+1)}{(b^{\rho}-a^{\rho})^{\alpha}} \quad {}^{\rho}I_{a+}^{\alpha} f(b^{\rho}) \right| \leq$$

$$\frac{\rho(b^{\rho}-a^{\rho})}{\alpha+1} \left(Z_4(\alpha,\rho,p) + Z_5(\alpha,\rho,p)\right)^{\frac{1}{p}} \times \left(\frac{|f'(a^{\rho})|^q \rho + |f'(b^{\rho})|^q}{q+1}\right)^{\frac{1}{q}}, \tag{5.9}$$

with $\alpha > 0$, $\rho > 0$ and $Z_4(\alpha,\rho,p), Z_5(\alpha,\rho,p)$ are the same as in Theorem14.

Proof. Similarly the proof of the Theorem 14, by using Lemma 3, Hölder inequality and convexity of $|f'|^q$, we have (5.9). ■

Remark 11. *In Theorem 15,*

(1) *If one takes $\rho \to 1$, one has the inequality proved in* [9, Theorem 6].
(2) *If one takes $\rho \to 1$, and after that if one takes $\alpha = 1$, one has the inequality*

[2, Theorem 2.3].

6. Competing Interests

The authors declare that they have no competing intersts.

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