ON INFINITE MATRICES AND SEQUENCE SPACES

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ABSTRACT

The purpose of this paper is to define the spaces \( \sigma_0 \) and \( \sigma \) by using de la Valée poussin and invariant mean. Furthermore we characterize certain matrices in \( \sigma_\infty \) which will up a gap in the existing literature.

Keywords: Infinite matrices, de la Vallée poussin, invariant mean, matrix transformations.

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1. INTRODUCTION

Let \( w \) denote the set of all real and complex sequences \( x = (x_k) \). By \( l_\infty \) and \( c \), we denote the Banach spaces of bounded and convergent sequences \( x = (x_k) \) normed by \( \| x \| = \sup_k |x_k| \), respectively. A linear functional \( L \) on \( l_\infty \) is said to be a Banach limit \([1]\) if it has the following properties:

1. \( L(x) \geq 0 \) if \( n \geq 0 \) (i.e. \( x_n \geq 0 \) for all \( n \)),
2. \( L(e) = 1 \) where \( e = (1,1,...) \),
3. \( L(Dx) = L(x) \), where the shift operator \( D \) is defined by \( D(x_n) = \{x_{n+1}\} \).

Let \( B \) be the set of all Banach limits on \( \ell_\infty \). A sequence \( x \in \ell_\infty \) is said to be almost convergent if all Banach limits of \( x \) coincide. Let \( \hat{c} \) denote the space of almost convergent sequences. Lorentz \([8]\) has shown that

\[ \hat{c} = \left\{ x \in \ell_\infty : \lim_{n} d_{m,n}(x) \text{ exists uniformly in } n \right\} \]
where
\[ d_{m,n}(x) = \frac{x_n + x_{n+1} + x_{n+2} + \cdots + x_{n+m}}{m+1}. \]

If \( p_k \) is real and \( p_k > 0 \), we define (see, Maddox [9])
\[ c_0(p) = \left\{ x : \lim_{k \to \infty} x_k^p_k = 0 \right\} \]
and
\[ c(p) = \left\{ x : \lim_{k \to \infty} x_k^p_k - l_p_k = 0, \text{for some } l \right\} \]

If \( p_m \) is real such that \( p_m > 0 \) and \( \sup p_m < \infty \), we define (see, Nanda [13])
\[ \hat{c}_0(p) = \left\{ x : \lim_{m \to \infty} d_{m,n}(x)^{p_m} = 0, \text{uniformly in } n \right\} \]
and
\[ \hat{c}(p) = \left\{ x : \lim_{m \to \infty} d_{m,n}(x) - le_l^{p_m} = 0, \text{for some } l, \text{uniformly in } n \right\}. \]

Shaefer [23] defined the \( \sigma \)-convergence as follows: Let \( \sigma \) be a one-to-one mapping from the set of natural numbers into itself. A continuous linear functional \( \phi \) on \( l_\infty \) is said to be an invariant mean or a \( \sigma \)-mean provided that

(i) \( \phi(x) \geq 0 \) when the sequence \( x = (x_k) \) is such that \( x_k \geq 0 \) for all \( k \),
(ii) \( \phi(e) = 1 \) where \( e = (1,1,1,\ldots) \), and
(iii) \( \phi(x) = \phi(x_{\sigma(k)}) \) for all \( x \in l_\infty \).

We denote by \( V_\sigma \) the space of \( \sigma \)-convergent sequences. It is known that \( x \in V_\sigma \) if and only if
\[ \frac{1}{m} \sum_{k=1}^{m} x_{\sigma^k(n)} \to \text{a limit} \]
as \( m \to \infty \), uniformly in \( n \). Here \( \sigma^k(n) \) denotes the \( k \) th iterate of the mapping \( \sigma \) at \( n \).

A \( \sigma \)-mean extends the limit functional on \( c \) in the sense that \( \phi(x) = \lim x \) for all \( x \in c \) if and only if \( \sigma \) has no finite orbits, that is to say, if and only if, for all \( n > 0, k \geq 1 \), \( \sigma^k(n) \neq n \).

In case \( \sigma \) is the translation mapping \( n \to n+1 \), a \( \sigma \)-mean reduces to the unique Banach limit and \( V_\sigma \) reduces to \( \hat{c} \).

In [23], Schaefer has defined the concept of \( \sigma \)-conservative, \( \sigma \)-regular and \( \sigma \)-coercive matrices and characterized matrix classes \( (c,V_\sigma) \), \( (c,V_\sigma)_{\text{reg}} \) and \( (l_\infty,V_\sigma) \) where \( V_\sigma \) denote
the set of all bound sequences all of whose invariant means (or \( \sigma \) – means) are equal. In [11], Mursaleen characterized the class \((c(p), V_\sigma), (c(p), V_\sigma)_{\text{reg}}\) and \((l_\infty(p), V_\sigma)\) matrices which generalized the results due to Schaefer [23].

Matrix transformations between sequence spaces have been discussed by Savas and Mursaleen [21], Basarir and Savas [2], Vatan [4], Vatan and Simsek [5], Savas ([14],[15], [16], [17], [18], [19],[20] ) and many others.

Recently, Khan and Rahman [3] studied the sequence space \(ces[(p_r), (q_r)]\) and investigated some properties. Let \((q_r)\) be positive sequence of real numbers for \(p = (p_j)\) with \(\inf p_j > 0\), we have

\[
ces[(p_r), (q_r)] = \left\{ x : \sum_{j=0}^{\infty} \left( \frac{1}{Q_{2j}} \sum_k q_k |x_k| \right)^{p_j} < \infty \right\}
\]

where

\[
Q_{2j} = q_{2j} + q_{2j+1} + q_{2j+2} + \ldots + q_{2j+1}
\]

and \(\sum_j\) denotes summation over the range \(2^j \leq k \leq 2^{j+1}\). It is easy to see that this space is paranormed space under the paranorm

\[
g(x) = \left( \sum_{j=0}^{\infty} \left( \frac{1}{Q_{2j}} \sum_k q_k |x_k| \right)^{p_j} \right)^{\frac{1}{M}}
\]

where

\[
H = \sup p_j < \infty \text{ and } M = \left(1, H\right).
\]

If we take \(q_r = 1\) for all \(r\), then the space \(ces[(p_r), (q_r)]\) reduces to the space \(ces(p_r)\) studied by [6]. Also, if \(p_r = p\) for all \(r\), then the space \(ces[(p_r), (q_r)]\) reduces to the space \(ces_p\) due to Lim [7].

It is easy to show that \(ces[(p_r), (q_r)]\) is complete with paranorm (1.1) and it has Köthe-Toeplitz dual \(ces^+[((p_r), (q_r))]\) defined by

\[
 ces^+[(p_r), (q_r)] = \left\{ \alpha = (a_k) : \sum_{j=0}^{\infty} \left( Q_{2j} \max_j \left( \frac{kq_k}{q_k} \right) \right)^{tj} B^{-t}j < \infty \text{ for some } B > 1 \right\}.
\]
It can be shown that $\text{ces}^+(\mathbf{(p_r),(q_r)})$ is isomorphic to $\text{ces}\mathbf{(p_r),(q_r)}$ which is the dual space of $\text{ces}\mathbf{(p_r),(q_r)}$, i.e., the space of all continuous linear functional on $\text{ces}\mathbf{(p_r),(q_r)}$.

We write the following inequality which will be used later. For any $B > 0$ and any two complex numbers $a$ and $b$, we have

$$|ab| \leq B \left( |a|^\frac{1}{t} B^{-t} + |b|^p \right)$$

(1.2)

where $p > 1$ and $\frac{1}{t} + \frac{1}{p} = 1$ (see, Maddox [9]).

2. $(\sigma, \lambda)$-CONVERGENCE

We define the following:

Let $\lambda = (\lambda_m)$ be a non-decreasing sequence of positive numbers tending to $\infty$ such that

$$\lambda_{m+1} \leq \lambda_m + 1, \lambda_1 = 1.$$ 

A sequence $x = (x_k)$ of real numbers is said to be $(\sigma, \lambda)$-convergent to a number $L$ if and only if $x \in V_\sigma^\lambda$, where

$$V_\sigma^\lambda = \{ x \in l_\infty : \lim_{m \to \infty} t_{nm}(x) = L \text{ uniformly in } n; L = (\sigma, \lambda) - \lim x \} ,$$

$$t_{nm}(x) = \frac{1}{\lambda_m} \sum_{i \in I_n} x \sigma^i(n) ,$$

and $I_n = [n - \lambda_n + 1, n]$ (see, [11]). Note that $c \subseteq V_\sigma^\lambda \subseteq l_\infty$. For $\sigma(n) = n + 1, V_\sigma^\lambda$ is reduced to the space $\hat{V}_\lambda$ of almost $\lambda$-convergent sequences and if we take $\sigma(n) = n + 1$ and $\lambda = n, V_\sigma^\lambda$ reduce to $\hat{c}$ (see, [8]).

It is quite natural to expect that the sequence $V_\lambda^\sigma$ and $V_0^\sigma$ can be extended to $V_\lambda^\sigma(p)$ and $V_0^\sigma(p)$ just as $\hat{c}$ and $\hat{c}_0$ were extended to $\hat{c}(p)$ and $\hat{c}_0(p)$ respectively.

The main object of this paper is to characterize matrix transformations between some sequence spaces. We first define the sequence spaces $V_\lambda^\sigma(p)$ and $V_0^\sigma(p)$ (the definitions are given below) and characterize certain matrices in $(V_\lambda^\sigma)_\infty$. 

If $p_m$ is real such that $p_m > 0$ and $\sup p_m < \infty$, we define
\[ V^\sigma_{\lambda_0}(p) = \left\{ x : \lim_{m \to \infty} \left| t_{m,n}(x) \right|^p_m = 0, \text{uniformly in } n \right\} \]

\[ V^\sigma_{\lambda}(p) = \left\{ x : \lim_{m \to \infty} \left| t_{m,n}(x - t_e) \right|^p_m = 0, \text{for some } t, \text{uniformly in } n \right\}. \]

and

\[ (V^\sigma_{\lambda})_\infty(p) = \left\{ x : \sup_{m,n} \left| t_{m,n}(x) \right|^p_m < \infty \right\}. \]

In particular, if \( p_m = p > 0 \) for all \( m \), we have \( V^\sigma_{\lambda_0}(p) = V^\sigma_{\lambda_0}, \quad V^\sigma_{\lambda}(p) = V^\sigma_{\lambda} \) and \( (V^\sigma_{\lambda})_\infty(p) = (V^\sigma_{\lambda})_\infty. \)

3. MAIN RESULTS

Let \( X \) and \( Y \) be two nonempty subsets of the space \( W \) of complex sequences. Let \( A = (a_{nk}) \), \( n, k = 1, 2, \ldots \) be an infinite matrix of complex numbers. We write \( A x = (A_n(x)) \) if \( A_n(x) = \sum k a_{nk} x_k \) converges for each \( n \). (Throughout \( \sum_k \) denotes summation over \( k \) from \( k = 1 \) to \( k = \infty \).) If \( x = (x_k) \in X \Rightarrow A x = (A_n(x)) \in Y \) we say that \( A \) defines a (matrix) transformation from \( X \) to \( Y \) and we denote it by \( A : X \to Y \). By \( (X,Y) \) we mean the class of matrices \( A \) such that \( A : X \to Y \).

We now characterize the matrices in the class \( \left( \text{ces } \left[ (p_r), (q_r) \right], (V^\sigma_{\lambda})_\infty \right) \). We write

\[ t_{m,n}(x) = t_{m,n}(Ax) = \sum_k a(m,n,k) x_k \]

where

\[ a(m,n,k) = \frac{1}{\lambda_m} \sum_{i \in I_n} a^i(n,k). \]

**Theorem 3.1** Let \( 1 < p_j < \sup_j p_j < \infty \) and \( \frac{1}{p_j} + \frac{1}{t_j} = 1 \) for \( j = 0, 1, 2, \ldots \) \( A \in \left( \text{ces } \left[ (p_r), (q_r) \right], (V^\sigma_{\lambda})_\infty \right) \) if and only if there exists an integer \( B > 1 \) such that

\[ W(B) = \sup_{m,n,j=0}^\infty \left( Q_{2j} A_j(m,n) \right)^{t_j} B^{-t_j} < \infty \]  

(3.1)
where \( A_j(r,n) = \max_j \left( \frac{a(m,n,k)}{q_k} \right) \) and for every \( m, \max_j \) means maximum over \( \left[ 2^j, 2^{j+1} \right] \).

**Proof.** Sufficiency: Suppose that there exists an integer \( B > 1 \) such that \( W(B) < \infty \). Then by inequality (1.2), we have

\[
\sum_{k=0}^{\infty} \left| a(m,n,k)x_k \right| = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \left| a(m,n,k)x_k \right|
\]

\[
\leq \sum_{j=0}^{\infty} Q_{2^j} \max_j \left( \frac{a(m,n,k)}{q_k} \right) \frac{1}{Q_{2^j}} \sum_{j=0}^{\infty} \left| x_k \right|
\]

\[
\leq B \left[ \sup_{m,n,j=0}^{\infty} \left( Q_{2^j} A_j(m,n) \right)^{t_j} \right] B^{-t_j} + \sum_{j=0}^{\infty} \left( \frac{1}{Q_{2^j}} \sum_{j=0}^{\infty} \left| x_k \right| \right)^{p_j}
\]

\[
\leq B \left[ \sup_{r,n,j=0}^{\infty} \left( Q_{2^j} A_j(m,n) \right)^{t_j} \right] B^{-t_j} + \sum_{j=0}^{\infty} \left( \frac{1}{Q_{2^j}} \sum_{j=0}^{\infty} \left| x_k \right| \right)^{p_j}
\]

Therefore \( A \in \left( ces \left( (p_r) \right), (q_r) \right), (V_{\sigma})_{\infty} \).

Necessity: Suppose that \( A \in \left( ces \left( (p_r) \right), (q_r) \right), (V_{\sigma})_{\infty} \) but

\[
\sup_{m,n,j=0}^{\infty} \left( Q_{2^j} A_j(m,n) \right)^{t_j} B^{-t_j} = \infty
\]

for all \( B > 1 \). Then \( \sum_{k=1}^{\infty} a(m,n,k)x_k \) converges uniform in \( n \) for all \( m \) and \( x \in ces \left( (p_r) \right), (q_r) \), hence \( a(r,n,k)_{k=1,2,...} \in ces^+ \left( (p_r) \right), (q_r) \) for all \( m \) and \( n \). It is easy to see that each \( t_{m,n} \) defined by \( t_{m,n}(x) = \sum_{k=1}^{\infty} a(m,n,k)x_k \) is an element of \( ces^+ \left( (p_r) \right), (q_r) \). Since \( ces \left( (p_r) \right), (q_r) \) is complete and since \( \sup_{m,n} \left| t_{m,n}(x) \right| < \infty \) on \( ces \left( (p_r) \right), (q_r) \), by the uniform boundedness principle, there exists a number \( L \) independent \( m,n,x \) and a number \( \delta > 1 \) such that

\[
\left| t_{m,n}(x) \right| < L
\]

(3.2)
for all \( n, m \) and \( x \in S \left[ 0, \delta \right] \) where \( S \left[ 0, \delta \right] \) is the closed sphere in \( \text{ces} \left[ \left( p_r \right), \left( q_r \right) \right] \)
with center at the origin 0 and radius \( \delta \). We now choose integer \( E > 1 \) such that \( E \delta^M > L \).

Since

\[
\sup_{m,n,j=0}^\infty \left( 2^j A_j \left( m,n \right) \right)^{t_j} E^{-t_j} = \infty,
\]

there exists \( m_0 > 1 \) such that

\[
R = \sum_{j=0}^{m_0} \left( 2^j A_j \left( m,n \right) \right)^{t_j} E^{-t_j} > 1
\]

Define a sequence

\[
x_k = 0 \text{ if } k \geq 2^{m_0+1}
\]

and

\[
x_N(j) = 2^j [\text{sup} \left( m,n, N \left( j \right) \right)]^{p_j} \left[ a \left( m,n, N \left( j \right) \right) \right]^{-t_j} E^{-t_j},
\]

\[
x_k = 0 \text{ if } 0 \leq j \leq m_0 \text{ and } k \neq N \left( j \right)
\]

where \( N \left( j \right) \) is the smallest integer such that

\[
\left| a \left( m,n, N \left( j \right) \right) \right| = \max_{j} \left( \left| a \left( m,n,k \right) \right| \right).
\]

So we get \( g \left( x \right) < \delta \) but \( t_{mn} \left( x \right) > L \), which contradicts by (3.2). This completes the proof. \( \square \)

By specializing the sequences \( \left( p_r \right) \) and \( \left( q_r \right) \) of the spaces \( \text{ces} \left[ \left( p_r \right), \left( q_r \right) \right] \) in Theorem 1. We get the spaces \( \text{ces} \left( p_r \right) \) and \( \text{ces} p \) defined by [6] and Lim [7].

We have

**Corollary 3.1** Let \( 1 < p_j < \sup p_j < \infty \). Then \( A \in \left( \text{ces} \left( p \right), \left( V_{\sigma}^2 \right)_{p_j} \right) \) if and only if there exists an integer \( B > 1 \) such that \( W \left( B \right) < \infty \), where

\[
W \left( B \right) = \sup \sum_{m,n,j=0}^\infty \left( 2^j A_j \left( m,n \right) \right)^{t_j} B^{-t_j} \text{ and } \frac{1}{p_j} + \frac{1}{t_j} = 1 \left( j = 0, 1, 2, \ldots \right).
\]

**Proof.** If we take \( q_r = 1 \) for every \( r \) in Theorem 1, then we obtain the result. \( \square \)
**Corollary 3.2** Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{t} = 1$. Then $A \in \left( \text{ces}_p \left( V_{\lambda}^\sigma \right) \right)$ if and only if

$$\sup_{m,n} \left( \sum_{j=0}^{\infty} \left( 2^j A_j (m,n) \right)^{t_j} \right)^{\frac{1}{t_j}} < \infty.$$ 

**Proof.** If we take $q_r = 1$ and $p_r = p$ for every $r$ in Theorem 1, then we obtain the proof of Corollary.

**Theorem 3.2** Let $1 < p_j < \sup p_j < \infty$ and

$$\frac{1}{p_j} + \frac{1}{t_j} = 1, (j = 1, 2, \ldots). A \in \left( \text{ces} \left( (p_r), (q_r) \right) \right), \left( V_{\lambda}^\sigma \right)$$. Then

(i) $\lim_{m \to \infty} a(m,n,k) = a_k$ uniformly in $n$ and for fixed $k$,

(ii) there exists $B > 1$ such that $W(B) < \infty$.

where

$$W(B) = \sup_{r,n} \left( \sum_{j=0}^{\infty} Q_{2^j A_j (m,n)}^{t_j} B^{-t_j} \right).$$

**Proof.** Suppose that $A \in \left( \text{ces} \left( (p_r), (q_r) \right) \right), \left( V_{\lambda}^\sigma \right)$. Then $t_{m,n}(x) = \sum_{k=1}^{\infty} a(m,n,k)x_k$ exists for every $m \geq 1$ and $\lim_{m \to \infty} t_{mn}(x)$ uniformly in $n$ exists for every $x \in \text{ces} \left( (p_r), (q_r) \right)$. Therefore by a similar argument to that in Theorem 1 we have the condition (i) is obtained by taking $x = e_k \in \text{ces} \left( (p_r), (q_r) \right)$, where $e_k$ is a sequence with 1 at the $k^{th}$ place and zero elsewhere.

**Sufficiency:** The conditions (i)-(ii) hold. From (i), we have

$$\sum_{j=0}^{\infty} \left( Q_{2^j A_j (m,n)}^{t_j} B^{-t_j} \right) \leq W(B) < \infty$$

(3.3)

By using (3.3) it is easy to check that $\sum_{k=1}^{\infty} a_k x_k$ is absolutely convergent for each $x \in \text{ces} \left( (p_r), (q_r) \right)$. Moreover for each $x \in \text{ces} \left( (p_r), (q_r) \right)$ and $\epsilon > 0$, we choose integer number $m_0 > 1$ such that
Define the matrix \( b(m,n,k) \) where 
\[
\left( b(m,n,k) \right)_{r=1}^{\infty} = \left( a(m,n,k) - a_k \right)
\]
for all \( n \). By the condition (ii) and inequality (1.2), we have, for all \( n \)
\[
\sum_{k=m_0+1}^{\infty} |b(m,n,k)x_k| \leq B \left[ \sum_{j=m_0}^{\infty} \left( Q_{2j}W_j(m,n) \right)^{t_j} B^{-t_j} + 1 \right] \left( g_{m_0}(x) \right) \frac{1}{M}
\]
where
\[
W_j(m,n) = \max_j \left( \frac{|a(m,n,k)|}{q_k} - a_k \right)
\]
By inequality above, we get
\[
\sum_{j=m_0}^{\infty} \left( Q_{2j}W_j(m,n) \right)^{t_j} B^{-t_j} \leq 2W(B) < \infty.
\]
Therefore
\[
\lim_{m \to \infty} \sum_{k=1}^{\infty} a(m,n,k)x_k = \sum_{k=1}^{\infty} a_k x_k \text{ uniformly in } n.
\]
This shows that \( A \in \left( ces \left( p_r \right), (q_r) \right), \left( V_{\lambda}^\sigma \right)_\infty \) which proves the Theorem. \( \square \)

**Corollary 3.3** Let \( 1 < p_j < \sup p_j < \infty \). Then \( A \in \left( ces(p), (V_{\lambda}^\sigma)_{\infty} \right) \) if and only if

(i) \( \lim_{r \to \infty} a(m,n,k) = a_k \) uniformly in \( n \) and for fixed \( k \),

(ii) there exists \( B > 1 \) such that \( W(B) < \infty \),

where
\[
W(B) = \sup_{r, \lambda} \sum_{j=0}^{\infty} \left( 2^j A_j(m,n) \right)^{t_j} B^{-t_j}
\]

**Proof.** If \( q_r = 1 \) for every \( r \) in Theorem 2, then we get the conditions (i)-(ii). \( \square \)

**Corollary 3.4** Let \( 1 < p < \infty \) and \( \frac{1}{p_j} + \frac{1}{t_j} = 1 \). Then \( A \in \left( ces(p), (V_{\lambda}^\sigma) \right) \) if and only if

(i) \( \lim_{r \to \infty} a(m,n,k) = a_k \) uniformly in \( n \) and for fixed \( k \),
(ii) \[ \sup_{r,n} \left( \sum_{j=0}^{\infty} \left( 2^j A_j (m,n) \right)^t - \frac{1}{j} \right) < \infty. \]

**Proof.** If \( q_r = 1 \) and \( p_r = p \) for all \( r \) in Theorem 2, then we get the proof of the corollary. \( \square \)

**Theorem 3.3** Let \( 1 < p_j < \sup p_j < \infty \) and \( \frac{1}{p_j} + \frac{1}{\tau_j} = 1, (j = 1, 2, \ldots) \). Then

\[ A \in \left[ \text{ces} \left( p_r, \{q_r\}, (V_0^\sigma) \right) \right] \text{ if and only if} \]

(i) \[ \lim_{r \to \infty} a(m,n,k) = 0 \text{ uniformly in } n \text{ and for fixed } k, \]

(ii) there exists \( B > 1 \) such that \( W(E) < \infty, \)

where

\[ W(B) = \sup_{r,n} \sum_{j=0}^{\infty} \left( Q_{2^j A_j (m,n)} B^{-t_j} \right). \]

**Proof.** Theorem 3 can be provided by using an argument similar to that in Theorem 2. \( \square \)

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