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Research Article

FAN-GOTTESMAN COMPACTIFICATIONS AND STONE SPACE

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ABSTRACT

A Stone space is a zero dimensional, compact and T_0 −topological space. Also it is called Boolean space. There are a lot of methods for obtainnig compact space. Aleksandrov, Stone-Cech, Wallman and Fan-Gottesman compactification are four of them. In this paper, it is discussed Fan-Gottesman compactification of regular space with normal base and necessary and sufficient conditions for Fan-Gottesman compactification of T_3 space to be a Stone space are given by considering relation between the Stone space and compact space.
Keywords: Stone space, Fan-Gottesman compactification.

1. INTRODUCTION

The notion of compactness or compactification is an important concept in general topology as well as in a branch of mathematics and quantum physic. The notions of the compact spaces and compactification in fuzzy topological spaces and general topological spaces are highly developed and are used extensively in many practical and engineering problems, computational topology for geometric design, computer-aided geometric design, engineering design research and mathematical sciences.

There are well known compactification methods such as Aleksandrov compactification, Stone-Cech compactification, Wallman compactification but Fan-Gottesman compactification methods is studied infrequently by mathematicians. In 1929, Aleksandrov proved that all local compact Hausdorff spaces may be completed to a compact Hausdorff by the addition of one point [1]. In 1937, Cech constructed the compactification of a space by using the diagonal product of all continuous functions [4], while Stone was using Boolean algebras and rings of continuous functions [8] to do this. Afterwards it is called that Stone-Cech compactification. In 1938, Henry Wallman introduced compactification of T_1 −spaces (A space X is a T_1 −space or Frechet space if it satisfies the T_1 axioms, i.e. for each $x, y \in X$ such that $x \neq y$ there is an open set $U \subset X$ so that $x \in U$ but $y \notin U$) having a normal base which is also called Wallman compactification [9]. In 1952, Ky Fan and Noel Gottesman defined a compactification method obtained similar to the Wallman compactification and afterwards called Fan-Gottesman compactification of regular spaces (A space X is a regular space if for each $x \in X$ and each closed $C \subset X$ such that $x \notin C$

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there are open sets $U, V \subset X$ so that $x \in U$, $C \subset V$ and $U \cap V = \emptyset$) with a normal base [7]. This compactification method was defined via ultra-open filter in [5].

A topological space X is called a zero dimensional if X has a basis consisting of clopen sets. In Bourbaki [3] the condition Hausdorff (A space X is a T_2 −space or Hausdorff space, for each $x, y \in X$ such that $x \neq y$ there are open sets $U, V \subset X$ so that $x \in U, y \in V$ and $U \cap V = \emptyset$) is added for a space X to be a zero dimensional space. A totally disconnected space is a topological space that is maximally disconnected, in the sense that it has no non-trivial connected subsets. Hence a topological space X is a totally disconnected space if and only if for any $x \neq y$ in X, there exists a clopen set U of X such that $x \in U$ and $y \notin U$. The concepts of zero-dimensionality and total disconnectedness are closely related. Indeed, every zero-dimensional T_1 −space is totally disconnected. A Stone space, also called a Boolean space, is a topological space that is zero-dimensional, T_0 −space (A space X is a T_0 −space if it satisfies the T_0 axioms, i.e. for each $x, y \in X$ such that $x \neq y$ there is an open set $U \subset X$ so that U contains one of x and y but not the other) and compact.

Our paper consist of four part. The first section of our paper is introduction.This section contains historical background of compactification methods and stone space. The second section of our paper contains some preliminaries about Fan-Gottesman compactification and Stone space. Also in second section of our paper contains some remarks and proposition of clopen sets on a compactification of a space. The third section of our paper contains main result. We obtained under which conditions Fan-Gotttesman compactification of a T_3 −space (A space X is a T_3 −space if it satisfies regular and T_1 −spaces axioms) is a Stone space. The last section of our paper is conclusion. Conclusion of the paper consist of our results and their contributions to science.

2. SOME PRELIMINARIES ABOUT FAN-GOTTESMAN COMPACTIFICATION AND STONE SPACE

Let X be a topological space. By a *compactification* of X, we shall mean a pair (C, e) where C is compact Hausdorff, $e: X \to C$ is an embedding and $e[X]$ is dense in C. In this case, we may simply refer to C as being a compactification of X. Two compactifications (C, e), and $(\acute{C}, \acute{e})$, are regarded as equivalent if there is a homeomorphism $h: C \to \acute{C}$ for which $e = \acute{e}h$.

Recall the Fan-Gottesman compactification is defined by Ky Fan and Noel Gottesman. In 1952, they constructed the compactification for a regular space X with a base β for the open sets, containing $\emptyset$ and satisfying the following conditions:

1. $A, B \in \beta$ implies $A \cap B \in \beta$
2. $A \in \beta$ implies $X - cl_X A \in \beta$, where $cl_X A$ is a closure of A in X.
3. For every open set U in X and every $A \in \beta$ such that $cl_X A \subset U$, there exists a set $B \in \beta$ such that $cl_X A \subset B \subset cl_X B \subset U$. Then β is called normal base.

The compactification associated with β is obtained as follows. It is considered that is a regular space having a normal base for open set, which satisfies the above three properties of normal base. A centered system is defined as a family of elements in β such that

$$\bigcap_{i=1}^{m} cl_X A_i \neq \emptyset$$

for every finite family $A_1, A_2, \dots, A_m$. Every centered system on β is contained in at least one maximal centered system on β from Zorn lemma. For $A \in \beta$, A^* is defined as the set of all maximal centered systems x^*such that $B \in x^*$ for some $B \in \beta$ where $cl_X B \subset A$. The set $X^*$of all maximal centered systems is equipped with a topology having as a base for its open sets the class $\beta^*$of all sets A^*, $A \in \beta$. X^* is compact and Hausdorff space with this topology.(X^*, β^*) is called Fan-Gottesman compactification of X [7].

We gived definition of this compactification via ultra-open filter in [5].

$$X^* = X \cup \{cl_X(G): G \in y, y \text{ is a nonconvergent ultra} - \text{open filter in } X\}$$

where "ultra-open" means maximal among all filters, having a base consisting of open sets.The sets

$$S(G) = G \cup \{cl_X(G): y \in X^* - X, cl_X(G) \in y\}$$

where G is open in X, constitute an open base of X^*.

We determined the Fan-Gottesman compactification via open ultrafilters.

Definition 2.1. Let X be a T_3 $-$space and FX the subcollection of all closed ultrafilters on X. For each open set $O \subset X$ define $O^* \subset FX$ to be the set

$$O^* = \{\hat{G} \in FX: cl_X O \in \hat{G}\}$$

Let ϕ is the $\{O^*: O \text{ is open subset of } X\}$ set. It is clear that ϕ is the base for open sets of topology on FX. FX is a compact space and the Fan-Gottesman compactifications of X.

In order to avoid the confusion between FX and (X^*, β^*), we will use FX when it regarded as Fan-Gottesman compactification of X.

On the other hand, for each closed set $D \subset X$, we define $D^* \subset FX$ by

$$D^* = \{\hat{G} \in FX: G \subseteq D \text{ for some } G \text{ in } \hat{G}\}$$

The following properties of FX are useful

a) If $U \subset X$ is open, then $FX - U^* = (FX - U)^*$
b) If $D \subset X$ is closed, then $FX - D^* = (FX - D)^*$
c) If U_1 and U_2 are open in X, then $(U_1 \cap U_2)^* = U_1^* \cap U_2^*$ and $(U_1 \cup U_2)^* = U_1^* \cup U_2^*$

Properties We consider the map $f_X: X \to FX$ be defined by $f_X(x) = \hat{G}_x$, the closed ultrafilter converging to x in X. Then the following properties hold.

1) If U is open in X, then $cl_{FX}(f_X(U)) = U^*$. In particular $f_X(X)$ is dense in FX.
2) f_X is continuous and it is an embedding of X in FX if and only if X is a T_3 $-$space.
3) If U_1 and U_2 are open subsets of X, then $cl_{FX}(f_X(U_1 \cap U_2)) = cl_{FX}(f_X(U_1)) \cap cl_{FX}(f_X(U_2))$.
4) FX is a compact T_2 $-$space.

Now we will give definition for recalling stone space.

A Stone space is a totally disconnected, compact topological space. Equivalently a Stone space is a zero dimensional, compact and T_0 $-$topological space. Also it is called Boolean space [3].

Al-Hajri and others get some results about open, closed and clopen sets on a compactification in [2]. Now,we will give these results.

Proposition 2.1. Let X be a topological space, $K(X)$ be a compactification of X, F be a closed set of $K(X)$ and O be a open set of $K(X)$. If $F \cap X = O \cap X$ then $O \subseteq F$.

Corollary 2.1. If C be a clopen set of X such that there exists two clopen sets H_1 and H_2 of $K(X)$ with $C = H_1 \cap X = H_2 \cap X$, then $H_1 = H_2$

Proposition 2.2. Let X be a topological space, $K(X)$ be a compactification of X and C is a clopen subset of X. If there exists a clopen set H of $K(X)$ such that $H \cap X = C$, then $H = cl_{K(X)}(C)$.

Proposition 2.3. Let X be a topological space, $K(X)$ be a compactification of X . C is a clopen subset of X if and only if $C = X \cap cl_{K(X)}(C) = X \cap int\left(cl_{K(X)}(C)\right)$.

Proposition 2.4. Let X be a topological space, $K(X)$ be a compactification of X . If $K(X)$ is a Stone space, then the following properties hold

1. X is totally disconnected
2. For each open set O of X and $x \in O$ there exists a clopen set C such that $x \in C \subseteq O$.

3. MAIN RESULTS

Proposition 3.1. Let X be a local compact Hausdorff space and C be a clopen subset of X. Then there exists a clopen set N of the Fan-Gottesman compactification FX such that $N \cap X = C$.

Proof: It is well known that studied spaces X is locally compact, $X - C$ or $cl_X C$ is compact for C all open subsets of X. If $cl_X C$ is compact, then $cl_X C$ is clopen set of FX. If $X - C$ is compact, then $X - C$ is clopen set of FX. We can choose $a^* = \{A \in \beta : X - A \text{ is compact}\}$ as a centered system. Then, it was shown that if X is a local compact Hausdorff space, then Fan-Gottesman compactification of X is homeomorfic to the Aleksandrov compactification of X in [6]. We can choose $N = C \cup \{\infty\}$. So N is clopen set of FX such that $N \cap X = C$.

Corollary: Let X be a local compact Hausdorff space. Then the Aleksandrov compactification of X is Stone space.

Theorem 3.1. Let X be a T_3 space. Then the Fan-Gottesman compactification FX is a Stone space if and only if for each disjoint closed sets D_1 and D_2 of X, there exists a clopen set C such that $D_1 \subseteq C$ and $D_2 \cap C = \emptyset$.

Proof: (⇒)if Q is a clopen set of FX, then there exists a clopen set C of X such that $Q = C^*$. In deed, let ϑ be a collection of open sets of X such that $Q = \bigcup\{V^* : V \in \vartheta\}$. Since Q is a closed set of FX, Q is a compact set of FX. Hence there exists a finite subcollection ϑ' of ϑ such that $Q = \bigcup\{V^* : V \in \vartheta'\}$. Thus $Q = C^*$ with $C = \bigcup\{V : V \in \vartheta'\}$. That C is a clopen set of X fellow at once from the fact that $C = Q \cap X$.

Let D_1 and D_2 are two disjoint closed sets of X. Then D_1^* and D_2^* are two disjoint closed sets of FX. Let $\mathcal{D}_1 \in D_1^*$ and $\mathcal{D}_2 \in D_2^*$ Since FX is a Stone space, there exists a clopen set C of X such that $\mathcal{D}_1 \in C^*$ and $\mathcal{D}_2 \notin C^*$. So, for each $\mathcal{D}_2 \in D_2^*$, there exists a collection ϑ of clopen set of X such that $D_1^* \subseteq \bigcup\{V^* : V \in \vartheta\}$ and $\mathcal{D}_2 \notin \bigcup\{V^* : V \in \vartheta\}$. For D_1^* is compact closed set of FX, there exists a finite subcollection ϑ' of ϑ such that $D_1^* \subseteq \bigcup\{V^* : V \in \vartheta'\}$ and $\mathcal{D}_2 \notin \bigcup\{V^* : V \in \vartheta'\}$. Set $F_{\mathcal{D}_2} = \bigcap\{FX - V^* : V \in \vartheta'\}$. Hence $F_{\mathcal{D}_2}$ is a clopen neighborhood of $\mathcal{D}_2$ such that $F_{\mathcal{D}_2} \cap D_1^* = \emptyset$. Set $\mathcal{F} = \left\{F_{\mathcal{D}_2} : \mathcal{D}_2 \in D_2^*\right\}$. For D_1^* is compact closed set of FX, there exists a finite subcollection $\mathcal{F}'$ of $\mathcal{F}$ such that $D_2^* \subseteq \bigcup\left\{F_{\mathcal{D}_2} : F_{\mathcal{D}_2} \in \mathcal{F}\right\}$. Thus

$Q = \bigcup\{F_{D_2} : F_{D_2} \in \mathcal{F}\}$ is a clopen set of FX such that ${D_2}^* \subseteq Q$ and ${D_1}^* \subseteq FX - Q$. Then there exists a a clopen set C of X such that $C^* = FX - Q$. Therefore $D_1 \subseteq C$ and $D_2 \cap C = \emptyset$.

($\Leftarrow$)$\mathcal{D}_1$ and $\mathcal{D}_2$ are two distinct element of FX. Then there exists closed sets D_1 and D_2 of X such that $D_1 \in \mathcal{D}_1$, $D_2 \in \mathcal{D}_2$ and $D_1 \cap D_2 = \emptyset$. Hence there exists a clopen set C of X such that $D_1 \subseteq C$ and $D_2 \cap C = \emptyset$. Thus ${D_1}^* \subseteq C^*$ and ${D_2}^* \cap C^* = \emptyset$. Since $D_1 \in \mathcal{D}_1$ and $D_2 \in \mathcal{D}_2$, $\mathcal{D}_1 \in {D_1}^*$and $\mathcal{D}_2 \in {D_2}^*$. So C^* is a clopen set of FX such that $\mathcal{D}_1 \in C^*$ and $\mathcal{D}_2 \in C^*$.Therefore FX is a Stone space.

4. CONCLUSION

In this paper we get following results

- ✓ Clopen sets are characterized by Fan-Gottesman compactification of local compact Hausdorff space.
- ✓ Necessary and sufficient conditions for Fan-Gottesman compactification of T_3 space are given to be a Stone space

It is know that there are a lot of applications of compactification not only mathematics but also other scientific fields. For example, quantum physic and computer-aided, geometric design are some of them. So, our results can be used by scientists working in this fields.

In future, it is aimed to get new application to image clasification problem by using properties of Stone space and Fan-Gottesman compactification. Also we will investigate the relation between obtained new results and other compactification methods

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