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Research Article

LACUNARY STATISTICAL CONVERGENCE OF COMPLEX UNCERTAIN SEQUENCE

Ömer KİŞİ*[1], Hatice Kübra ÜNAL[2]

[1]*Bartın University, Deparment of Mathematics, BARTIN;* ORCID: 0000-0001-6844-3092
[2]*Bartın University, Deparment of Mathematics, BARTIN;* ORCID: 0000-0002-8668-8330

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ABSTRACT

This study introduces the concepts of lacunary statistical convergence of complex uncertain sequences: lacunary statistical convergence almost surely (S_θ.a.s.), lacunary statistical convergence in measure, lacunary statistical convergence in mean, lacunary statistical convergence in distribution and lacunary statistically convergence uniformly almost surely (S_θ.u.a.s). In addition, decomposition theorems and relationships among them are discussed.
Keywords: Lacunary convergence, uncertainty theory, complex uncertain variable.

1. INTRODUCTION

Freedman and Sember [1] introduced the concept of lower asymptotic density and defined the concept of convergence in density. Taking this definition, we can give the definition of statistical convergence which has been formally introduced by Fast [2]. Schoenberg [3] reintroduced this concept independently. A number sequence (x_k) is statistically convergent to L provided that for every $\varepsilon > 0$, $d(\{k \in N: |x_k - L| \geq \varepsilon\}) = 0$ or equivalently there exists a subset $K \subset N$ with $d(K) = 1$ and $n_0(\varepsilon)$ such that $k > n_0(\varepsilon)$ and $k \in K$ imply that $|x_k - L| < \varepsilon$. In this case, we write $st - lim x_k = L$. From the definition, we can easily show that any convergent sequence is statistically convergent, but not conversely.

By a lacunary sequence we mean an increasing integer sequence $\theta = \{k_r\}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. Throughout this paper the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$.

The concept of lacunary statistical convergence was defined by Fridy and Orhan [4]. A sequence $x = (x_k)$ is said to be lacunary statistically convergent to the number L if for every $\varepsilon > 0$, $lim_r \frac{1}{h_r} |\{k \in I_r: |x_k - L| \geq \varepsilon\}| = 0$. In this case, we write $S_\theta - lim x_k = L$ or $x_k \to L(S_\theta)$.

However, in our daily life, we often encounter the case that there are lack of or no observed data about the events, not only for economic reasons or technical difficulties, but also for influence of unexpected events.

* Corresponding Author: e-mail: okisi@bartin.edu.tr, tel: (378) 501 10 00

In order to deal with belief d egree, an uncertainty theory was founded by Liu [5] and redefined by Liu [6] which based on an uncertain measure which satisfies normality, duality, subadditivity, and product axioms. Liu [5] first introduced convergence in measure, convergence in mean, convergence almost surely (a.s.) and convergence in distribution and their relationships were also discussed. Thereafter, a concept of uncertain variable was proposed to represent the uncertain quantity and a concept of uncertainty distribution to describe uncertain variables. Up to now, uncertainty theory has successfully been applied to uncertain programming (Liu [7], Liu and Chen [8]), uncertain risk analysis and uncertain reliability analysis (Liu [9]), uncertain logic (Liu [10]), uncertain differential equations (Yao and Chen [11]), uncertain graphs (Gao and Gao [12], Zhang and Peng [13]), uncertain finance (Chen [14], Liu [15]), etc.

In real life, uncertainty not only appears in real quantities but also in complex quantities. In order to model complex uncertain quantities, Peng [16] presented the concepts of complex uncertain variable and complex uncertainty distribution, and also the expected value was proposed to measure a complex uncertain variable. Since sequence convergence plays an important role in the fundamental theory of mathematics, there are also many convergence concepts in uncertainty theory. You [17] introduced another type of convergence named convergence uniformly almost surely and showed the relationships among those convergence concepts. Zhang [18] proved some theorems on the convergence of uncertain sequence. After that, Guo and Xu [19] gave the concept of convergence in mean square for uncertain Tripathy and Nath [20] introduced statistical convergence of complex uncertain sequences. A lot of developments have been made in this area after the various studies of researchers [21-28]. Lacunary convergence Inspired by these, we study the convergence concepts of lacunary statistically convergence of complex uncertain sequences and discuss the relationships among them in this study.

2. MAIN RESULTS

Definition 1. The complex uncertain sequence $\{\zeta_n\}$ is said to be lacunary statistically convergent almost surely $(S_\theta.a.s)$ to ζ if for every $\varepsilon > 0$ there exists an event Λ with $M(\Lambda) = 1$ such that

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{k\in I_r \ : \ \|\zeta_k(\gamma)-\zeta(\gamma)\|\geq\varepsilon\right\}\right|=0,$$

for every $\gamma \in \Lambda$. In this case, we write $\zeta_n \to \zeta \ (S_\theta.a.s)$.

Definition2. The complex uncertain sequence $\{\zeta_n\}$ is said to be lacunary statistically convergent in measure to ζ if

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{k\in I_r \ : \ M\left(\|\zeta_k-\zeta\|\geq\varepsilon\right)\geq\delta\right\}\right|=0,$$

for every $\varepsilon, \delta > 0$.

Definition3. The complex uncertain sequence $\{\zeta_n\}$ is said to be lacunary statistically convergent in mean to ζ if

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{k\in I_r \ : \ E\left(\|\zeta_k-\zeta\|\right)\geq\varepsilon\right\}\right|=0,$$

for every $\varepsilon > 0$.

Definition4. Let $\Phi, \Phi_1, \Phi_2, \ldots$ be the complex uncertainty distributions of complex uncertain variables $\zeta, \zeta_1, \zeta_2, \ldots$, respectively. We say the complex uncertain sequence $\{\zeta_n\}$ be lacunary statistically converges in distribution to ζ if for every $\varepsilon > 0$,

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{k\in I_r \ : \ \|\Phi_k(c)=\Phi(c)\|\geq\varepsilon\right\}\right|=0,$$

for all c at which $\Phi(c)$ is contin uous.

Definition5. The complex uncertain sequence $\{\zeta_n\}$ is said to be lacunary statistically convergent uniformly almost surely $(S_\theta.u.a.s)$ to ζ if for every $\varepsilon > 0$, $\exists\delta > 0$ and a sequence of events $\{E_k{}'\}$ such that

$$\lim_{r\to\infty} \tfrac{1}{h_r}\left|\left\{k \in I_r \ : \ \left|M\left(E_k^{'}\right) - 0\right| \geq \varepsilon\right\}\right| = 0$$

$$\Rightarrow \lim_{r\to\infty} \tfrac{1}{h_r}\left|\left\{k \in I_r \ : \ \left|\zeta_k(x) - \zeta(x)\right| \geq \delta\right\}\right| = 0.$$

Definition6. A complex uncertain sequence $\{\zeta_n\}$ is said to be lacunary statistically bounded or S_θ-bounded if there exists a real number $M>0$ such that

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{k \in I_r \ : \ \|\zeta_k\| \geq M\right\}\right| = 0,$$

i.e.,

$$\delta^\theta\left(k \in I_r \ : \ \|\zeta_k\| \geq M\right) = 0.$$

Definition7. A complex uncertain sequence $\{\zeta_n\}$ is said to be lacunary statistically convergent to ζ if for every $\varepsilon > 0$,

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{k \in I_r \ : \ \|\zeta_k(\gamma) - \zeta(\gamma)\| \geq \varepsilon\right\}\right| = 0,$$

for every $\gamma \in \Lambda$.

Now, we give the relationships among the convergence concepts of complex uncertain sequences.

Theorem 1. If the complex uncertain sequence $\{\zeta_n\}$ lacunary statistically converges in mean to ζ, then, $\{\zeta_n\}$ lacunary statistically converges in measure to ζ.

Proof. It follows from the Markov inequality that for any given $\varepsilon, \delta > 0$, we have

$$\lim_{r\to\infty} \tfrac{1}{h_r}\left|\left\{k \in I_r \ : \ M\left(\|\zeta_k - \zeta\| \geq \varepsilon\right) \geq \delta\right\}\right|$$

$$\leq \lim_{r\to\infty} \tfrac{1}{h_r}\left|\left\{k \in I_r \ : \ \left(\tfrac{E(\|\zeta_k - \zeta\|)}{\varepsilon}\right) \geq \delta\right\}\right|.$$

Thus, $\{\zeta_n\}$ lacunary statistically converges in measure to ζ and the theorem is proved.

Remark1. Converse of above theorem is not true. i.e., $\{\zeta_n\}$ lacunary statistical convergence in measure does not imply $\{\zeta_n\}$ lacunary statistical convergence in mean. Following example illustrates this.

Example1. Consider the uncertaintly space (Γ, L, M) to be γ_1, γ_2, ... with

$$M\{\Lambda\} = \begin{cases} \sup_{\gamma_n \in \Lambda} \frac{1}{n+1}, & \text{if } \sup_{\gamma_n \in \Lambda} \frac{1}{n+1} < 0.5, \\ 1 - \sup_{\gamma_n \in \Lambda^c} \frac{1}{n+1}, & \text{if } \sup_{\gamma_n \in \Lambda^c} \frac{1}{n+1} < 0.5, \\ 0.5, & \text{otherwise,} \end{cases}$$

and the complex uncertain variables be defined by

$$\zeta_n(\gamma) = \begin{cases} (n+1)i, & \text{if } \gamma = \gamma_n, \\ 0, & \text{otherwise.} \end{cases}$$

for $n=1,2,\ldots$ and $\zeta \equiv 0$. For some small numbers $\varepsilon, \delta > 0$ and $n \geq 2$, we have

$$\lim_{r\to\infty}\tfrac{1}{h_r}\left|\left\{k\in I_r\ :\ M\left(\|\zeta_k-\zeta\|\geq\varepsilon\right)\geq\delta\right\}\right|$$
$$=\lim_{r\to\infty}\tfrac{1}{h_r}\left|\left\{k\in I_r\ :\ M\left(\gamma\ :\ \|\zeta_k(\gamma)-\zeta(\gamma)\|\geq\varepsilon\right)\geq\delta\right\}\right|$$
$$=\lim_{r\to\infty}\tfrac{1}{h_r}\left|\left\{k\in \mathrm{N}:\ M\left\{\gamma_n\right\}\geq\delta\right\}\right|=0.$$

Thus, the sequence $\{\zeta_n\}$ lacunary statistically converges in measure to ζ.

However, for $n\geq 2$, we have the uncertainty distribution of uncertain variable $\|\xi_n-\xi\|=\|\xi_n\|$.

$$\Phi_n(x)=\begin{cases}0, & \text{if } x<0,\\ 1-\frac{1}{n+1}, & \text{if } 0\leq x<n+1,\\ 1, & x\geq n+1.\end{cases}$$

Hence, for each $n\geq 2$, we have

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{k\in I_r\ :\ E\left(\|\zeta_n-\zeta\|-1\right)\right\}\right|=\left[\int_0^{n+1}1-\left(1-\frac{1}{n+1}\right)dx\right]-1=0,$$

which is impossible. That is, the sequence $\{\zeta_n\}$ does not lacunary statistically converges in mean to ζ.

Theorem2. Assume complex uncertain sequence $\{\zeta_n\}$ with real part $\{\xi_n\}$ and imaginary part $\{\gamma_n\}$, respectively, for *n=1,2,…*. If uncertain sequences $\{\xi_n\}$ and $\{\gamma_n\}$ lacunary statistically convergent in measure to ξ and γ, respectively, then, complex uncertain sequence $\{\zeta_n\}$ lacunary statistically convergent in measure to $\zeta=\xi+i\gamma$.

Proof. It follows from the definition of lacunary statistically convergence in measure of uncertain sequence that for any small numbers $\varepsilon,\delta>0$,

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{k\in I_r\ :\ M\left(\|\xi_k-\xi\|\geq\frac{\varepsilon}{\sqrt{2}}\right)\geq\delta\right\}\right|=0$$

and

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{k\in I_r\ :\ M\left(\|\gamma_k-\gamma\|\geq\frac{\varepsilon}{\sqrt{2}}\right)\geq\delta\right\}\right|=0.$$

Note that

$$\|\zeta_n-\zeta\|=\sqrt{|\xi_n-\xi|^2+|\gamma_n-\gamma|^2}.$$

Thus, we have

$$\left\{\|\zeta_n-\zeta\|\geq\varepsilon\right\}\subset\left\{\|\xi_n-\xi\|\geq\frac{\varepsilon}{\sqrt{2}}\cup\|\gamma_n-\gamma\|\geq\frac{\varepsilon}{\sqrt{2}}\right\}.$$

Using the subadditivity axiom of uncertain measure, we obtain

$$\lim_{r\to\infty}\tfrac{1}{h_r}\left|\left\{k\in I_r\ :\ M\left(\|\zeta_k-\zeta\|\geq\varepsilon\right)\geq\delta\right\}\right|$$
$$\leq\lim_{r\to\infty}\tfrac{1}{h_r}\left|\left\{k\in I_r\ :\ M\left(\|\xi_k-\xi\|\geq\tfrac{\varepsilon}{\sqrt{2}}\right)\geq\delta\right\}\right|$$
$$+\lim_{r\to\infty}\tfrac{1}{h_r}\left|\left\{k\in I_r\ :\ M\left(\|\gamma_k-\gamma\|\geq\tfrac{\varepsilon}{\sqrt{2}}\right)\geq\delta\right\}\right|=0.$$

Hence, we have

$$\lim_{r\to\infty}\tfrac{1}{h_r}\left|\left\{k\in I_r\ :\ M\left(\|\zeta_k-\zeta\|\geq\varepsilon\right)\geq\delta\right\}\right|=0.$$

That is, $\{\zeta_n\}$ lacunary statistically converges in measure to ζ.

Theorem3. Assume complex uncertain sequence $\{\zeta_n\}$ with real part $\{\xi_n\}$ and imaginary part $\{\gamma_n\}$, respectively, for *n=1,2,…*. If uncertain sequences $\{\xi_n\}$ and $\{\gamma_n\}$ lacunary statistically convergent in measure to ξ and γ, respectively, then, complex uncertain sequence $\{\zeta_n\}$ lacunary statistically convergent in distribution to $\zeta = \xi + i\gamma$.

Proof. Let $c = a + ib$ be a given continuity point of the complex uncertainty distribution Φ. On the other hand, for any $\alpha > a, \beta > b$, we have

$$\begin{aligned}\{\xi_n \le a, \gamma_n \le b\} &= \{\xi_n \le a, \gamma_n \le b, \xi \le \alpha, \gamma \le \beta\} \cup \{\xi_n \le a, \gamma_n \le b, \xi > \alpha, \gamma > \beta\} \\ &\cup \{\xi_n \le a, \gamma_n \le b, \xi \le \alpha, \gamma > \beta\} \cup \{\xi_n \le a, \gamma_n \le b, \xi > \alpha, \gamma \le \beta\} \\ &\subset \{\xi \le a, \gamma \le b\} \cup \{|\xi_n - \xi| \ge \alpha - a\} \cup \{|\gamma_n - \gamma| \ge \beta - b\}.\end{aligned}$$

It follows from the subadditivity axiom that

$$\Phi_n(c) = \Phi_n(a+ib) \le \Phi(\alpha + i\beta) + M\{|\xi_n - \xi| \ge \alpha - a\} + M\{|\gamma_n - \gamma| \ge \beta - b\}.$$

Since $\{\xi_n\}$ and $\{\gamma_n\}$ lacunary statistically convergent in measure to ξ and γ, respectively, hence, for any small numbers $\varepsilon > 0$, we have

$$\lim_{r\to\infty} \frac{1}{h_r} \left|\left\{k \in I_r \ : \ M\left(\|\xi_k - \xi\| \ge \alpha - a\right) \ge \varepsilon\right\}\right| = 0$$

and

$$\lim_{r\to\infty} \frac{1}{h_r} \left|\left\{k \in I_r \ : \ M\left(\|\xi_k - \xi\| \ge \beta - b\right) \ge \varepsilon\right\}\right| = 0.$$

Thus, we obtain

$$\lim\sup_{n\to\infty} \Phi_n(c) \le \Phi(\alpha + i\beta)$$

for any $\alpha > a, \beta > b$. Letting $\alpha + i\beta \to a + ib$, we get

$$\limsup_{n\to\infty} \Phi_n(c) \le \Phi(c). \qquad (1)$$

On the other hand, for any $x < a, y < b$ we have

$$\begin{aligned}\{\xi \le x, \gamma \le y\} &= \{\xi_n \le a, \gamma_n \le b, \xi \le x, \gamma \le y\} \cup \{\xi_n \le a, \gamma_n \le b, \xi \le x, \gamma \le y\} \\ &\cup \{\xi_n > a, \gamma_n \le b, \xi \le x, \gamma \le y\} \cup \{\xi_n > a, \gamma_n > b, \xi \le x, \gamma \le y\} \\ &\subset \{\xi_n \le a, \gamma_n \le b\} \cup \{|\xi_n - \xi| \ge a - x\} \cup \{|\gamma_n - \gamma| \ge b - y\}.\end{aligned}$$

This implies,

$$\Phi(x + iy) \le \Phi_n(a + ib) + M\{\|\xi_n - \xi\| \ge a - x\} + M\{\|\gamma_n - \gamma\| \ge b - y\}.$$

Since

$$\lim_{r\to\infty} \frac{1}{h_r} \left|\left\{k \in I_r \ : \ M\left(\|\xi_k - \xi\| \ge a - x\right) \ge \varepsilon\right\}\right| = 0$$

and

$$\lim_{r\to\infty} \frac{1}{h_r} \left|\left\{k \in I_r \ : \ M\left(\|\xi_k - \xi\| \ge b - y\right) \ge \varepsilon\right\}\right| = 0,$$

we obtain

$$\Phi(x + iy) \le \liminf_{n\to\infty} \Phi_n(a + ib)$$

for any $x < a, y < b$. Taking $x + iy \to a + ib$, we get

$$\Phi(c) \le \liminf_{n\to\infty} \Phi_n(c) \qquad (2)$$

It follows from *(1)* and *(2)* that $\Phi_n(c) \to \Phi(c)$ as $n \to \infty$. That is the complex uncertain sequence $\{\zeta_n\}$ is lacunary statistically convergent in distribution to $\zeta = \xi + i\gamma$.

Remark2. Converse of the above theorem is not necessarily true. *i.e.* lacunary statistically convergence in distribution does not imply lacunary statistically convergence in measure. Following example illustrates this.

Example2. Consider the uncertaintly space (Γ, L, M) to be $\{\gamma_1, \gamma_2\}$ with $M(\gamma_1) = M(\gamma_2) = \frac{1}{2}$. We define a complex uncertain variable as

$$\zeta(\gamma) = \begin{cases} i, & \text{if } \gamma = \gamma_1, \\ -i, & \text{if } \gamma = \gamma_2. \end{cases}$$

We also define $\zeta_n = -\zeta$ for *n=1,2,…*. Then, ζ_n and ζ have the same distribution

$$\Phi_n(c) = \Phi_n(a+ib) = \begin{cases} 0, & \text{if } a < 0, -\infty < b < +\infty, \\ 0, & \text{if } a \geq 0, b < -1, \\ \frac{1}{2}, & \text{if } a \geq 0, -1 \leq b < 1, \\ 1, & \text{if } a \geq 0, b \geq 1. \end{cases}$$

Then, $\{\zeta_n\}$ is lacunary statistically in distribution to ζ. However, for a given *ε>0,* we have

$$\lim_{r\to\infty} \tfrac{1}{h_r} \left| \left\{ k \in I_r \ : \ M\left(\|\xi_k - \xi\| \geq \varepsilon \right) \geq 1 \right\} \right|$$
$$= \lim_{r\to\infty} \tfrac{1}{h_r} \left| \left\{ k \in I_r \ : \ M\left(\gamma \ : \ \|\xi_k(\gamma) - \xi(\gamma)\| \geq \varepsilon \right) \geq 1 \right\} \right| = 0.$$

That is the sequence $\{\zeta_n\}$ does not lacunary statistically convergence in measure to ζ. *By Theorem 3,* the real part and imaginary part of $\{\zeta_n\}$ also not lacunary statistically convergent in measure. In addition, since $\zeta_n = -\zeta$ *for n=1,2,…*, the sequence $\{\zeta_n\}$ does not is lacunary statistically convergence a.s to ζ. This completes the proof.

Lacunary statistically convergence a.s. does not imply is lacunary statistically convergence in measure.

Example3. Consider the uncertaintly space (Γ, L, M) to be $\gamma_1, \gamma_2, \ldots$ with

$$M\{\Lambda\} = \begin{cases} \sup_{\gamma_n \in \Lambda} \frac{n}{(2n+1)}, & \text{if } \sup_{\gamma_n \in \Lambda} \frac{n}{(2n+1)} < 0.5, \\ 1 - \sup_{\gamma_n \in \Lambda^c} \frac{n}{(2n+1)}, & \text{if } \sup_{\gamma_n \in \Lambda^c} \frac{n}{(2n+1)} < 0.5, \\ 0.5, & \text{otherwise.} \end{cases}$$

and we define a complex uncertain variable as

$$\zeta_n(\gamma) = \begin{cases} in, & \text{if } \gamma = \gamma_n, \\ 0, & \text{otherwise.} \end{cases}$$

for *n=1,2,…* and $\zeta \equiv 0$. Then, the sequence $\{\zeta_n\}$ lacunary statistically converges *a.s. to* ζ. However for some small numbers $\varepsilon > 0$, we have

$$\lim_{r\to\infty} \tfrac{1}{h_r} \left| \left\{ k \in I_r \ : \ M\left(\|\zeta_k - \zeta\| \geq \varepsilon \right) \geq \tfrac{1}{2} \right\} \right|$$
$$= \lim_{r\to\infty} \tfrac{1}{h_r} \left| \left\{ k \in I_r \ : \ M\left(\gamma \ : \ \|\zeta_k(\gamma) - \zeta(\gamma)\| \geq \varepsilon \right) \geq \tfrac{1}{2} \right\} \right|$$
$$= \lim_{r\to\infty} \tfrac{1}{h_r} \left| \left\{ k \in \mathrm{N} : \ M\{\gamma_n\} \geq \tfrac{1}{2} \right\} \right| = 0,$$

That is the sequence $\{\zeta_n\}$ does not lacunary statistically converge in measure to ζ.

Remark3. Lacunary statistically convergence in measure does not imply lacunary statistically convergence *a.s.*

Example4. Consider the uncertaintly space (Γ, L, M) to be $[0,1]$ with Borel algebra and Lebesque measure. For any positive integer n, there is an integer p such that $n = 2^p + k$, where k is an integer between *0* and $2^p - 1$. Then, we define a complex uncertain variable by

$$\zeta_n(\gamma) = \begin{cases} i, & \text{if } \frac{k}{2^p} \le \gamma \le \frac{k+1}{2^p}, \\ 0, & \text{otherwise.} \end{cases}$$

for *n=1,2,…*and $\zeta \equiv 0$. For some small numbers $\varepsilon > 0$ and $n \ge 2$, we have

$$\begin{aligned} &\lim_{r\to\infty} \tfrac{1}{h_r}\left|\left\{k \in I_r \,:\, M\left(\|\zeta_k - \zeta\| \ge \varepsilon\right) \ge \tfrac{1}{2}\right\}\right| \\ &\quad = \lim_{r\to\infty} \tfrac{1}{h_r}\left|\left\{k \in I_r \,:\, M\left(\gamma \,:\, \|\zeta_k(\gamma) - \zeta(\gamma)\| \ge \varepsilon\right) \ge \tfrac{1}{2}\right\}\right| \\ &\quad = \lim_{r\to\infty} \tfrac{1}{h_r}\left|\left\{k \in \mathrm{N} : M\{\gamma_n\} \ge \tfrac{1}{2}\right\}\right| = 0, \end{aligned}$$

Thus, the sequence $\{\zeta_n\}$ lacunary statistically converges in measure to ζ. In addition for every $\varepsilon > 0$, we have

$$\lim_{r\to\infty} \frac{1}{h_r}\left|\left\{k \in I_r \,:\, E\left(\|\zeta_k - \zeta\|\right) \ge \varepsilon\right\}\right| = 0$$

Hence, the sequence $\{\zeta_n\}$ also lacunary statistically converges in mean to ζ. However, for any $\gamma \in [0,1]$, there is an infinite number of intervals of the form $\left[\frac{k}{2^p}, \frac{k+1}{2^p}\right]$ containing γ. Thus, $\zeta_n(\gamma)$ does not lacunary statistically converge to *0*. In other words, the sequence $\{\zeta_n\}$ does not lacunary statistically converge *a.s.* to ζ. This completes the proof.

Lacunary statistically convergence *a.s.* does not imply lacunary statistically convergence in mean.

Example5. Consider the uncertaintly space (Γ, L, M) to be $\{\gamma_1, \gamma_2\}$ with

$$M\{\Lambda\} = \sum_{\gamma_n \in \Lambda} \frac{1}{3^n}.$$

The complex uncertain variables are defined by

$$\zeta_n(\gamma) = \begin{cases} i3^n, & \text{if } \gamma = \gamma_n, \\ 0, & \text{otherwise.} \end{cases}$$

for *n=1,2,…*and $\zeta \equiv 0$. Then, the sequence $\{\zeta_n\}$ lacunary statistically converges *a.s.* to ζ. However, the uncertaintly distributions of $\|\zeta_n\|$ are

$$\Phi_n(x) = \begin{cases} 0, & \text{if } x < 0, \\ 1 - \frac{1}{3^n}, & \text{if } 0 \le x < 3^n, \\ 1, & \text{if } x \ge 3^n, \end{cases}$$

for *n=1,2,…,* respectively. Then, we have

$$\lim_{r\to\infty} \frac{1}{h_r}\left|\left\{k \in I_r \,:\, E\left(\|\zeta_k - \zeta\|\right) \ge 1\right\}\right| = 0.$$

Therefore, the sequence $\{\zeta_n\}$ does not lacunary statistically converge in mean to ζ.

From the example *5*, we can obtain that lacunary statistically convergence in mean does not imply lacunary statistically converge *a.s.*

Proposition1. Let $\zeta, \zeta_1, \zeta_2, \ldots$ be complex uncertain variables. Then, $\{\zeta_n\}$ lacunary statistical converges *a.s* to ζ if and only if for any $\varepsilon, \delta > 0$, we have

$$\lim_{r\to\infty} \frac{1}{h_r}\left|\left\{k \in I_r \,:\, M\left(\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} \|\zeta_k - \zeta\| \ge \varepsilon\right) \ge \delta\right\}\right| = 0.$$

Proof. By the definition of lacunary statistical converges *a.s*, we have that there exists an event Λ with $M(\Lambda) = 1$ such that

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{k\in I_r\ :\ \|\zeta_k-\zeta\|\geq\varepsilon\right\}\right|=0$$

for every $\varepsilon > 0$. Then, for any $\varepsilon > 0$, there exists a number *k* such that $\|\xi_n - \xi\| < \varepsilon$ where *n*>*k* and for any $\gamma \in \Lambda$, that is equivalent to

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{k\in I_r\ :\ M\left(\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}\|\zeta_k-\zeta\|<\varepsilon\right)\geq 1\right\}\right|=0.$$

It follows from the duality axiom of uncertain measure that

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{k\in I_r\ :\ M\left(\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}\|\zeta_k-\zeta\|\geq\varepsilon\right)\geq\delta\right\}\right|=0.$$

Proposition2. Let $\zeta, \zeta_1, \zeta_2, \ldots$ be complex uncertain variables. Then, $\{\zeta_n\}$ lacunary statistical converges uniformly *a.s* to ζ if and only if for any $\varepsilon, \delta > 0$, we have

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{k\in I_r\ :\ M\left(\bigcup_{n=k}^{\infty}\|\zeta_k-\zeta\|\geq\varepsilon\right)\geq\delta\right\}\right|=0.$$

Proof If $\{\zeta_n\}$ lacunary statistical converges uniformly *a.s* to ζ, then, for any $\vartheta > 0$ there exists a number K such that $M\{K\} < \vartheta$ and $\{\zeta_n\}$ lacunary statistical converges to ζ on $\Gamma - K$. Thus, for any $\varepsilon > 0$, there exists a number *k*>*0* such that $\|\xi_n - \xi\| < \varepsilon$; where *n*>*k* and for any $\gamma \in \Gamma - K$. That is

$$\bigcup_{n=k}^{\infty}\left\{\|\zeta_n-\zeta\|\geq\varepsilon\right\}\subset K.$$

It follows from the subadditivity axiom that

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{k\in I_r\ :\ M\left(\bigcup_{n=k}^{\infty}\|\zeta_n-\zeta\|\geq\varepsilon\right)\right\}\right|\leq\delta^{\theta}\left(M\{K\}\right)<\delta.$$

Then,

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{k\in I_r\ :\ M\left(\bigcup_{n=k}^{\infty}\|\zeta_n-\zeta\|\geq\varepsilon\right)\geq\delta\right\}\right|=0.$$

On the contrary, if

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{k\in I_r\ :\ M\left(\bigcup_{n=k}^{\infty}\|\zeta_k-\zeta\|\geq\varepsilon\right)\geq\delta\right\}\right|=0,$$

for any $\varepsilon, \delta > 0$, then, for given $\delta > 0$ and $m \geq 1$, there exists m_k such that

$$\delta^{\theta}\left(M\left(\bigcup_{n=m_k}^{\infty}\|\zeta_k-\zeta\|\geq\frac{1}{m}\right)\right)<\frac{\delta}{2^m}.$$

Let

$$K=\bigcup_{m=1}^{\infty}\bigcup_{n=m_k}^{\infty}\left\{\|\zeta_n-\zeta\|\geq\tfrac{1}{m}\right\}.$$

Then,

$$\delta^{\theta}\left(M\{K\}\right)\leq\sum_{m=1}^{\infty}\delta^{\theta}\left(M\left(\bigcup_{n=m_k}^{\infty}\left\{\|\zeta_n-\zeta\|\geq\frac{1}{m}\right\}\right)\right)\leq\sum_{m=1}^{\infty}\frac{\delta}{2^m}.$$

In addition, we get

$$\sup_{\gamma\in\Gamma-K}\|\zeta_n-\zeta\|<\frac{1}{m}$$

for any *m=1,2,3,…* and $n > m_k$. The proposition is thus proved.

Theorem4 If the complex uncertain sequence $\{\zeta_n\}$ lacunary statistical converges uniformly *a.s* to ζ, then, $\{\zeta_n\}$ lacunary statistical converges to ζ.

Proof. It follows from above Proposition that $\{\zeta_n\}$ lacunary statistical converges uniformly *a.s* to ζ, then,

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{k\in I_r\ :\ M\left(\bigcup_{n=k}^{\infty}\|\zeta_n-\zeta\|\geq\varepsilon\right)\geq\delta\right\}\right|=0.$$

Since

$$\delta^{\theta}\left(M\left(\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}\|\zeta_n-\zeta\|\geq\varepsilon\right)\right)\leq\delta^{\theta}\left(M\left(\bigcup_{n=k}^{\infty}\|\zeta_n-\zeta\|\geq\varepsilon\right)\right)$$

taking the limit as $n\to\infty$ on both side of above inequality, we obtain

$$\delta^{\theta}\left(M\left(\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}\|\zeta_n-\zeta\|\geq\varepsilon\right)\right)=0.$$

By the first proposition, $\{\zeta_n\}$ lacunary statistical converges to ζ.

3. CONCLUSION

In this paper, we give lacunary statistically convergence of complex uncertain sequence. In further studies, the lacunary *I*-statistically convergence by using double sequences can be defined and examined for complex uncertain sequence.

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