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Research Article

FUZZY CONE B-METRIC SPACES

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ABSTRACT

In this article, we present the theory of fuzzy cone b-metric space as a new type of generalized metric spaces. We give some basic properties of this new space as Hausdorffness, convergence, completeness etc. In addition to, we introduce fuzzy cone b-metric Banach contraction theorem using our results.
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1. INTRODUCTION

Firstly, the theory of cone metric space was defined by Huang and Zhang in 2007 [3]. They handled ordering Banach space in lieu of the R as follows:

Consider a real Banach space E. When the following conditions are satisfied, the set $P \subset E$ is defined as a cone: for $a, b \in R^+ \cup \{0\}$;

1) P is nonempty, $P \neq \{\theta\}$,
2) P is closed,
3) $ax_1 + bx_2 \in P,$ if $x_1, x_2 \in P,$
4) $x_1 = \theta$, if $x_1 \in P$ and $-x_1 \in P,$

When $P \subset E$ is a cone, a partial ordering $\preceq$ according to P is found where $x_1 \preceq x_2$ means $x_2 - x_1 \in P$. Moreover, the followings will be used:

- $x_1 \prec x_2 \Leftrightarrow x_1 \preceq x_2$ and $x_1 \neq x_2$,
- $x_1 << x_2 \Leftrightarrow x_2 - x_1 \in \operatorname{int} P$ ($\operatorname{int} P$ is the set of interior points of P).

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If there exists a $K>0$ which holds $\theta \preceq x_1 \preceq x_2 \Rightarrow \|x_1\| \le K\|x_2\|$ for each $x_1, x_2 \in E$, in that case P is called a normal cone. Also, the normal constant of P is K that is the smallest positive number satisfying above inequality.

Definition 1.1 [3] Let $X \neq \varnothing$ be an arbitrary set and a mapping d be defined from $X \times X$ to E. When the followings are hold, d is defined as cone metric on X . Also, (X, d) is called cone metric space: for each $x_1, x_2, x_3 \in X$,

d1. $\theta \prec d(x_1, x_2)$ and $d(x_1, x_2) = \theta \Leftrightarrow x_1 = x_2$,
d2. $d(x_1, x_2) = d(x_2, x_1)$,
d3. $d(x_1, x_2) \preceq d(x_1, x_3) + d(x_3, x_2)$.

Obviously, cone metric spaces are a generalization of metric spaces.

Example 1.2 [3] Let $E = R^2$, $P = \{(x_1, x_2):\ x_1, x_2 \ge 0\} \subset E$ and $X = R$. Let d be defined from $X \times X$ to E where $d(x_1, x_2) = (|x_1 - x_2|, \alpha |x_1 - x_2|)$ for a constant $\alpha \ge 0$. In that case, (X, d) is a cone metric space.

In 2011, the structure of cone b-metric space was presented by Hussain and Shah [4]. They examined some basic properties of this space.

Definition 1.3 [4] Let $X \neq \varnothing$ be an arbitrary set, P be a cone of E and $D:\ X \times X \to P$ be a vector-valued function. If the following conditions are hold, then D is said to be a cone b-metric on X with the constant $k \ge 1$. Also, (X, D) is called a cone b-metric space: for each $x_1, x_2, x_3 \in X$,

M1. $\theta \preceq D(x_1, x_2)$ and $D(x_1, x_2) = \theta \Leftrightarrow x_1 = x_2$,
M2. $D(x_1, x_2) = D(x_2, x_1)$,
M3. $D(x_1, x_3) \preceq k\left[D(x_1, x_2) + D(x_2, x_3)\right]$.

Lemma 1.4 [4] Let d be a cone b-metric on X. For each $t_1 >> \theta$ and $t_2 >> \theta$, $t_1, t_2 \in E$, there exists a $t \in E$, $t >> \theta$ satisfying $t << t_1$ and $t << t_2$.

The notion of fuzzy sets was defined by Zadeh [8]. Later, the theory of fuzzy metric space given by Kramosil and Michalek was modified by George and Veeramani and they give basic properties of this space [1, 5].

Definition 1.5 [7] A continuous $t-$norm $*: [0,1] \times [0,1] \to [0,1]$ is a binary operation if the followings are satisfied: for all $x, y, z, t \in [0,1]$,

(1) $*$ is associative and commutative,
(2) $*$ is continuous,

(3) $x*1=x$,
(4) $x*y\le z*t$ whenever $x\le z$ and $y\le t$.

The following equalities which are given by the symbols $*_M, *_P$ and $*_L$ respectively are the three basic continuous t-norms:

- $x *_M y=\min\{x,y\}$,
- $x *_P y = x.y$,
- $x *_L y=\max\{x+y-1,0\}$.

Definition 1.6 Let X be an arbitrary set and $*$ be a continuous $t-$norm. A fuzzy set M on $X^2\times(0,\infty)$ is called a fuzzy metric on X if for each $x_1,x_2,x_3\in X$ and $t,s>0,$ the following axioms are hold:

FM1. $M(x_1,x_2,t)>0,$
FM2. $M(x_1,x_2,t)=1 \Leftrightarrow x_1=x_2,$
FM3. $M(x_1,x_2,t)=M(x_2,x_1,t),$
FM4. $M(x_1,x_3,t+s)\ge M(x_1,x_2,t)*M(x_2,x_3,s),$
FM5. $M(x_1,x_2,\cdot):(0,\infty)\to[0,1]$ is continuous.

The ordered triple $(X,M,*)$ is called a fuzzy metric space.

In 2015, the theory of fuzzy cone metric space was defined by Oner et al. [6].

Definition 1.7 [6] Let X be an arbitrary set, E be a real Banach space and P be a cone of E. A fuzzy set M on $X^2\times\operatorname{int}(P)$ is called a fuzzy cone metric on X if for each $x_1,x_2,x_3\in X$ and $t,s\in\operatorname{int}(P),$ the following axioms are hold:

FCM1. $M(x_1,x_2,t)>0,$
FCM2. $M(x_1,x_2,t)=1 \Leftrightarrow x_1=x_2,$
FCM3. $M(x_1,x_2,t)=M(x_2,x_1,t),$
FCM4. $M(x_1,x_3,t+s)\ge M(x_1,x_2,t)*M(x_2,x_3,s),$
FCM5. $M(x_1,x_2,\cdot):\ \operatorname{int}(P)\to[0,1]$ is continuous.

The ordered triple $(X,M,*)$ is called a fuzzy cone metric space.

2. FUZZY CONE B-METRIC SPACE

We introduce a new concept of generalized metric space called fuzzy cone b-metric space. Also, we give some basic properties of this new space as Hausdorffness, convergence, completeness etc.

Definition 2.1 Let X be an arbitrary set, E be a real Banach space, P be a cone of E and $*$ be a continuous t-norm. A fuzzy set M on $X^2 \times \operatorname{int}(P)$ is said to be fuzzy cone $b-$ metric with the constant $b \geq 1$ on X if for each $x_1, x_2, x_3 \in X$ and $t, s \in \operatorname{int}(P)$, the following axioms are hold:

FCB1. $M(x_1, x_2, t) > 0,$

FCB2. $M(x_1, x_2, t) = 1 \Leftrightarrow x_1 = x_2,$

FCB3. $M(x_1, x_2, t) = M(x_2, x_1, t),$

FCB4. $M(x_1, x_3, b(t+s)) \geq M(x_1, x_2, t) * M(x_2, x_3, s),$

FCB5. $M(x_1, x_2, \cdot): \operatorname{int}(P) \to [0,1]$ is continuous.

The ordered triple $(X, M, *)$ is said to be fuzzy cone $b-$ metric space.

Note that if we take $b = 1$ in the definition of fuzzy cone b-metric space, then condition FCM4 in the definition of fuzzy cone metric space is satisfied. So, every fuzzy cone metric space is a fuzzy cone b-metric space. Also the family of fuzzy cone b-metric spaces is larger than that of the fuzzy cone metric spaces. If we take $E = R,\ P = (0, \infty)$ and $x *_P y = x.y$ for all $x, y \in [0,1]$ in the definition of fuzzy cone metric space, then fuzzy cone metric space becomes a fuzzy metric space. So, every fuzzy metric space is a fuzzy cone metric space. Also the family of fuzzy cone metric spaces is larger than that of the fuzzy metric spaces. Consequently, if we take $b = 1,\ E = R,\ P = (0, \infty)$ and $x *_P y = x.y$ for all $x, y \in [0,1]$ in the definition of fuzzy cone b-metric space, it becomes a fuzzy metric space. Namely, every fuzzy metric space is a fuzzy cone b-metric space.

$$\begin{pmatrix}\text{Fuzzy cone b-metric}\\ \text{space}\end{pmatrix} \overset{b=1}{\Rightarrow} \begin{pmatrix}\text{Fuzzy cone metric}\\ \text{space}\end{pmatrix} \overset{E=R,\ P=(0,\infty),\ x*_P y = x.y}{\Rightarrow} \begin{pmatrix}\text{Fuzzy metric}\\ \text{space}\end{pmatrix}$$

Example 2.2 Let $E = R^2$, $X = R$ and $m *_P n = m.n$ for all $m, n \in [0,1]$. Take a normal cone $P = \{(k_1, k_2): k_1, k_2 \geq 0\} \subset E$ such that $K = 1$ [2]. Let M be defined from $X^2 \times \operatorname{int}(P)$ to $[0,1]$ by

$$M(x_1, x_2, t) = e^{-\frac{|x_1 - x_2|}{\|t\|}}$$

for all $x_1, x_2 \in X$ and $t >> \theta$. In that case, M is a fuzzy cone b-metric on X.

First three conditions can be easily verified.

FCB4. For each $x_1, x_2, x_3 \in X$,

$$|x_1 - x_3| \leq |x_1 - x_2| + |x_2 - x_3|.$$

$s \preceq t+s$ and $t \preceq t+s$ imply $\|s\| \le \|t+s\|$ and $\|t\| \le \|t+s\|$ for all $t >> \theta$ and $s >> \theta$, respectively because of normal cone P. Then, $\frac{\|t+s\|}{\|s\|} \ge 1$ and $\frac{\|t+s\|}{\|t\|} \ge 1$. Thus, we have

$$|x_1 - x_3| \le \frac{\|t+s\|}{\|t\|}|x_1 - x_2| + \frac{\|t+s\|}{\|s\|}|x_2 - x_3|$$

$$\frac{|x_1 - x_3|}{\|t+s\|} \le \frac{|x_1 - x_2|}{\|t\|} + \frac{|x_2 - x_3|}{\|s\|}$$

and for $b \ge 1$,

$$\frac{|x_1 - x_3|}{b\|t+s\|} \le \frac{|x_1 - x_3|}{\|t+s\|} \le \frac{|x_1 - x_2|}{\|t\|} + \frac{|x_2 - x_3|}{\|s\|}.$$

Hence,

$$e^{\frac{|x_1 - x_3|}{b\|t+s\|}} \le e^{\frac{|x_1 - x_2|}{\|t\|}} e^{\frac{|x_2 - x_3|}{\|s\|}}$$

$$e^{-\frac{|x_1 - x_3|}{b\|t+s\|}} \ge e^{-\frac{|x_1 - x_2|}{\|t\|}} e^{-\frac{|x_2 - x_3|}{\|s\|}}.$$

Thus the condition is satisfied.

FCB5. Let n be defined from $\text{int}(P)$ to $(0,\infty)$ by $n(t) = \|t\|$ and f be defined from $(0,\infty)$ to $[0,1]$ by $f(u) = e^{-\frac{|x_1 - x_2|}{u}}$. Then, M can be thought as composition of f and n. Since both n and f are continuous functions, M is also a continuous function.

In that case, $(X, M, *)$ is a fuzzy cone b-metric space.

Example 2.3 Let d be a cone b-metric on X. Take a normal cone P with $K = 1$ and $m *_P n = m.n$ for all $m, n \in [0,1]$. Define $M : X^2 \times \text{int}(P) \to [0,1]$ by

$$M(x_1, x_2, t) = \frac{\|t\|}{\|t\| + \|d(x_1, x_2)\|}$$

for each $x_1, x_2 \in X$ and $t >> \theta$. In that case, M is a fuzzy cone b-metric on X. Also, M is said to be the standard fuzzy cone b-metric induced by a cone b-metric.

First three conditions and FCB5 can be easily verified.

FCB4. Since d is a cone b-metric on X, for each $x_1, x_2, x_3 \in X$,

$$d(x_1, x_3) \le b\left[d(x_1, x_2) + d(x_2, x_3)\right]$$

and we have

$$\|d(x_1,x_3)\| \le \|b[d(x_1,x_2)+d(x_2,x_3)]\|$$
$$\le b\|d(x_1,x_2)\| + b\|d(x_2,x_3)\|.$$

$s \preceq t+s$ and $t \preceq t+s$ imply $\|s\| \le \|t+s\|$ and $\|t\| \le \|t+s\|$ for each $t >> \theta$ and $s >> \theta$, respectively because of normal cone P. Then, $\frac{\|t+s\|}{\|s\|} \ge 1$ and $\frac{\|t+s\|}{\|t\|} \ge 1$. So, we get

$$\|d(x_1,x_3)\| \le \frac{b\|t+s\|}{\|t\|}\|d(x_1,x_2)\| + \frac{b\|t+s\|}{\|s\|}\|d(x_2,x_3)\|$$

$$\frac{\|d(x_1,x_3)\|}{b\|t+s\|} \le \frac{\|d(x_1,x_2)\|}{\|t\|} + \frac{\|d(x_2,x_3)\|}{\|s\|}$$
$$= \frac{\|s\|\|d(x_1,x_2)\| + \|t\|\|d(x_2,x_3)\|}{\|s\|\|t\|}$$

and we have

$$1+\frac{\|d(x_1,x_3)\|}{b\|t+s\|} \le 1+\frac{\|s\|\|d(x_1,x_2)\| + \|t\|\|d(x_2,x_3)\|}{\|s\|\|t\|}$$
$$\le \frac{\|s\|\|t\| + \|s\|\|d(x_1,x_2)\| + \|t\|\|d(x_2,x_3)\|}{\|s\|\|t\|}$$
$$\le \frac{\|s\|\|t\| + \|s\|\|d(x_1,x_2)\| + \|t\|\|d(x_2,x_3)\| + \|d(x_1,x_2)\|\|d(x_2,x_3)\|}{\|s\|\|t\|}$$
$$= \frac{\left(\|t\| + \|d(x_1,x_2)\|\right)\left(\|s\| + \|d(x_2,x_3)\|\right)}{\|s\|\|t\|}.$$

Then,

$$\frac{b\|t+s\| + \|d(x_1,x_3)\|}{b\|t+s\|} \le \frac{\|t\| + \|d(x_1,x_2)\|}{\|t\|} + \frac{\|s\| + \|d(x_2,x_3)\|}{\|s\|}$$

and we have

$$\frac{b\|t+s\|}{b\|t+s\| + \|d(x_1,x_3)\|} \ge \frac{\|t\|}{\|t\| + \|d(x_1,x_2)\|} . \frac{\|s\|}{\|s\| + \|d(x_2,x_3)\|}.$$

Thus, FCB4 is satisfied. As a result, M is a fuzzy cone b-metric on X .

Example 2.4 Let M_1 be a fuzzy cone b-metric on X and M_2 be a fuzzy cone b-metric on Y. Let M be defined from $(X\times Y)^2 \times \operatorname{int}(P)$ to $[0,1]$ by

$$M((x_1,x_2),(y_1,y_2),t) = M_1(x_1,y_1,t) * M_2(x_2,y_2,t)$$

for all $(x_1,x_2),(y_1,y_2)\in X\times Y$ and $t \gg \theta$. In that case, M is a fuzzy cone b-metric on X.

First three conditions and FCB5 can be easily verified.

FCB4. For all $(x_1,x_2),(y_1,y_2),(z_1,z_2)\in X\times Y$,

$$\begin{aligned} M((x_1,x_2),(z_1,z_2),b(t+s)) &= M_1(x_1,z_1,b(t+s)) * M_2(x_2,z_2,b(t+s)) \\ &\geq M_1(x_1,y_1,t) * M_1(y_1,z_1,s) * M_2(x_2,y_2,t) * M_2(y_2,z_2,s) \\ &= M((x_1,x_2),(y_1,y_2),t) * M((y_1,y_2),(z_1,z_2),s) \end{aligned}$$

Thus, FCB4 is satisfied. As a result, M is a fuzzy cone b-metric on $X\times Y$.

Proposition 2.5 Let M be a fuzzy cone b-metric on a set X. In that case it is nondecreasing mapping for each $x_1,x_2\in X$.

Proof Showing that M is a nondecreasing mapping according to $t\in int(P)$ is easy. Firstly, assume that $M(x_1,x_2,t) > M(x_1,x_2,t_0)$ for $t_0 \gg t \gg \theta.$ For $b\geq 1,$

$$\begin{aligned} M(x_1,x_2,bt_0) &\geq M(x_1,x_2,t) * M(x_2,x_2,t_0-t) \\ &= M(x_1,x_2,t) \\ &> M(x_1,x_2,t_0). \end{aligned}$$

So, we obtain a contradiction. Then, $M(x_1,x_2,\cdot)$ is nondecreasing.

Remark 2.6 (1) Let M be a fuzzy cone b-metric on a set X. If $M(x_1,x_2,bt) > 1-\rho$ for all $x_1,x_2\in X$, $t\gg\theta$ and $0<\rho<1,$ then there exists a $s,$ $\theta \ll s \ll t$ such that $M(x_1,x_2,s) > 1-\rho.$

(2) If $\rho_1 > \rho_2,$ then a ρ_3 such that $\rho_1 * \rho_3 \geq \rho_2$ can be found. Also, a ρ_5 such that $\rho_5 * \rho_5 \geq \rho_4$ for any ρ_4 can be found $(\rho_1,\rho_2,\rho_3,\rho_4,\rho_5\in(0,1)).$

Definition 2.7 Let $(X,M,*)$ be a fuzzy cone b-metric space and $x_1\in X.$ Then, for any $0<\rho<1$ and $t\gg\theta$, the set

$$B(x_1,\rho,bt) = \{\, x_2\in X \;:\; M(x_1,x_2,bt) > 1-\rho \}$$

is defined as an open ball.

Definition 2.8 A subset G of a fuzzy cone b-metric space $(X,M,*)$ is called open if given any point x_1 in G, there exist a $1>\rho>0$ and a $t>>\theta$ such that $M(x_1,x_2,bt)>1-\rho$. There x_2 also belongs to G.

Lemma 2.9 For any $x_1 \in X$, $0<\rho<1$ and $t>>\theta$, $B(x_1,r,bt)$ is an open set in fuzzy cone $b-$metric space.

Proof Let $B(x_1,\rho,bt)$ be an open ball. Then,

$$x_2 \in B(x_1,\rho,bt) \Rightarrow M(x_1,x_2,bt)>1-\rho.$$

If we consider Remark 2.6 (1), since $M(x_1,x_2,bt)>1-\rho$, a s for $\theta<<s<<t$ which satisfies $M(x_1,x_2,s)>1-\rho$ can be found. Assume that $\rho_0=M(x_1,x_2,s)$. Since $\rho_0>1-\rho$, a t_0 for $0<t_0<1$ such that $\rho_0>1-t_0>1-\rho$ can be found. If we consider Remark 2.6 (2), for ρ_0 and t_0 such that $\rho_0>1-t_0$, a ρ_1 such that $\rho_0*\rho_1\geq 1-t_0$ can be found. Take into consideration the ball $B(x_2,1-\rho_1,b(t-s))$. We claim that

$$B(x_2,1-\rho_1,b(t-s))\subset B(x_1,\rho,bt).$$

Take $x_3 \in B(x_2,1-\rho_1,b(t-s))$. Then, $M(x_2,x_3,b(t-s))>1-(1-\rho_1)=\rho_1$ for $b\geq 1$. For this reason,

$$\begin{aligned} M(x_1,x_3,bt) &\geq M(x_1,x_2,s)*M(x_2,x_3,t-s) \\ &> \rho_0*\rho_1 \\ &\geq 1-t_0 \\ &> 1-\rho. \end{aligned}$$

Then, $x_3 \in B(x_1,\rho,bt)$. So, the proof is completed.

Proposition 2.10
$\tau_b=\{G\subset X \ : \ x_1\in G \text{ iff there exist } t>>\theta \text{ and } \rho\in(0,1) \text{ such that } B(x_1,\rho,bt)\subset G\}$ is a topology in fuzzy cone b-metric space.

Proof i) If $x_1\in\phi$, so $\phi=B(x_1,r,bt)\subset\phi$. Therefore, $\phi\in\tau_b$. Since $B(x_1,\rho,bt)\subset X$ for any $x_1\in X$, $\rho\in(0,1)$ and $t>>\theta$, $X\in\tau_b$.

ii) Let $U,V\in\tau_b$ and $x_1\in U\cap V$. In that case, $x_1\in U$ and $x_1\in V$. Since $x_1\in U$ and $U\in\tau_b$, there exist a $t_1\in E$, $t_1>>\theta$ and $\rho_1\in(0,1)$ such that

$B(x_1,\rho_1,bt_1)\subset U.$ Similarly, since $x_1\in V$ and $V\in\tau_b,$ there exist a $t_2\in E,\ t_2>>\theta$ and $\rho_2\in(0,1)$ such that $B(x_1,\rho_2,bt_2)\subset V.$ From Lemma 1.4, for $t_1>>\theta$ and $t_2>>\theta$, there exists a $t\in E,\ t>>\theta$ such that $t<<t_1$ and $t<<t_2$. Take $\rho=\min\{\rho_1,\rho_2\}.$ In that case, $B(x_1,\rho,bt)\subset B(x_1,\rho_1,bt_1)\subset U$ and

$B(x_1,\rho,bt)\subset B(x_1,\rho_2,bt_2)\subset V.$

So, $B(x_1,\rho,bt)\subset B(x_1,\rho_1,bt_1)\cap B(x_1,\rho_2,bt_2)\subset U\cap V.$

Consequently, $U\cap V\in\tau_b.$

iii) For all $i\in I$, let $U_i\in\tau$ and $x_1\in\bigcup_{i\in I}U_i.$ Then, for $\exists i_0\in I$, $x_1\in U_{i_0}.$ Since $U_{i_0}\in\tau_b,$ there exist a $t\in E,\ t>>\theta$ and $\rho\in(0,1)$ such that $B(x_1,\rho,bt)\subset U_{i_0}.$ In this case,

$$B(x_1,\rho,bt)\subset U_{i_0}\subset\bigcup_{i\in I}U_i\in\tau_b.$$

Hence, (X,τ_b) is a topological space.

Theorem 2.11 Let M be a fuzzy cone b-metric on a set X. In that case, (X,τ_b) is a Hausdorff space.

Proof Suppose that $x_1\neq x_2$ for $x_1,x_2\in X.$ It is obvious that $1>M(x_1,x_2,b^2t)>0.$ Consider $M(x_1,x_2,bt)=\rho$ for some $\rho,\ 1>\rho>0.$ From Remark 2.6 (2), for each ρ_0 such that $1>\rho_0>\rho,$ there exists a $\rho_1\in(0,1)$ such that $\rho_1*\rho_1\geq\rho_0.$ Take into consideration the open sets $B(x_1,1-\rho_1,\frac{bt}{2})$ and $B(x_2,1-\rho_1,\frac{bt}{2}).$ We claim that

$$B(x_1,1-\rho_1,\frac{bt}{2})\cap B(x_2,1-\rho_1,\frac{bt}{2})=\phi.$$

Suppose that $B(x_1,1-\rho_1,\frac{bt}{2})\cap B(x_2,1-\rho_1,\frac{bt}{2})\neq\phi.$ Then, we can find a x_3 such that

$$x_3\in B(x_1,1-\rho_1,\frac{bt}{2})\cap B(x_2,1-\rho_1,\frac{bt}{2}).$$

So, $x_3\in B(x_1,1-\rho_1,\frac{bt}{2})$ and $x_3\in B(x_2,1-\rho_1,\frac{bt}{2}).$ Therefore,

$$M(x_1,x_3,\frac{bt}{2})>1-(1-\rho_1)=\rho_1$$

and

$$M(x_2, x_3, \frac{bt}{2}) > 1-(1-\rho_1) = \rho_1.$$

For $b \geq 1$,

$$\begin{aligned}\rho &= M(x_1, x_2, b^2 t) \\ &\geq M(x_1, x_3, \frac{bt}{2}) * M(x_3, x_2, \frac{bt}{2}) \\ &> \rho_1 * \rho_1 \\ &\geq \rho_0 \\ &> \rho.\end{aligned}$$

Thus, we obtain a contradiction. As a result, the proof is completed.

Theorem 2.12 Let $(X, M, *)$ be a fuzzy cone b-metric space, then X is a first countable space.

Proof Let $x_1 \in X$ and $t >> \theta.$ Also, take

$$\beta_{x_1} = \{B(x_1, \frac{1}{n}, \frac{bt}{n}) : \ n \in N\}$$

where $B\left(x_1, \frac{1}{n}, \frac{bt}{n}\right)$ denotes the open ball of x_1 in X. It suffices to show that β_{x_1} is a local basis at x_1. Then, let $G \in \tau_b$ and $x_1 \in G.$ By the definition of an open set, there exist $0 < \rho < 1$ and $t \in E, t >> \theta$ which satisfies $B(x_1, \rho, bt) \subset G.$ Take $n \in N$ such that $\frac{1}{n} < \rho.$ Since $\frac{1}{n} < 1,$ $\frac{bt}{n} << bt.$ Now, we must show $B(x_1, \frac{1}{n}, \frac{bt}{n}) \subset B(x_1, \rho, bt).$ Let $x_2 \in B(x_1, \frac{1}{n}, \frac{bt}{n}).$ In this case, $M(x_1, x_2, \frac{bt}{n}) > 1 - \frac{1}{n} > 1 - \rho.$ Since $\frac{bt}{n} << bt,$ by Proposition 2.5, we get $M(x_1, x_2, bt) > M(x_1, x_2, \frac{bt}{n}) > 1 - \rho.$ So, $x_2 \in B(x_1, \rho, bt)$ which implies $B(x_1, \frac{1}{n}, \frac{bt}{n}) \subset B(x_1, \rho, bt) \subset G.$ As a result, x_1 has a countable local basis as β_{x_1}. The proof is completed.

Let $(X, M, *)$ be a fuzzy cone b-metric space and take a sequence $\{x_k\}$ in this space. In that case, definitions of convergent sequence and Cauchy sequence are as follows:

Definition 2.13 If for each $\varepsilon \in (0,1)$ and $t >> \theta,$ there exists a $k_0 \in N$ which satisfies $M(x_k, x, bt) > 1 - \varepsilon$ for each $k \geq k_0,$ then $\{x_k\}$ is said to be convergent to x in X.

Also, x is said to be the limit of $\{x_k\}$ and this is denoted by $\lim_{k\to\infty} x_k = x$ or $x_k \to x$ as $k \to \infty.$

In other words, $\{x_k\}$ converges to x if and only if $M(x_k, x, bt) \to 1$ as to $k \to \infty$ for each $t >> \theta.$

Definition 2.14 If for each $\varepsilon \in (0,1)$ and $t >> \theta,$ there exists a $k_0 \in N$ such that $M(x_k, x_m, bt) > 1-\varepsilon$ for each $k,m \geq k_0$, then $\{x_k\}$ is said to be Cauchy sequence in this space.

In other words, $\{x_k\}$ is a Cauchy sequence if and only if $M(x_k, x_m, bt) \to 1$ as to $k,m \to \infty$ for each $t >> \theta.$

Also, one can say that a complete fuzzy cone b-metric space is a fuzzy cone b-metric space in which every Cauchy sequence is convergent.

Lemma 2.15 Let $(X, M, *)$ is a fuzzy cone b-metric space. Then, every convergent sequence in X has a unique limit.

Proof. Suppose that $x_k \to x_1,$ $x_k \to x_2$ and $x_1 \neq x_2.$ Since $\{x_k\}$ converges to x_1 and x_2, for any $t >> \theta$ and $\varepsilon_1 \in (0,1),$ there exist $k_1, k_2 \in N$ such that $M(x_k, x_1, bt) > 1-\varepsilon_1$ for each $k \geq k_1$ and $M(x_k, x_2, bt) > 1-\varepsilon_1$ for each $k \geq k_2.$ If we set $k_0 = \max\{k_1, k_2\}$, then for each $k \geq k_0$, $t >> \theta$ and $s >> \theta,$

$$M(x_1, x_2, bt) \geq M(x_1, x_k, s) * M(x_k, x_2, t-s)$$
$$> (1-\varepsilon_1) * (1-\varepsilon_1).$$

From Remark 2.6(2), for $1-\varepsilon_1,$ we can find $1-\varepsilon$ such that

$$(1-\varepsilon_1) * (1-\varepsilon_1) \geq 1-\varepsilon.$$

Thus,

$$M(x_1, x_2, bt) > 1-\varepsilon.$$

Then, $M(x_1, x_2, t) = 1 \Leftrightarrow x_1 = x_2.$ So, the proof is completed.

Lemma 2.16 Let $(X, M, *)$ be a fuzzy cone b-metric space. Then, every convergent sequence is a Cauchy sequence.

Proof Since $\{x_k\}$ converges to x, for any $t >> \theta$ and $\varepsilon_1 \in (0,1),$ there exists a $k_0 \in N$ which satisfies $M(x_k, x, bt) > 1-\varepsilon_1$ for each $k \geq k_0.$ Then for each $k,m \geq k_0$, $t >> \theta$ and $s >> \theta,$

$$M(x_k, x_m, bt) \geq M(x_k, x, s) * M(x, x_m, t-s)$$
$$> (1-\varepsilon_1) * (1-\varepsilon_1).$$

From Remark 2.6(2), for $1-\varepsilon_1,$ we can find $1-\varepsilon$ such that

$$(1-\varepsilon_1) * (1-\varepsilon_1) \geq 1-\varepsilon.$$

Hence $M(x_k, x_m, bt) > 1-\varepsilon$. So, the proof is completed.

3. FUZZY CONE B-METRIC BANACH CONTRACTION THEOREM

The fuzzy Banach contraction theorem was given by Grabiec [2] in 1988. We extend it to the complete fuzzy cone $b-$ metric space.

Theorem 3.1 Let M be a complete fuzzy cone b-metric on a set X which satisfies

$$\lim_{t\to\infty} M(x, y, t) = 1 \tag{3.1.1}$$

for each $x, y \in X$. Let $T : X \to X$ be a mapping such that

$$M(Tx, Ty, qt) \geq M(x, y, t) \tag{3.1.2}$$

for each $x, y \in X$ where $0 < q < 1.$ In that case, there exists a unique fixed point of $T.$

Proof Take $x \in X$ and $x_k = T^k x$ for each $k \in N.$ Let us use the method of induction. Then, we have

$$M(x_k, x_{k+1}, qt) \geq M(x, x_1, \frac{t}{q^{k-1}}) \tag{3.1.3}$$

for each $k \in N$ and $t >> \theta.$ For any $p \in Z^+$, we get

$$M(x_k, x_{k+p}, bt) \geq M(x_k, x_{k+1}, \frac{t}{p}) * ... * M(x_{k+p-1}, x_{k+p}, \frac{t}{p})$$
$$\geq M(x, x_1, \frac{t}{p.q^k}) * ... * M(x, x_1, \frac{t}{p.q^{k+p-1}})$$

by (3.1.3). According to (3.1.1), we have

$$\lim_{k\to\infty} M(x_k, x_{k+p}, bt) \geq 1 * ... * 1 = 1.$$

Thus, $\{x_k\}$ is a Cauchy sequence. Also, since X is complete, $\{x_k\}$ is a convergent sequence. Then, assume that $\{x_k\}$ converges to $y \in X.$ So, we obtain

$$M(Ty, y, bt) \geq M(Ty, Tx_k, \frac{t}{2}) * M(Tx_k, y, \frac{t}{2})$$
$$= M(Ty, Tx_k, \frac{t}{2}) * M(T^{k+1}x, y, \frac{t}{2}).$$

From (3.1.2),

$$M(Ty, y, bt) \geq M(y, x_k, \frac{t}{2q}) * M(x_{k+1}, y, \frac{t}{2})$$
$$\geq 1 * 1 = 1.$$

By FCB2, we obtain $Ty = y$, a fixed point. Finally, to verify uniqueness of the fixed point, suppose that $Tz = z$ for some $z \in X$. In this case,

$$1 \geq M(z, y, t) = M(Tz, Ty, t)$$
$$\geq M(z, y, \frac{t}{q}) = M(Tz, Ty, \frac{t}{q})$$
$$\geq M(z, y, \frac{t}{q^2})$$
$$\vdots$$
$$\geq M(z, y, \frac{t}{q^k}) \to 1 \text{ as } k \to \infty.$$

By FCB2, $z = y$.

Consequently, T has a unique fixed point.

Example 3.2: We consider Example 2.3 and define $T : X \to X$ by $Tx = \frac{x}{5}$. In that case, $(X, M, *)$ is a complete fuzzy cone b-metric space which satisfies (3.1.1) and T satisfies (3.1.2) with $q = \frac{1}{5} \in (0,1)$. Thus, there exists a unique fixed point of T which is 0.

4. CONCLUSIONS

In this paper, we introduce theory of fuzzy cone b-metric space and examine basic properties of this space. Also, we extend Banach contraction theorem to the complete fuzzy cone b-metric spaces.

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