Sigma J Eng & Nat Sci 10 (3), 2019, 363-368

Publications Prepared for the Sigma Journal of Engineering and Natural Sciences
2019 International Conference on Applied Analysis and Mathematical Modeling
Special Issue was published by reviewing extended papers

Research Article

ON SOME BETA-FRACTIONAL INTEGRAL INEQUALITIES

Deniz UÇAR[1], Veysel Fuat HATİPOĞLU*[2]

[1] *Usak University, USAK;* ORCID: 0000-0002-5709-1535
[2] *Mugla Sıtkı Koçman University, MUGLA;* ORCID: 0000-0003-2156-8781

Received: 10.10.2019 Accepted: 22.11.2019

ABSTRACT

In this paper, we obtain some new integral inequalities using beta-fractional integrals in the case of two synchronous functions. For this purpose we state and prove several theorems. Our results are pioneer for the literature of integral inequalities in beta-fractional integral sense.
Keywords: Integral inequalities, beta-fractional integral.

1. INTRODUCTION

Integral inequality based on fractional derivatives is a rising trend in mathematics. There are several different fractional integral and derivative definitions. In 2014 [1], authors proposed a relatively new fractional derivative definition so called beta-derivative which is the modified version of ∝-derivative defined by [2]. Some articles have been focused on analytical or numerical solutions of the fractional differential equations involving beta-fractional derivative [3, 4, 5, 6, 7, 8]. Also interested reader can obtain more information on fractional integral inequalities from recent articles [9, 10, 11]. Next, we give the definition of beta-fractional derivative.

Definition 1.1. [3] Let f be a function, such that, $f:[\alpha,\infty)\to\mathbb{R}$. Then the beta-derivative is defined as:

$$D_x^{\beta}(f(x)) = \lim_{\epsilon\to 0} \frac{f\left(x+\epsilon\left(x+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\right)-f(x)}{\epsilon}$$

for all $x \geq \alpha, \beta \in (0,1]$. Then if the limit of the above exists, f is says to be beta-differentiable.

Some useful properties of this definition [3] are as follows: Assuming that $g \neq 0$ and f are two functions β-differentiable with $\beta \in (0,1]$ then, the following relations can be satisfied

$$ {}_{0}^{A}\mathcal{D}_{t}^{\beta}\left(a f(t)\right) + bg(t)\right) = \alpha\, {}_{0}^{A}\mathcal{D}_{t}^{\beta}\left(f(t)\right) + b\, {}_{0}^{A}\mathcal{D}_{t}^{\beta}\left(g(t)\right), \tag{1.1}$$

for all α and b are real numbers.

$$ {}_{0}^{A}\mathcal{D}_{t}^{\beta}(c) = 0, \tag{1.2}$$

* Corresponding Author: e-mail: veyselfuat.hatipoglu@mu.edu.tr, tel: (252) 211 54 07

for c any given constant.

$${}_{0}^{\mathrm{A}}\mathcal{D}_{t}^{\beta}\big(f(\mathrm{t})+g(\mathrm{t})\big)=g(t)\,{}_{0}^{\mathrm{A}}\mathcal{D}_{t}^{\beta}\big(f(t)\big)+f(t)\,{}_{0}^{\mathrm{A}}\mathcal{D}_{t}^{\beta}\big(g(t)\big). \tag{1.3}$$

$${}_{0}^{\mathrm{A}}\mathcal{D}_{t}^{\beta}\left(\frac{f(t)}{g(t)}\right)=\frac{g(t)\,{}_{0}^{\mathrm{A}}\mathcal{D}_{t}^{\beta}\big(f(t)\big)-f(t)\,{}_{0}^{\mathrm{A}}\mathcal{D}_{t}^{\beta}\big(g(t)\big)}{g^{2}(t)}\ . \tag{1.4}$$

Now, we can give the definition of beta-fractional integral.

Definition 1.2. [3] Let $f:[\alpha,b]\to\mathbb{R}$ be a continuous function on the opened interval (a,b), then the beta-integral of f is given as:

$${}_{0}^{A}I_{t}^{\beta}\big(f(t)\big)=\int_0^t\left(x+\frac{1}{\Gamma(\beta)}\right)^{\beta-1}\ f(x)dx.$$

for all $x\le a,\beta\in(0,1]$. Then if the limit of the above exists, f is says to be beta-differentiable.

2. MAIN RESULTS

In this section we present our results using β-fractional integrals.

Theorem 2.1. *Let f and g be two synchronous functions on* $[0,\infty)$. *Then the following inequality holds*

$${}_{0}^{A}I_{t}^{\beta}(fg)(t)\ge\frac{1}{{}_{0}^{A}I_{t}^{\beta}\ (1)}\,{}_{0}^{A}I_{t}^{\beta}(f)(t)\,{}_{0}^{A}I_{t}^{\beta}(g)(t) \tag{2.1}$$

for all $t\ge 0,\beta>0$.

Proof Since f and g are two synchronous functions on $[0,\infty)$, we have

$$\big(f(\tau)-f(\rho)\big)\big(g(\tau)-g(\rho)\big)\ge 0$$

and

$$f(\tau)g(\tau)+f(\rho)g(\rho)\ge\ f(\tau)g(\rho)+f(\rho)g(\tau) \tag{2.2}$$

for all $\tau\le 0,\rho\le 0$. If we multiply both sides of the inequality (2.2) by $\tau+\frac{1}{\Gamma(\beta)}^{\beta-1}$ and integrate with respect to τ from 0 to t, we obtain

$$\int_0^t\left(\tau+\frac{1}{\Gamma(\beta)}\right)^{\beta-1}\big(f(\tau)g(\tau)\big)\,d\tau+\int_0^t\left(\tau+\frac{1}{\Gamma(\beta)}\right)^{\beta-1}\ \big(f(\rho)g(\rho)\big)d\rho\ge$$
$$\int_0^t\left(\tau+\frac{1}{\Gamma(\beta)}\right)^{\beta-1}f(\tau)g(\rho)d\tau+\int_0^t\left(\tau+\frac{1}{\Gamma(\beta)}\right)^{\beta-1}\ f(\rho)g(\sigma)\,d\tau.$$

Using the following equality

$$\int_0^t f(\tau)d_\beta\tau=\int_0^t f(\tau)\left(\tau+\frac{1}{\Gamma(\beta)}\right)^{\beta-1}d\tau$$

we have

$${}_{0}^{A}I_{t}^{\beta}\big(f(t)g(t)\big)+f(\rho)g(\rho)\,{}_{0}^{A}I_{t}^{\beta}(1)\ge g(\rho)\,{}_{0}^{A}I_{t}^{\beta}\big(f(t)+f(\rho)\big)\,{}_{0}^{A}I_{t}^{\beta}\big(g(t)\big). \tag{2.3}$$

Multiplying both sides of the inequality (2.3) by $\rho+\frac{1}{\Gamma(\beta)}^{\beta-1}$ and integrating with respect to ρ on $[0,t]$, we have

$$\left(\rho+\frac{1}{\Gamma(\beta)}\right)^{\beta-1}{}_{0}^{A}I_{t}^{\beta}(fg)(t)+\left(\rho+\frac{1}{\Gamma(\beta)}\right)^{\beta-1}f(\rho)g(\rho)\,{}_{0}^{A}I_{t}^{\beta}(1)\ \ge$$
$$\left(\rho+\frac{1}{\Gamma(\beta)}\right)^{\beta-1}{}_{0}^{A}I_{t}^{\beta}(f)(t)g(\rho)+\left(\rho+\frac{1}{\Gamma(\beta)}\right)^{\beta-1}f(\rho)\,{}_{0}^{A}I_{t}^{\beta}(g)(t)$$

then

$${}_0^A I_t^\beta(fg)(t)\int_0^t\left(\rho+\frac{1}{\Gamma(\beta)}\right)^{\beta-1}d\rho+{}_0^A I_t^\beta(1)\int_0^t\left(\rho+\frac{1}{\Gamma(\beta)}\right)^{\beta-1}f(\rho)g(\rho)\,d\rho\geq{}_0^A I_t^\beta(f)(t)\int_0^t\left(\rho+\frac{1}{\Gamma(\beta)}\right)^{\beta-1}d\rho+{}_0^A I_t^\beta(g)(t)\left(\rho+\frac{1}{\Gamma(\beta)}\right)^{\beta-1}f(\rho)d\rho$$

$${}_0^A I_t^\beta(fg)(t)\,{}_0^A I_t^\beta(1)+{}_0^A I_t^\beta(1)\,{}_0^A I_t^\beta(fg)(t)\geq{}_0^A I_t^\beta(f)(t)\,{}_0^A I_t^\beta(g)(t)+{}_0^A I_t^\beta(g)(t)\,{}_0^A I_t^\beta(f)(t)$$

$$2\,{}_0^A I_t^\beta(fg)(t)\,{}_0^A I_t^\beta(1)\geq 2\,{}_0^A I_t^\beta(f)(t)\,{}_0^A I_t^\beta(g)(t)$$

$${}_0^A I_t^\beta(fg)(t)\geq\frac{1}{{}_0^A I_t^\beta(1)}\,{}_0^A I_t^\beta(f)(t)\,{}_0^A I_t^\beta(g)(t).$$

This completes the proof.

Theorem 2.2. *Let f and g be two synchronous functions on $[0,1)$. Then we have the following inequality*

$${}_0^A I_t^\beta(fg)(t)\,{}_0^A I_t^\beta(1)+{}_0^A I_t^\beta(1)\,{}_0^A I_t^\beta(fg)(t)\geq{}_0^A I_t^\beta(f)(t)\,{}_0^A I_t^\beta(g)(t)+{}_0^A I_t^\beta(f)(t)\,{}_0^A I_t^\beta(g)(t) \tag{2.4}$$

for all $t>0,\alpha>0,$ and $\beta>0$.

Proof Using the same way in the proof of Theorem 2.1, we can obtain (2.3). Multiplying both sides of the inequality (2.3) by $\rho+\frac{1}{\Gamma(\alpha)}^{\alpha-1}$ and integrating with respect to ρ from 0 to t we have

$${}_0^A I_t^\beta(fg)(t)\int_0^t\left(\rho+\frac{1}{\Gamma(\alpha)}\right)^{\alpha-1}d\rho+$$

$${}_0^A I_t^\beta(1)\int_0^t\left(\rho+\frac{1}{\Gamma(\alpha)}\right)^{\alpha-1}f(\rho)g(\rho)d\rho\geq$$

$${}_0^A I_t^\beta(f)(t)\int_0^t\left(\rho+\frac{1}{\Gamma(\alpha)}\right)^{\alpha-1}g(\rho)d\rho+{}_0^A I_t^\beta(g)(t)\int_0^t\left(\rho+\frac{1}{\Gamma(\alpha)}\right)^{\alpha-1}f(\rho)d\rho$$

$${}_0^A I_t^\beta(fg)(t)\,{}_0^A I_t^\alpha(1)+{}_0^A I_t^\beta(1)\,{}_0^A I_t^\alpha(fg)(t)\geq{}_0^A I_t^\alpha(f)(t)\,{}_0^A I_t^\beta(g)(t)+{}_0^A I_t^\beta(f)(t)\,{}_0^A I_t^\alpha(g)(t)$$

$$2\,{}_0^A I_t^\beta(fg)(t)\,{}_0^A I_t^\beta(1)\geq 2\,{}_0^A I_t^\beta(f)(t)\,{}_0^A I_t^\beta(g)(t)$$

$${}_0^A I_t^\beta(fg)(t)\geq\frac{1}{{}_0^A I_t^\beta(1)}\,{}_0^A I_t^\beta(f)(t)\,{}_0^A I_t^\beta(g)(t)$$

and this ends the proof.

Remark 2.1. If the functions f and g are asynchronous on $[0,\infty)$, then the inequalities (2.2) and (2.3) are reversed.

Theorem 2.3. *We assume the functions f_i for $i=1,2,\dots,n$ are positive increasing functions on $[0,\infty)$. Then the following inequality holds*

$${}_0^A I_t^\beta\left(\prod_{i=1}^n f_i\right)(t)\geq\frac{1}{\left({}_0^A I_t^\beta(1)\right)^{n-1}}\prod_{i=1}^n\left({}_0^A I_t^\beta(f_i)\right)(t) \tag{2.5}$$

for any $t>0,\ \beta>0$.

Proof We use induction method to prove the theorem. We can easily see that for n = 1, we have

$${}_0^A I_t^\beta f_1(t)\geq{}_0^A I_t^\beta f_1(t) \tag{2.6}$$

for all $t>0,\ \beta>0$. For $n=2$, we use Theorem 2.1 and we have

$${}_0^A I_t^\beta(f_1f_2)(t)\geq\frac{1}{{}_0^A I_t^\beta(1)}\,{}_0^A I_t^\beta(f_1)(t)\,{}_0^A I_t^\beta(f_2)(t). \tag{2.7}$$

We suppose that

$${}_0^A I_t^\beta \left(\prod_{i=1}^{n-1} f_i\right)(t) \geq \frac{1}{\left({}_0^A I_t^\beta (1)\right)^{n-2}} \left(\prod_{i=1}^{n-1} {}_0^A I_t^\beta (f_i)\right)(t) \tag{2.8}$$

holds. Since the functions f_i, $i = 1,2,\ldots,n$, are positive increasing functions, then $\left(\prod_{i=1}^{n-1} f_i\right)(t)$ is also an increasing function. If we choose $g(\mathrm{t}) \coloneqq \left(\prod_{i=1}^{n-1} f_i\right)(t)$, $f_n = f$ and use theorem 2.1, we obtain

$${}_0^A I_t^\beta \left(\prod_{i=1}^{n} f_i\right)(t) = {}_0^A I_t^\beta (gf)(t) \geq \frac{1}{{}_0^A I_t^\beta (1)} {}_0^A I_t^\beta (g)(t) {}_0^A I_t^\beta (f)(t). \tag{2.9}$$

Using the inequality (2.8) we have

$${}_0^A I_t^\beta \left(\prod_{i=1}^{n} f_i\right)(t) \geq \frac{1}{{}_0^A I_t^\beta (1)} \frac{1}{\left({}_0^A I_t^\beta (1)\right)^{n-2}} \left(\prod_{i=1}^{n-1} {}_0^A I_t^\beta (f_i)\right)(t) {}_0^A I_t^\beta (f_n)(t) \tag{2.10}$$

and this ends the proof.

Theorem 2.4. *Let f and g are two functions defined on $[0,\infty)$ such that f is increasing, g is differentiable. Assume that there exists a real number $m \coloneqq inf_{t\geq 0}\, \mathrm{g}'(t)$. Then for all $t > 0$ and $\beta > 0$, following inequality holds*

$${}_0^A I_t^\beta (fg)(t) \geq \frac{1}{{}_0^A I_t^\beta (1)} {}_0^A I_t^\beta \big(f(t)\big) {}_0^A I_t^\beta \big(g(t)\big) - \frac{m}{{}_0^A I_t^\beta (1)} {}_0^A I_t^\beta \big(f(t)\big) {}_0^A I_t^\beta (t) + m\, {}_0^A I_t^\beta \big(tf(t)\big) \tag{2.11}$$

Proof We define the function *h(t) := g(t) - mt*. It is easy to see that the function h is increasing and differentiable on $[0,\infty)$. Using Theorem 2.1, we have

$${}_0^A I_t^\beta \big((g(t) - mt)f(t)\big) \geq \frac{1}{{}_0^A I_t^\beta (1)} {}_0^A I_t^\beta \big(g(t) - m(t)\big) {}_0^A I_t^\beta \big(f(t)\big)$$

$${}_0^A I_t^\beta (fg)(t) - m\, {}_0^A I_t^\beta \big(tf(t)\big) \geq \frac{1}{{}_0^A I_t^\beta (1)} {}_0^A I_t^\beta \big(f(t)\big) \left[{}_0^A I_t^\beta \big(g(t)\big) - m\, {}_0^A I_t^\beta (t)\right]$$

$${}_0^A I_t^\beta (fg)(t) \geq \frac{1}{{}_0^A I_t^\beta (1)} {}_0^A I_t^\beta \big(f(t)\big) {}_0^A I_t^\beta \big(g(t)\big) - \frac{m}{{}_0^A I_t^\beta (1)} {}_0^A I_t^\beta \big(f(t)\big) {}_0^A I_t^\beta (t) + m\, {}_0^A I_t^\beta \big(tf(t)\big).$$

This concludes the prof.

Theorem 2.5. *Let the functions f and g are defined on $[0,\infty)$. We assume the function f is decreasing and g is differentiable. If there exists a real number M :=$sup_{t\geq 0}$ g' (t) Then we have*

$${}_0^A I_t^\beta (fg)(t) \geq \frac{1}{{}_0^A I_t^\beta (1)} {}_0^A I_t^\beta (f)(t) {}_0^A I_t^\beta (g)(t) - \frac{M}{{}_0^A I_t^\beta (1)} {}_0^A I_t^\beta (f)(t) {}_0^A I_t^\beta t \tag{2.12}$$

for all $t > 0, \beta > 0$.

Proof We define $G(t) \coloneqq g(t) - Mt$. Since the function G is differentiable and decreasing on $[0,\infty)$, using Theorem 2.1 we have

$${}_0^A I_t^\beta (fG)(t) = {}_0^A I_t^\beta \big(f(t)(g(t) - Mt)\big) \geq \frac{1}{{}_0^A I_t^\beta (1)} \left[{}_0^A I_t^\beta \big(f(t)\big) {}_0^A I_t^\beta (g(t) - Mt)\right] \geq$$
$$\frac{1}{{}_0^A I_t^\beta (1)} {}_0^A I_t^\beta (f)(t) {}_0^A I_t^\beta (g)(t) - \frac{M}{{}_0^A I_t^\beta (1)} {}_0^A I_t^\beta (f)(t) {}_0^A I_t^\beta t.$$

Theorem 2.6. *Let the functions f and g are differentiable and there exists $m_1 : inf_{t\geq 0}\, g'(t)$. Then for all $t > 0$, $\beta > 0$ we have*

$$ {}_{0}^{A}I_{t}^{\beta}\left[\left(f(t)-m_1(t)\right)\left(g(t)-m_2(t)\right)\right] \geq \frac{1}{{}_{0}^{A}I_{t}^{\beta}(1)}\left[{}_{0}^{A}I_{t}^{\beta}\left(f(t)g(t)\right)-m_2\,{}_{0}^{A}I_{t}^{\beta}\left(tf(t)\right)-m_1\,{}_{0}^{A}I_{t}^{\beta}\left(tg(t)\right)+m_1m_2\,{}_{0}^{A}I_{t}^{\beta}\left(t^2\right)\right]. \qquad (2.13) $$

Proof We consider the functions $F(t)=f(t)-m_1 t\ and\ G(t):=g(t)-m_2 t.$ It is clear that the functions $F(t)$ and $G(t)$ are increasing on $[0,\infty)$. Using Theorem 2.1, we have

$$ {}_{0}^{A}I_{t}^{\beta}[(f(t)-m_1 t)(g(t)-m_2 t)] = {}_{0}^{A}I_{t}^{\beta}\left(f(t)g(t)\right)-m_2\,{}_{0}^{A}I_{t}^{\beta}(f(t).t)-m_1\,{}_{0}^{A}I_{t}^{\beta}\left(t.g(t)\right)+m_1m_2\left({}_{0}^{A}I_{t}^{\beta}(t)\right)^2 $$

$$ \geq \frac{1}{{}_{0}^{A}I_{t}^{\beta}(1)}\,{}_{0}^{A}I_{t}^{\beta}(f(t))\,{}_{0}^{A}I_{t}^{\beta}\left(g(t)\right)-\frac{m_2}{{}_{0}^{A}I_{t}^{\beta}(1)}\,{}_{0}^{A}I_{t}^{\beta}\left(f(t)\right){}_{0}^{A}I_{t}^{\beta}(t)-\frac{m_1}{{}_{0}^{A}I_{t}^{\beta}(1)}\,{}_{0}^{A}I_{t}^{\beta}(t)\,{}_{0}^{A}I_{t}^{\beta}\left(g(t)\right)+\frac{m_1m_2}{{}_{0}^{A}I_{t}^{\beta}(1)}\left({}_{0}^{A}I_{t}^{\beta}(t)\right)^2 $$

$$ \geq \frac{1}{{}_{0}^{A}I_{t}^{\beta}(1)}\left[{}_{0}^{A}I_{t}^{\beta}\left(f(t)\right){}_{0}^{A}I_{t}^{\beta}\left(g(t)\right)-m_2\,{}_{0}^{A}I_{t}^{\beta}\left(f(t)\right){}_{0}^{A}I_{t}^{\beta}(t)-m_1\,{}_{0}^{A}I_{t}^{\beta}(t)\,{}_{0}^{A}I_{t}^{\beta}\left(g(t)\right)+m_1m_2\left({}_{0}^{A}I_{t}^{\beta}(t)\right)^2\right]. $$

3. CONCLUSION

Fractional derivatives are an attraction point for several researchers. In this paper, we consider beta-fractional derivative. By using beta-fractional integrals, some new integral inequalities established in the case of two synchronous functions. As a main contribution to the literature, we prove six theorems. Our results are pioneer for the literature of integral inequalities in beta-fractional integral sense.

Acknowledgement

This paper is an extended version of a paper presented at International Conference on Computational Methods in Applied Sciences, including complete proofs.

REFERENCES

[1] Atangana, A., Doungmo Goufo, E. F. (2014). Extension of matched asymptotic method to fractional boundary layers problems, *Mathematical Problems in Engineering*, 2014.

[2] Khalil, R., Al Horani, M., Yousef, A. and Sababeh, M., (2014) A new definition of fractional derivative, *Journal of Computational and Applied Mathematics* 264, 65-70.

[3] Atangana, A. (2015) *Derivative with a new parameter: Theory, methods and applications,* Academic Press.

[4] Atangana, A., Alqahtani, R. (2016) Modelling the spread of river blindness disease via the caputo fractional derivative and the beta-derivative, *Entropy*, 18(2), 40.

[5] Atangana, A. (2017) *Fractional operators with constant and variable order with application to geohydrology*. Academic Press.

[6] Atangana, A., & Alkahtani, B. S. T., (2016) Modeling the spread of Rubella disease using the concept of with local derivative with fractional parameter: Beta-Derivative. *Complexity*, 21(6), 442-451.

[7] Hatipoğlu, V. F., (2019) A numerical algorithm for the solution of nonlinear fractional differential equations via beta-derivatives. *Matematical Methods in the Applied Sciences,* 42(16), 5258– 5265.

[8] Yépez-Martínez, H., Gómez-Aguilar, J. F., and Atangana, A., (2018) First integral method for non-linear differential equations with conformable derivative, *Mathematical Modelling of Natural Phenomena*, 13(1), 14.
[9] Sarikaya, M. Z., Set, E., Yaldiz, H., Basak, N., (2013) Hermite–Hadamard's inequalities for fractional integrals and related fractional inequalities, *Mathematical and Computer Modelling*, 57(9-10), 2403-2407.
[10] Dragomir, S. S., Bhatti, M. I., Iqbal, M., Muddassar, M., (2015) Some new Hermite-Hadamard's type fractional integral inequalities, *Journal of Computational Analysis & Applications*, 18(4), 655-661.
[11] Sarikaya, M. Z., Dahmani, Z., Kiris, M. E., and Ahmad, F., (2016) (k, s)-Riemann-Liouville fractional integral and applications, *Hacettepe Journal of Mathematics and Statistics*, 45(1), 77-89.