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Research Article

ON δ-PRIMARY HYPERIDEALS OF COMMUTATIVE SEMIHYPERRINGS

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ABSTRACT

In this paper, we study the mapping δ that assigns to each hyperideal I of the commutative semihyperring R, another hyperideal $\delta(I)$ of the same semihyperring. Also we introduced the notation of δ-zero divisor of commutative semihyperring and δ-semidomainlike semihyperring which is generalization to those in the semirings. Moreover we showed that if δ be a global hyperideal expansion then I is δ-primary if and only if $Z_{\delta}(R/I) \subseteq \delta(\{0_{R/I}\})$.
Keywords: Semihyperring, hyperideal, δ-primary, δ-semidomailike semihyperring**.**

1. INTRODUCTION

The first one who introduced the theory of algebraic hyperstructure as a generalization of ordinary algebraic structure was the French mathematician F. Marty in 1934. Firstly, Marty introduced the concepts of hypergroup[4]. In 1956 Marc Krasner introduced the concepts of the hyperrings and hyperfields and he generalized the additive structure of the hyperring as canonical hypergroup[2]. Since then, many researchers have studied the theory of hyperstructure and many results have been published on hyperrings, most of these results where extensions of the standard results in commutative algebra.

Marty put the definition of a hyperoperation $\circ$ as relation from a set H to $P^*(H)$ which the set of the nonempty subsets of H. It can be written as :

$$\circ: H \times H \longrightarrow P^*(H).$$

By the definition of hyperoperation, Marty defined the hypergroup as the following .If H is nonempty set and $\circ$ is a hyperoperation defined on H, such that $\circ$ satisfy the following conditions, For $x, y, z \in H$, we have $(x \circ y) \circ z = x \circ (y \circ z)$ and $x \circ H = H \circ x$. Then $(H, \circ)$ is a hypergroup where, $x \circ (y \circ z) = \bigcup_{r \in (y \circ z)} x \circ r$, $x \circ H = \bigcup_{h \in H} x \circ h$.

Definition 1.1: [4] In the sense of Krasner, the algebraic structure $(R, +, \cdot)$ is a hyperring if:

1. $(H, +)$ is canonical hype r group. i.e.

i. $\forall x, y, z \in R,\ x + (y + z) = (x + y) + z$.

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ii. $\forall x, y \in R,\ x + y = y + x$.
iii. $\exists$ a zero element $0 \in R$, such that $x + 0 = 0 + x = \{0\}, \forall x \in R$.
iv. For each $x \in R, \exists x' \in R$ such that $0 \in x + x'$, x' is called the opposite of x, and we write $x' = -x$.
v. If $z \in x + y$, then $y \in -x + z$ and $x \in -y + z$.
2. $(H, \cdot)$ is a semigroup have a zero as bilaterally absorbing element, i.e.
$0 \cdot x = x \cdot 0 = 0$.
3. The multiplication is distributive to the hyperaddition, so that
$x \cdot (y + z) = x \cdot y + x \cdot z$.
$(x + y) \cdot z = x \cdot y + x \cdot z$ for all $x, y, z \in R$.

Definition 1.2:[1] A semihyperring is an algebraic structure $(R, +, \cdot)$ which satisfies the following axioms:

1) $(R, +)$ is commutative hypermonoid , i.e.
(i) $(x + y) + z = x + (y + z)$ for all $x, y, z \in R$.
(ii) There exists $0 \in R$, such that $x + 0 = 0 + x = x$ for all $x \in R$.
(iii)$x + y = y + x$ for all $x, y \in R$.
2) (H, $\cdot$) is a semigroup have a zero as bilaterally absorbing element, i.e.
(i) $0 \cdot x = x \cdot 0 = 0$.
(ii) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all $x, y, z \in R$.
3) The multiplication is distributive to the hyperaddition, so that
$x \cdot (y + z) = x \cdot y + x \cdot z$, $(x + y) \cdot z = x \cdot y + x \cdot z$ for all $x, y, z \in R$.

Definition 1.3:[1] A semihyperring $(R, +, \cdot)$ is called:

(i) Commutative, if $x \cdot y = y \cdot x$ for all $x, y \in R$.
(ii) Semihyperring with identity if $1_R \in R$, such that $x \cdot 1_R = 1_R \cdot x = x$, for all $x \in R$.

Definition 1.4:[1] A left (right) hyperideal of a semihyperring R is a non-empty subset I of R, such that

(i) For all $x, y \in I, x + y \subseteq I$
(ii) For all $x \in I$, $r \in R$, $r \cdot x \in I$ ($x \cdot r \in I$), a hyperideal of a semihyperring is a non-empty subset of R, which is left and right hyperideal of R, and the set of all hyperideals of R is denoted by $Id(R)$.

Definition 1.5:[1] For $x \in R$, if there exist one and only one $-x \in R$ such that $0 \in x + (-x)$ then $-x$ is called the opposite of x. Denote to the set of all opposite elements of R, by $V(R)$,that is $V(R) = \{x \in R | \exists y \in R, 0 \in x + y\}$.

Definition 1.6:[2] Let I be a hyperideal of a semihyperring R, then I is called k-hyperideal if for $a \in I, x \in R$ we have $a + x \approx I \implies x \in I$, where $A \approx B$ mean $A \cap B \neq \phi$.

Definition 1.7:[1] A semihyperring R is called additively reversive if it satisfy if it satisfy the reversive property with respect to the hyperoperation of addition, i.e. for $a \in b + c$ implies $b \in a + (-c)$ and $c = a + (-b)$.

Definition 1.8: Let R be commutative semihyperring, and let I be a hyperideal of R, the closure of I which denoted by $cl(I)$ is defined by $cl(I) = \{a \in R : a + c \subseteq I$, for some $c \in I\}$.

Theorem 1.1: Let R be a semihyperring. Then R has at least one maximal k-hyperideal.
Proof: let Δ be the set of all k-hyperideals of R. Since $\{0\}$ is proper k-hyperideal of R then $\Delta \neq \phi$, and the relation of inclusion is partially order on Δ, then by Zorn's lemma we have a maximal element in Δ. So that R has at least one maximal k-hyperideal.■

In [2], Ameri showed that for a semihyperring R, with I is a hyperideal of R, the set $(R/I, \oplus, \odot)$ is a semihyperring, where $R/I = \{x + I, x \in R\}$, $x + I \oplus y + I = \{z + I : z \in x + y + I\}$ and $x + I \odot y + I = xy + I$.

2. NOTATION AND BASIC STRUCTURE

Definition: [5]A hyperideal expansion is a function δ, which assigns to each hyperideal I of a semihyperring R another hyperideal $\delta(I)$ of the same semihyperring such that $I \subseteq \delta(I)$ and if $I \subseteq J$ then $\delta(I) \subseteq \delta(J)$ for any hyperideal J of R.

Definition 2.1: [5] Let δ be a hyperideal expansion, a proper hyperideal I of a semihyperring R is called δ-primary if $ab \in I$, and $a \notin I$, then $b \in \delta(I)$. The definition can be stated as: If $b \in I, a \notin \delta(I)$, then $b \in I$.

Remarks:

1. Let I be a hyperideal of the semihyperring R, the identity function δ_0, such that $\delta_0(I) = I$ is an expansion of for each $I \in Id(R)$ A hyperideal I is δ_0-primary if and only if I is prime hyperideal.
2. Let $r(I)$ be the radical of hyperideal I of the semihyperring be defined the same as on the hyperideal of hyperring. Define $\delta_1(I) = r(I)$ for each $I \in Id(R)$ then δ_1 is a hyperideal expansion. A hyperideal I is δ_1-primary if and only if I is primary hyperideal.
3. Let I be a hyperideal of the semihyperring R, let $\delta_2(I) = cl(I)$. Then δ_2 is a hyperideal expansion.
4. Let B be the biggest hyperideal of the hyperring R, then $\delta_3(I) = B$ for all $I \in Id(R)$, then δ_3 is a hyperideal expansion.
5. Let δ_4 defined by the intersection of all maximal hyperideals containing I, and $\delta_4(R) = R$, then δ_4 is hyperideal expansion.
6. If δ is a hyperideal expansion, let $E_\delta: Id(R) \longrightarrow Id(R)$ defined by

   $E_\delta(I) = \cap\{J \in Id(R): I \subseteq J, J$ is δ-primary$\}$. Then E_δ is a hyperideal expansion.

Theorem 2.1: let R be commutative semihyperring, and let δ be a hyperideal expansion then a hyperideal P is δ-primary if and only if for any two hyperideals I and J if $IJ \subseteq P$ and $I \nsubseteq P \Rightarrow J \subseteq \delta(P)$.

Proof: Let P δ-primary hyperideal, and suppose that I, J hyperideals such that $I.J \subseteq P$ and $I \nsubseteq P$ but $J \nsubseteq \delta(P)$. Now choose $a \in I - P, b \in J - \delta(P)$, then $ab \subseteq I.J \subseteq P$ but $a \notin P, b \notin \delta(P)$ which is a contradiction because P is δ-primary. Therefore $J \subseteq \delta(P)$. Conversely, assume that for any two hyperideals I, J if $I.J \subseteq P$ and $I \nsubseteq P$, then $J \subseteq \delta(P)$. For any $a, b \in R$ suppose that $ab \subseteq P$ and $a \notin P$, then $< a >< b > \subseteq P$ and $< a > \nsubseteq P$. So $< b > \subseteq \delta(P)$ implies that, $b \in < b > \subseteq \delta(P)$, and P is δ-primary.■

Theorem 2.2: If I is δ-primary hyperideal of the commutative semihyperring R and J be any subset of R, then $(I:Q)$ is also δ-primary, and if J is a hyperideal with $J \nsubseteq \delta(I)$ then $(I:J) = I$.

Proof: Let $ab \in (I:Q)$ and $a \notin (I:Q)$, then $\exists q \in Q$, such that $aq \notin I$, but $abq = aqb \in I, aq \notin I$ then $b \in \delta(I) \subseteq \delta(I:Q)$. Hence $(I:Q)$ is δ-primary. For any $x \in I$, and for all $y \in J$, we have that $xy \in I$. Then $x \in (I:J)$ and so $I \subseteq (I:J)$. Conversely, for any $x \in J, y \in (I:J)$ we have $xy \in I$, therefore $J(I:J) \subseteq I$, and by the assumption $J \nsubseteq \delta(I)$, then $(I:J) \subseteq I$ so we have $(I:J) = I$■.

Theorem 2.3: Let δ be a hyperideal expansion of the commutative semihyperring R. If δ(I) ⊆ r(I) for every δ-primary hyperideal I, then $\delta(I) = r(I)$.

Proof: let $x \in r(I)$, then $x^n \in I$ for some $n \in \mathbb{N}$. If $= 1$, then $x \in I \subseteq \delta(I)$ and we have done. So we may assume that $n > 1$, such that $x^n \in I$ but $x^{n-1} \notin I$. Since I is δ-primary and $x^n = xx^{n-1} \in I, x^{n-1} \notin I$, then $x \in I$, therefore $\delta(I) = r(I)$. ■

We will define a δ-zero divisor of a semihyperring and a hyperring. This definition is an extension to the definition of a δ-zero divisor of a semiring [3].

Definition 2.2: Let δ be a hyperideal expansion on a semihyperring (hyperring) R, an element x is called δ-zero divisor in R, if there exists $y \in R$ with $y \notin \delta(\{0\})$ such that $xy \in \delta(\{0\})$. The set of all δ-zero divisor in R will be denoted by $Z_\delta(R)$.

Theorem 2.4: Let R be a semihyperring with identity and let δ be a hyperideal expansion such that $\delta(\{0\}) \neq R$, then

1. $nil_\delta(R)$ is a hyperideal of R with $nil_\delta(R) \subseteq Z_\delta(R)$.
2. If $Z_\delta(R)$ is a hyperideal, then $Z_\delta(R)$ is δ-primary.

Proof: 1. Let $x, y \in nil_\delta(R), r \in R$, then $x, y \in \delta(\{0\})$ which is a hyperideal. So that $x + y \subseteq \delta(\{0\})$ and $rx \in \delta(\{0\})$, therefore $nil_\delta(R)$ is a hyperideal. Now for $x \in nil_\delta(R)$, we have $x = x \cdot 1_R \in \delta(\{0\})$, but $\delta(\{0\}) \neq R$, so $1_R \notin \delta(\{0\})$ and $x \in Z_\delta(R)$.

2. Suppose that $Z_\delta(R)$ is a hyperideal of R, let $x, y \in R$ such that $xy \in Z_\delta(R)$, then there exists $r \in R$ with $xyr \in \delta(\{0\})$ but $r \notin \delta(\{0\})$. Now if $yr \in \delta(\{0\})$,then $y \in Z_\delta(R)$, and if $yr \notin \delta(\{0\})$, then $x \in Z_\delta(R)$. Hence $Z_\delta(R)$ is δ-primary hyperideal.■

Theorem 2.5: Let I be a hyperideal of a semihyperring R, and let δ be a global hyperideal expansion. Then I is δ-primary if and only if $Z_\delta(R/I) \subseteq \delta(\{0_{R/I}\})$.

Proof : Suppose that $\delta(I)$ is δ-primary, let $r_1 + I$ be an element of $Z_\delta(R/I)$, then $\exists\, r_2 + I \notin \delta(\{0_{R/I}\})$ such that $r_1 + I \odot r_2 + I = r_1r_2 + I \in \delta(\{0_{R/I}\})$. Since δ is global, then for the natural homomorphism $f: R \longrightarrow R/I$, which defined by $f(x) = x + R$ we have $f(I) = \{0_{R/I}\}$ and $\delta(\{0_{R/I}\}) = \delta(f(I)) = f(\delta(I)) = \delta(I)/I$. Then $r_1r_2 \in \delta(I), r_2 \notin \delta(I)$ so that $r_1 \in I$ and $r_1 + I \in \delta(\{0_{R/I}\})$. Conversely, Suppose that $Z_\delta(R/I) \subseteq \delta(\{0_{R/I}\})$, and let $x, y \in R$ such that $xy \in I, x \notin I$, then $x + I \odot y + I = xy + I = I$, and so $y + I \in Z_\delta(R/I)$ then $y + I \in \delta(\{0_{R/I}\})$ then $y \in \delta(I)$ and so I is δ-primary. ■

Definition 2.3: 1. Let R be a semihyperring with a hyperideal expansion δ, then R is called δ-semidomainlike semihyperring if $Z_\delta(R) \subseteq nil_\delta(R)$.

2. Let R be a hyperring with a hyperideal expansion δ, then R is called δ-domainlike hyperring if $Z_\delta(R) \subseteq nil_\delta(R)$.

Theorem 2.6: Let R be a semihyperring with a hyperideal expansion δ such that $\delta(\delta(I)) = \delta(I)$ for every hyperideal I in R, then

1. $\delta(\{0\})$ is δ-primary hyperideal if and only if $Z_\delta(R) = nil_\delta(R)$.
2. $\delta(\{0\})$ is δ-primary if and only if R is δ-semidomainlike semihyperring.
3. If R is δ-semidomainlike semihyperring then $Z_\delta(R)$ is the unique minimal δ-primary hyperideal.

Proof: 1. Suppose that $\delta(\{0\})$ is δ-primary hyperideal, by previous theorem we proved that $nil_\delta(R) \subseteq Z_\delta(R)$, so it is enough to prove that $Z_\delta(R) \subseteq nil_\delta(R)$. Let $x \in Z_\delta(R)$ then $\exists y \notin \delta(\{0\})$ such that $xy \in \delta(\{0\})$, and since $\delta(\{0\})$ is δ-primary hyperideal $y \notin \delta(\{0\})$ then $x \in \delta(\delta(\{0\})) = \delta(\{0\})$, and so $nil_\delta(R) \subseteq Z_\delta(R)$. Conversely, suppose that $Z_\delta(R) = nil_\delta(R)$, and let $xy \in \delta(\{0\})$, $y \notin \delta(\{0\})$, then $x \in Z_\delta(R) = nil_\delta(R)$, $x \in \delta(\{0\})$, therefore, $\delta(\{0\})$ is δ-primary. (2) follows from (1). To prove (3), Assume that R is δ-semidomainlike semihyperring, then by (1), (2) we have $Z_\delta(R) = nil_\delta(R)$, and since $\delta(\{0\})$ is δ-primary hyperideal then also $Z_\delta(R)$ is δ-primary hyperideal. Let J be any δ-primary hyperideal of R, since $\{0\} \subseteq J$, then $\delta(\{0\}) \subseteq \delta(J)$ and if $x \in Z_\delta(R)$, then $\exists y \notin \delta(\{0\})$, such that $xy \in \delta(\{0\}) \subseteq \delta(J)$ and since J is δ-primary, if $y \notin \delta(J)$, then $x \in J$, so that $Z_\delta(R) = nil_\delta(R) \subseteq J$. ■

Definition 2.4: 1. Let R be commutative semihyperring, let δ be a hyperideal expansion on R, then R is called δ-semidomain is $xy \in \delta(\{0\})$, then either $x \in \delta(\{0\})$ or $y \in \delta(\{0\})$.

2. 1. Let R be commutative hyperring, let δ be a hyperideal expansion on R, then R is called δ-domain is $xy \in \delta(\{0\})$, then either $x \in \delta(\{0\})$ or $y \in \delta(\{0\})$.

Theorem 2.7: Let R be a semihyperring R, and let δ be global expansion such that $\delta(\delta(I)) = \delta(I)$ for every hyperideal $I$of R. If $\delta(I)$ is δ-primary then R/I is δ-semidomainlike if and only if R/I is δ-semidomain.

Proof: Suppose that R/I is a δ-semidomainlike semihyperring, since δ is global hyperideal expansion, then $\delta(0_{R/I}) = \delta(I)/I$, now, let $a + I, b + I \in R/I$ such that $a + I \odot b + I = ab + I \in \delta(0_{R/I}) = \delta(I)/I$, then $ab \in \delta(I)$ and since $\delta(I)$ is δ-primary, then either $a \in \delta(I)$ or $b \in \delta(\delta(I)) = \delta(I)$. So that $a + I \in \delta(I)/I$ or $b + I \in \delta(I)/I$, therefore R/I δ-semidomain. Conversely, assume that R/I is δ-semidomain, then by previous theorem it is enough to prove that $Z_\delta(R/I) \subseteq nil_\delta(R/I)$, so that let $x + I \in Z_\delta(R/I)$, then $\exists y + I \notin \delta(0_{R/I}) = \delta(I)/I$ such that $x + I \odot y + I = xy + I \in \delta(0_{R/I})$. Since R/I is δ-semidomain, then $+I \in \delta(0_{R/I})$, and we have $Z_\delta(R/I) \subseteq nil_\delta(R/I)$. Hence R/I is δ-semidomainlike semihyperring.■

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