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Research Article

SOME SEQUENCE SPACES AND MATRIX TRANSFORMATIONS

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ABSTRACT

In this paper we characterize strongly lacunary invariant regular matrices and uniqueness of generalized limits and inclusion relations for some sequence spaces.

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1. INTRODUCTION

Let w be the set of all sequences real or complex and ℓ_∞ denote the Banach space of bounded sequences $x = \{x_n\}_{n=0}^{\infty}$ normed by $\|x\| = \sup_{n\geq 0}|x_n|$.

Let $\theta = (k_r)$ be the sequence of positive integers such that

i) $k_0 = 0$ and $0 < k_r < k_{r+1}$

ii) $h_r = (k_r - k_{r-1}) \to \infty$ as $r \to \infty$.

Then θ is called a lacunary sequence. The intervals determined by θ are denoted by $I = (k_r - k_{r-1}]$. The ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r (see, Freedman et al [1]).

Let σ be a one-to-one mapping of the set of positive integers into itself. A continuous linear functional φ on l_∞ is said to be an invariant mean or a σ - mean if and only if

1) $\varphi \geq 0$ when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n.

2) $\varphi(e) = 1$, where $e = (1,1,\ldots)$ and

3) $\varphi(x_{\sigma(n)}) = \varphi(x)$ for all $x \in l_\infty$.

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For a certain kinds of mapping σ every invariant mean φ extends the limit functional on space c, in the sen be that $\varphi(x)=\lim x$ for all $x\in c$. Consequently, $c\subset V_\sigma$ where V_σ is the bounded sequences all of whose σ-means are equal, (see, [14]).

If $x=(x_k)$, set $Tx=(Tx_k)=(x_{\sigma(k)})$ it can be shown that (see, Schaefer [10]) that

$$V_\sigma=\left\{x\in l_\infty : \lim t_{km}(x)=Le \text{ uniformly in m for some L} = \sigma\text{-}\lim x\right\} \tag{1.1}$$

where

$$t_{km}(x)=\frac{x_m+Tx_m+\ldots+T^k x_m}{k+1} \textbf{ and } t_{-1,m}=0$$

We say that a bounded sequence $x=(x_k)$ is σ-convergent if and only if $x\in V_\sigma$ such that $\sigma^k(n)\neq n$ for all $n\geq 0$, $k\geq 1$.

Just as the concept of almost convergence lead naturally to the concept of strong almost convergence, σ-convergence leads naturally to the concept of strong σ-convergence. A sequence $x=(x_k)$ is said to be strongly σ-convergent (see Mursaleen [7]) if there exists a number L such that

$$\frac{1}{k}\sum_{i=1}^{k}\left|x_{\sigma^i(m)}-L\right|\to 0, \tag{1.2}$$

as $k\to\infty$ uniformly in m. We write $[V_\sigma]$ as the set of all strong σ-convergent sequences.

When (1.2) holds we write $[V_\sigma]-\lim x=\mathrm{L}$. Taking $\sigma(m)=m+1$, we obtain $[V_\sigma]=[\hat{c}]$, which is the space of strong almost convergence. Note that

$$[V_\sigma]\subset V_\sigma\subset l_\infty.$$

The strongly summable sequences have been systematically investigated by Hamilton and Hill [2], Kuttner [3] and some others. The invariant summable sequences have been discussed by Schafer [14]. Mursaleen [8] have studied absolute invariant convergent and absolute invariant summable sequences. Further the strongly invariant summable sequences was studied by Saraswat and Gupta[9]. The spaces of strongly summable sequences were introduced and studied by Maddox [4,6]. Some works related to invariant summable sequences can be found in [10, 11, 12, 13].

Let $A=(a_{nk})$ be an infinite matrix of nonnegative real numbers and $p=(p_k)$ be a sequence such that $p_k>0$. (These assumptions are made throughout.) We write $Ax=\{A_n(x)\}$ if $A_n(x)=\sum_k a_{nk}\left|x_k\right|^{p_k}$ converges for each n. We write.

$$d_{rn}(x)=\frac{1}{h_r}\sum_{i\in I_r}A_{\sigma^n(i)}(x)=\sum_k a(n,k,r)\left|x_k\right|^{p_k}$$

where

$$a(n,k,r)=\frac{1}{h_r}\sum_{i\in I_r}a_{\sigma^n(i),k}.$$

If $\theta=2^r$

$$d_{rn}(x)=\frac{1}{h_r}\sum_{i\in I_r}A_{\sigma^n(i)}(x)=\sum_k a(n,k,r)\left|x_k\right|^{p_k}$$

and

$$a(n,k,r)=\frac{1}{h_r}\sum_{i\in I_r}a_{\sigma^n(i),k}$$

reduces to

$$t_{rn}(x)=\frac{1}{r+1}\sum_{i=0}^{m}A_{\sigma^n(i)}(x)=\sum_k a(n,k,r)\left|x_k\right|^{p_k}$$

where

$$a(n,k,r)=\frac{1}{r+1}\sum_{i=0}^{m}a_{\sigma^n(i),k}.$$

The following sequence spaces were defined in [6].

$$\left[A_{(\theta,\sigma)},p\right]_0=\{x\ :\ d_{rn}(x)\to 0 \text{ uniformly in } n\};$$

$$\left[A_{(\theta,\sigma)},p\right]=\{x\ :\ d_{rn}(x-le)\to 0 \text{ for some } l \text{ uniformly in } n\}$$

and

$$\left[A_{(\theta,\sigma)},p\right]_\infty=\left\{x\ :\ \sup_n d_{rn}(x)<\infty\right\}.$$

The sets $[A_{(\theta,\sigma)},p]_0$, $[A_{(\theta,\sigma)},p]$ and $[A_{(\theta,\sigma)},p]_\infty$ will be respectively called the spaces of strongly lacunary invariant summable to zero, strongly lacunary invariant summable and strongly lacunary invariant bounded sequences. If

($\theta=2^r$), the above spaces reduce to the following sequence spaces.

$$\left[A_\sigma,p\right]_0=\{x\ :\ t_{rn}(x)\to 0 \text{ uniformly in } n\};$$

$$\left[A_\sigma,p\right]=\{x\ :\ t_{rn}(x-le)\to 0 \text{ for some } l \text{ uniformly in } n\}$$

and

$$\left[A_\sigma,p\right]_\infty=\left\{x\ :\ \sup_n t_{rn}(x)<\infty\right\}.$$

If x is strongly lacunary invariant summable to l we write $x_k\to l[A_{(\theta,\sigma)},p]$. A pair (A,p) will be called strongly lacunary invariant regular if

$$x_k\to l\Rightarrow x_k\to l[A_{(\theta,\sigma)},p].$$

2. MATRIX TRANSFORMATIONS

Let X and Y be two nonempty subset of the space w of sequences. If $x = \{x_k\} \in X$ implies that $\left\{\sum_k a_{nk} x_k\right\} \in Y$, we say that A defines a (matrix) transformations from X into Y, and we write $A : X \to Y$. Let c_0 and $(N_{(\theta,\sigma)})_0$ respectively denote the linear spaces of null sequences and sequences lacunary invariant convergent to zero.

We now characterize the class of strongly lacunary invariant regular matrices.

Theorem 2.1. *Let* $0 < \phi \le p_k \le H < \infty$. *Then* (A, p) *is strongly lacunary invariant regular if and only if*

$$A \in (c_0, (N_{(\theta,\sigma)})_0),$$

where

$$(N_{(\theta,\sigma)})_0 = \left\{ x \; : \; \lim_{m\to\infty} \frac{1}{h_r} \sum_{i\in I_r} x_{\sigma^n}(i) = 0, \right\} \text{ uniformly in } n.$$

To prove Theorem 2.1 we need the following result.

Lemma 2.2 *(see Maddox [4]). If* $p_k, q_k > 0$, *then* $c_0(q) \subset c_0(p) \Leftrightarrow \liminf \frac{p_k}{q_k} > 0$

Proof. Necessity. Suppose that (A, p) is strongly lacunary invariant regular. Therefore

$$|x_k - l|^{1/p_k} \to 0 \Rightarrow \sum_k a(n,k,r)|x_{k-l}| \to 0$$

uniformly in n. $1/p_k \ge \frac{1}{H} > 0$, by Lemma 2.2,

$$x_k \to l \Rightarrow |x - l|^{1/p_k} \to 0.$$

Thus

$$x_k \to l \Rightarrow \sum_k a(n,k,r)(x_k - l) \to 0$$

uniformly in n and therefore $A \in (c_0, (N_{(\theta,\sigma)})_0)$.

Sufficiency. Since $p_k \ge \theta > 0$, by Lemma 2.2,

$$x_k \to l \Rightarrow |x_k - l|^{p_k} \to 0.$$

Again we have $A \in (c_0, (N_{(\theta,\sigma)})_0)$. Therefore $x_k \to l[A_{(\theta,\sigma)}, p]$ and this concludes the proof. Note that $p_k \le H$ is superfluous in the sufficient and $\phi \le p_k$ is superfluous in the necessity.

We next consider the uniqueness of generalized limits.

Theorem 2.3. Suppose that $A \in (c_0, (N_{(\theta,\sigma)})_0)$ and $p = \{p_k\}$ converges to a positive limit. Then $x = \{x_k\} \to l \Rightarrow x_k \to l[A_{(\theta,\sigma)}, p]$ uniquely if and only if

$$\sum_k a(n,k,r) \not\to 0 \text{ uniformly in } n \tag{2.1}$$

Proof Necessity. Suppose that $A \in (c_0, (N_{(\theta,\sigma)})_0)$ and $\{p_k\}$ be bounded. Let $x_k \to l$ imply that $x_k \to l[A_{(\theta,\sigma)}, p]$ uniquely. We have $e \to 1[A_{(\theta,\sigma)}, p]$. Therefore the condition (2.1) must hold. For, otherwise $e \to 0[A_{(\theta,\sigma)}, p]$ which contradicts the uniqueness of l.

Note that the restriction on $\{p_k\}$ (expect boundedness) is superfluous for the necessity.

Sufficiency. Suppose that the condition (2.1) holds and $A \in (c_0, (V_{(\theta,\sigma)})_0)$ and that $p_k \to \gamma > 0$. Further assume that $x_k \to l$ imply that $x_k \to l[A_{(\theta,\sigma)}, p]$ and $x_k \to l'[A_{(\theta,\sigma)}, p]$ where $|l - l'| = a > 0$. Then we get

$$\lim_{r\to\infty} \sum_k a(n,r,k) u_k = 0 \text{ (uniformly in } n) \tag{2.2}$$

where

$$u_k = |x_{k-l}|^{p_k} + |x_k - l'|^{p_k}$$

By the assumption we have $u_k \to a^{\gamma}$. Since $A \in (c_0, (N_{(\theta,\sigma)})_0), u_k \to a^{\gamma}$ implies that

$$\sum_k a(n,k,r) \left| u_k - a^{\gamma} \right| \to 0 \text{ (uniformly in } n). \tag{2.3}$$

But we have

$$a^r \sum_k (n,k,r) \le \sum_k a(n,k,r) u_k + \sum_k a(n,k,r) \left| u_k - a^{\gamma} \right| \tag{2.4}$$

Now by $(2.2), (2.3)$ and (2.4) it follows that

$$\lim_{r\to\infty} \sum_k a(n,r,k) = 0 \text{ (uniformly in } n).$$

Since this contradicts (2.1), we must have $l = l'$. This completes the proof.

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