In this work, we define 2-absorbing $\delta$-primary fuzzy ideals which is the generalization of 2-absorbing fuzzy ideal and 2-absorbing primary fuzzy ideals. Furthermore, we give some fundamental results concerning these notions.

Keywords: 2-absorbing fuzzy ideals, 2-absorbing $\delta$-primary fuzzy ideals.

1. INTRODUCTION

The notion of fuzzy subset is defined by Zadeh in[9] and studied to apply fuzzy theory on algebraic structures. Then many researchers have studied on fuzzy ring theory. Liu [4] investigated the concept of fuzzy ideal of a ring. In fuzzy commutative algebra, prime ideals are the most significant structures. Mukherjee and Sen defined prime and primary fuzzy ideals in [5,6]. Also in [7] Siddky and Khatab studied on the notion of nil radical of fuzzy ideals.

The concept of 2-absorbing ideal, which is a generalization of prime ideal, was introduced by Badawi in [1]. At present, work on the 2-absorbing ideal theory is progressing rapidly. It has studied extensively by many authors. Recently, Badawi et al. [2] introduced the concept of 2-absorbing primary ideals in commutative rings with $1 \neq 0$ and gave some characterizations related to it. Also in [8], Sönmez et al. introduced the notion of 2-absorbing primary fuzzy ideal, which is a generalization of prime and primary fuzzy ideals in commutative rings. Furthermore in [11], Zhao introduced the concept of expansion of ideals of R. In this paper, we introduce the notion of 2-absorbing $\delta$-primary fuzzy ideal which is a generalization of 2-absorbing primary fuzzy ideal. Also recall that from [8], a nonconstant fuzzy ideal is said to be 2-absorbing primary if if for any fuzzy points $x_r, y_s, z_t$ of $R$, $x_r y_s z_t \in \mu$ implies that either $x_r y_s \in \mu$ or $x_r z_t \in \sqrt{\mu}$.
Based on this definition, a nonconstant fuzzy ideal \( \mu \) of \( R \) is said to be a \( 2 \)-absorbing \( \delta \)-primary fuzzy ideal of \( R \) if for any fuzzy points \( x_r, y_s, z_t \) of \( R \), 
\[ x_r y_s z_t \notin \mu \] implies that either \( x_r y_s \in \mu \) or \( x_r z_t \in \delta(\mu) \) or \( y_s z_t \in \delta(\mu) \).

2. PREMILINARIES

We assume throughout that all rings are commutative with \( 1 \neq 0 \). Unless stated otherwise \( L = [0,1] \) stands for a complete lattice. Let \( I \) be a proper ideal of \( R \), the set 
\[ \{ I : I \text{ is an ideal of } R \} \] will be denoted by \( I(R) \). \( Z \) denotes the ring of integers, \( L(R) \) denotes the set of fuzzy sets of \( R \) and \( LI(R) \) denotes the set of fuzzy ideals of \( R \).

For \( \mu, \xi \in L(R) \), we say \( \mu \subseteq \xi \) if and only if \( \mu(x) \leq \xi(x) \) for all \( x \in R \). When \( r \in L \), \( x, y \in R \) we define \( x_r \in L(R) \) as follows:
\[
x_r(y) = \begin{cases} 
  r & \text{if } x = y, \\
  0 & \text{otherwise}, 
\end{cases}
\]
and \( x_r \) is referred to as fuzzy point of \( R \).

Also, for \( \mu \in L(R) \) and \( t \in L \), define \( \mu_t \) as follows:
\[
\mu_t = \{ x \in R : \mu(x) \geq t \}
\]

**Definition 2.1** [4] A fuzzy subset \( \mu \) of a ring \( R \) is called a fuzzy ideal of \( R \) if \( x, y \in R \) the following conditions are satisfied:

- \( \mu(x - y) \geq \mu(x) \wedge \mu(y), \forall x, y \in R \)
- \( \mu(xy) \geq \mu(x) \vee \mu(y), \forall x, y \in R \)

Let \( \mu \) be any fuzzy ideal of \( R \); \( x, y \in R \), and \( 0 \) be the additive identity of \( R \). Then it is easy to verify the following:

(i) \( \mu(0) \geq \mu(x) \), \( \mu(x) = \mu(-x) \) and \( \mu_t \subseteq \mu_s \) where \( s, t \in \text{Im}(\mu) \) and \( t > s \).

(ii) If \( \mu(0) = \mu(x - y) \), then \( \mu(x) = \mu(y) \), \( \mu(x) = s \) iff \( x \in \mu_s \) and \( x \notin \mu_t \), \( \forall t > s \).

**Definition 2.2** [3] Let \( \mu \) be any fuzzy ideal of \( R \). The ideal \( \mu_t \), \( (\mu(0) \geq t) \) is called level ideal of \( \mu \).

**Definition 2.3** [6] A fuzzy ideal \( \mu \) of \( R \) is called a prime fuzzy ideal if for any two fuzzy points \( x_r, y_s \) of \( R \), \( x_r y_s \in \mu \) implies either \( x_r \in \mu \) or \( y_s \in \mu \).

**Definition 2.4** [5] Let \( \mu \) be a fuzzy ideal of \( R \). Then \( \sqrt{\mu} \), called the radical of \( \mu \), is defined by \( \sqrt{\mu}(x) = \vee_{n \geq 1} \mu(x^n) \).
Definition 2.5 [5] A fuzzy ideal \( \mu \) of \( R \) is called a primary fuzzy ideal if for \( x, y \in R \), \( \mu(xy) > \mu(x) \) implies \( \mu(xy) \leq \mu(y^n) \) for some positive integer \( n \).

Theorem 2.6 [5] Let \( \mu \) be a fuzzy ideal of a ring \( R \). Then \( \sqrt{\mu} \) is a fuzzy ideal of \( R \).

Definition 2.7 [6] Let \( R \) be a ring. Then a nonconstant fuzzy ideal \( \mu \) is said to be a weakly completely prime fuzzy ideal iff for \( x, y \in R \), \( \mu(xy) = \max\{\mu(x), \mu(y)\} \).

Theorem 2.8 [7] If \( \mu \) and \( \xi \) are two fuzzy ideals of \( R \), then \( \mu \cap \xi = \sqrt{\mu} \cap \sqrt{\xi} \).

Theorem 2.9 [7] Let \( f : R \to S \) be a ring homomorphism and let \( \mu \) be a fuzzy ideal of \( R \) such that \( \mu \) is constant on \( \text{Ker} f \) and \( \xi \) be a fuzzy ideal of \( S \). Then,

\[
\begin{align*}
\bullet \sqrt{f(\mu)} &= f(\sqrt{\mu}), \\
\sqrt{f^{-1}(\xi)} &= f^{-1}(\sqrt{\xi}).
\end{align*}
\]

Definition 2.10 [1] A nonzero proper ideal \( I \) of a commutative ring \( R \) with \( 1 \neq 0 \) is called a 2-absorbing ideal if whenever \( a, b, c \in R \) with \( abc \in I \), then either \( ab \in I \) or \( ac \in I \) or \( bc \in I \).

Definition 2.11 [2] A proper ideal \( I \) of \( R \) is called a 2-absorbing primary ideal of \( R \) if whenever \( a, b, c \in R \) with \( abc \in I \), then either \( ab \in I \) or \( ac \in \sqrt{I} \) or \( bc \in \sqrt{I} \).

Theorem 2.12 [2] If \( I \) is a 2-absorbing primary ideal of \( R \), then \( \sqrt{I} \) is a 2-absorbing ideal of \( R \).

Definition 2.13 [8] A nonconstant fuzzy ideal is called a 2-absorbing primary fuzzy ideal if if for any fuzzy points \( x_r, y_s, z_t \) of \( R \), \( x_r y_s z_t \in \mu \) implies that either \( x_r y_s \in \mu \) or \( x_r z_t \in \sqrt{\mu} \) or \( y_s z_t \in \sqrt{\mu} \)

Definition 2.14 [11] A function \( \delta : I(R) \to I(R) \) is called an expansion function of ideals of \( R \) if whenever \( P, Q \) are ideals of \( R \) with \( P \subseteq Q \), then \( P \subseteq \delta(P) \) and \( \delta(P) \subseteq \delta(Q) \).

Definition 2.15 [11] A proper ideal \( I \) of \( R \) is said to be a \( \delta \)-primary ideal of \( R \) if \( xy \in I \) implies that \( x \in I \) or \( y \in \delta(I) \) for any \( x, y \in R \).

Theorem 2.16 [11] Let \( \delta_0 \) be an identity function, where \( \delta(I) = I \) for every \( I \in Id(R) \). Then an ideal \( I \) is \( \delta_0 \)-primary ideal if and only if \( I \) is a prime ideal.

Theorem 2.17 [11] Let \( \delta_1 \) be a function, which is defined \( \delta_1(I) = \sqrt{I} \) for every \( I \in Id(R) \). Then an ideal \( I \) is a \( \delta_1 \)-primary if and only if it is primary ideal.
3. 2-ABSORBING $\delta$—PRIMARy FUZZy IDEALS

**Definition 3.1** Let $\mu$ be a nonconstant fuzzy ideal of $R$. Then $\mu$ is called a 2-absorbing $\delta$—primary fuzzy ideal of $R$ if for any fuzzy points $x_r, y_s, z_t$ of $R$, $x_r y_s z_t \in \mu$ implies that either $x_r y_s \in \mu$ or $x_r z_t \in \delta(\mu)$ or $y_s z_t \in \delta(\mu)$.

**Example 3.2** Let $\delta_0$ be identity function where $\delta_0(\mu) = \mu$ for every $\mu \in LI(R)$. Then $\mu$ is a 2-absorbing $\delta_0$—primary fuzzy ideal if and only if it is a 2-absorbing fuzzy ideal.

**Example 3.3** Let $\delta_1$ be a function such that $\delta_1$ is defined by $\delta_1(\mu) = \sqrt{\mu}$, the fuzzy radical of fuzzy ideal $\mu$. Then $\mu$ is a 2-absorbing $\delta_1$—primary fuzzy ideal if and only if it is a 2-absorbing primary fuzzy ideal.

**Theorem 3.4** Let $\mu$ be a fuzzy ideal of $R$. Assume that $\delta$ is a fuzzy ideal expansion and ideal expansion of $R$ such that $\delta(\mu_t) = \delta(\mu)$, for $t \in [0, \mu(0)]$. If $\mu$ is a 2-absorbing $\delta$—primary fuzzy ideal, then the level ideal $\mu_t$, $t \in [0, \mu(0)]$, is a 2-absorbing $\delta$—primary ideal of $R$.

**Proof** Assume that $\mu$ is a 2-absorbing $\delta$—primary fuzzy ideal of $R$ such that $\delta(\mu_t) = \delta(\mu)$, for $t \in [0, \mu(0)]$. Let $xyz \in \mu_t$ where $t \in [0, \mu(0)]$. Then $x_r y_s z_t \in \mu$. Since $\mu$ is a 2-absorbing $\delta$—primary fuzzy ideal of $R$, then we have $x_r y_s \in \mu$ or $x_r z_t \in \delta(\mu)$ or $y_s z_t \in \delta(\mu)$. Thus we conclude that $xy \in \mu_t$ or $xz \in \delta(\mu)$, $\delta(\mu_t)$ or $yz \in \delta(\mu)$, $\delta(\mu_t)$. Hence, $\mu_t$ is a 2-absorbing $\delta$—primary ideal of $R$.

**Theorem 3.5** Let $\gamma$ and $\delta$ be two fuzzy ideal expansions of $R$. If $\delta(\mu) \subseteq \gamma(\mu)$ for each fuzzy ideal $\mu$, then every 2-absorbing $\delta$—primary fuzzy ideal of $R$ is also 2-absorbing $\gamma$—primary fuzzy ideal of $R$. Thus, in particular, a 2-absorbing fuzzy ideal of $R$ is a 2-absorbing $\delta$—primary fuzzy ideal for every fuzzy ideal expansion $\delta$.

**Proof** Assume that $\mu$ is a 2-absorbing $\delta$—primary fuzzy ideal of $R$. We show that $\mu$ is a 2-absorbing $\gamma$—primary fuzzy ideal of $R$. Let $x_r y_s z_t \in \mu$. Since $\mu$ is a 2-absorbing $\delta$—primary fuzzy ideal and $\delta(\mu) \subseteq \gamma(\mu)$, then we conclude that $x_r y_s \in \mu$ or $x_r z_t \in \delta(\mu) \subseteq \gamma(\mu)$ or $y_s z_t \in \delta(\mu) \subseteq \gamma(\mu)$. Hence $\mu$ is a 2-absorbing $\gamma$—primary fuzzy ideal.

For each $\delta$ fuzzy ideal expansion, if $\mu$ is a 2-absorbing fuzzy ideal, then we have $\mu$ is a 2-absorbing $\delta$—primary fuzzy ideal by the definition of 2-absorbing fuzzy ideals.

**Definition 3.6** A fuzzy ideal expansion $\delta$ is called intersection preserving if it satisfies $\delta(\mu \cap \xi) = \delta(\mu) \cap \delta(\xi)$ for any $\mu, \xi \in LI(R)$.
Definition 3.7 An expansion is said to be global if for any ring homomorphism \( f : R \to S \),
\[ \delta(f^{-1}(\mu)) = f^{-1}(\delta(\mu)) \] for all \( \mu \in \text{LI}(S) \).

Note that the expansions \( \delta_0 \) and \( \delta_1 \) are both intersection preserving and global.

Theorem 3.8 Let \( \delta \) be an intersection preserving fuzzy ideal expansion of \( R \). If \( \mu_1, \ldots, \mu_n \) are 2-absorbing \( \delta \)-primary fuzzy ideals of \( R \) and \( \xi = \delta(\mu_i) \) for all \( i \), then
\[ \mu = \bigcap_{i=1}^{n} \mu_i \]
is a \( \delta \)-primary fuzzy ideal.

Proof Suppose that \( x_r, y_s, z_t \in \mu \) and \( x_r, y_s \notin \mu \). Then \( x_r, y_s \notin \mu_j \) for some \( n \geq j \geq 1 \) and \( x_r, y_s, z_t \in \mu_j \) for all \( n \geq j \geq 1 \). Since \( \mu_j \) is a 2-absorbing \( \delta \)-primary fuzzy ideal, then we have
\[ y_s z_t \in \delta(\mu_j) = \bigcap_{i=1}^{n} \delta(\mu_i) = \delta(\bigcap_{i=1}^{n} \mu_i) = \delta(\mu) \]
or
\[ x_r z_t \in \delta(\mu_j) = \bigcap_{i=1}^{n} \delta(\mu_i) = \delta(\bigcap_{i=1}^{n} \mu_i) = \delta(\mu). \]

Thus \( \mu \) is a 2-absorbing \( \delta \)-primary ideal of \( R \).

Theorem 3.9 If \( \delta \) is a global and \( f : R \to S \) is a ring homomorphism, then for any 2-absorbing \( \delta \)-primary fuzzy ideal \( \mu \) of \( S \), \( f^{-1}(\mu) \) is a 2-absorbing \( \delta \)-primary fuzzy ideal of \( R \).

Proof Assume that \( x_r, y_s, z_t \in f^{-1}(\mu) \) where \( x_r, y_s, z_t \) are any fuzzy points of \( R \). Then
\[ r \wedge s \wedge t \leq f^{-1}(\mu)(xyz) = \mu(f(xy)) = \mu(f(x)f(y)f(z)). \]

Let \( f(x) = a, \ f(y) = b, \ f(z) = c \in S \). Thus we get that \( r \wedge s \wedge t \leq \mu(abc) \) and \( a_r b_s c_t \in \mu \). Since \( \mu \) is a 2-absorbing primary fuzzy ideal, then we get
\[ a_r b_s c_t \in \delta(\mu) \mathrm{ or } a_r c_t \in \delta(\mu) \mathrm{ or } b_s c_t \in \delta(\mu). \]

If \( a_r b_s \in \mu \), then we have
\[ r \wedge s \leq \mu(ab) = \mu(f(x)f(y)) = \mu(f(xy)) = f^{-1}(\mu)(xy). \]

Hence we conclude that \( x_r y_s \in f^{-1}(\mu) \).

If \( a_r c_t \in \delta(\mu) \), then we get
\[ r \wedge t \leq \delta(\mu)(ac) = \delta(\mu)(f(x)f(z)) = \delta(\mu)f(xz) = f^{-1}(\delta(\mu))(xz). \]

Since \( \delta \) is a global, then we have
By the similar way, it can be seen that \( y_s z_t \in \delta(f^{-1}(\mu)) \).

**Theorem 3.10** Let \( f : R \rightarrow S \) be a surjective ring homomorphism and let \( \delta \) be a global fuzzy ideal expansion. Then a fuzzy ideal \( \mu \) of \( R \) which is constant on \( \text{Ker} f \) is a 2-absorbing \( \delta \)-primary if and only if \( f(\mu) \) is a 2-absorbing \( \delta \)-primary fuzzy ideal of \( S \).

**Proof** Suppose that \( a_s b_s c_i \in f(\mu) \) where \( a_s, b_s, c_i \) are any fuzzy points of \( S \). Since \( f \) is a surjective ring homomorphism, we have there exist \( x, y, z \in R \) such that \( f(x) = a_s, f(y) = b_s, f(z) = c_i \).

Thus, \( a_s b_s c_i = r s t \leq f(\mu)(a b c) = f(\mu)(f(x)f(y)f(z)) = f(\mu)(f(xyz)) = \mu(xyz) \) because \( \mu \) is constant on \( \text{Ker} f \). Then we get \( x_s y_s z_t \in \mu \). Since \( \mu \) is a 2-absorbing \( \delta \)-primary fuzzy ideal, we conclude \( x_s y_s \in \mu \) or \( x_s z_t \in \delta(\mu) \) or \( y_s z_t \in \delta(\mu) \). Thus, \( r s t \leq \mu(xy) = f(\mu)(f(x y)) = f(\mu)(f(x)f(y)) = f(\mu)(a b) \) so \( a_s b_s \in f(\mu) \) or 
\[ r s t \leq \delta(\mu)(x y) = \delta(f(\mu))(f(x y)) = \delta(f(\mu)f(x)f(y)) = \delta(f(\mu))(a c) \] 
so \( a_s c_i \in \delta(f(\mu)) \).

By the similar way it is easy to see that \( b_s c_i \in \delta(f(\mu)) \) if \( y_s z_t \in \delta(\mu) \).

**Theorem 3.11** Let \( \mu \) be a fuzzy ideal of \( R \). If \( \delta(\mu) \) is a prime fuzzy ideal of \( R \), then \( \mu \) is a 2-absorbing primary fuzzy ideal of \( R \).

**Proof** Assume that \( x_s y_s z_t \in \mu \) and \( x_s y_s \notin \mu \) for any \( x, y, z \in R \) and \( r, s, t \in [0, 1] \). Since \( x_s y_s z_t \in \mu \) and \( R \) is a commutative ring then
\[ x_s y_s z_t = (x_s z_t)(y_s z_t) \in \mu \subseteq \delta(\mu). \]

Thus \( x_s z_t \in \delta(\mu) \) or \( y_s z_t \in \delta(\mu) \) since \( \delta(\mu) \) is a prime fuzzy ideal of \( R \). Hence, we conclude that \( \mu \) is a 2-absorbing primary fuzzy ideal of \( R \).

4. **CONCLUSION**

In this work the theoretical point of view of 2-absorbing primary fuzzy ideals is discussed. We introduce the definition of 2-absorbing \( \delta \)-primary fuzzy ideal which is a generalization of 2-absorbing primary fuzzy ideal. Furthermore, under a ring homomorphism, these ideals are investigated. In order to extend this study, one could study other algebraic structures and do some further study on properties them.

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