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Research Article

ON SOME NEW I-CONVERGENT DOUBLE SEQUENCE SPACES OF INVARIANT MEANS DEFINED BY IDEAL AND MODULUS FUNCTION

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ABSTRACT

The sequence space BV_σ was introduced and studied by Mursaleen [Houston J. Math. 9, 505-509 (1983; Zbl 0542.40003)]. The main aim of this paper is to study some new double sequence spaces of invariant means defined by ideal and modulus function. Furthermore, we also study several properties relevant to topological structures and inclusion relations between these spaces.

Keywords: Bounded variation, invariant mean, σ-Bounded variation, ideal, filter, Modulus function, I-Convergence, I-null, solid space, sequence algebra, symmetric space, convergence free space.

1. INTRODUCTION

Let $\mathbb{N}, \mathbb{R}, \mathbb{C}$ be the sets of all natural, real, and complex numbers respectively. We denote

$$\omega = \{x = (x_k) : x_k \in \mathbb{R} \text{ or } \mathbb{C}\},$$

showing the space of all real or complex double sequences, and

$$2\omega = \{x = (x_{ij}) : x_{ij} \in \mathbb{R} \text{ or } \mathbb{C}\},$$

showing the space of all real or complex double sequences.

Definition.1.1: A double sequence of complex numbers is defined as a function $X: \mathbb{N} \times \mathbb{N} \to \mathbb{C}$. We denote a double sequence as (x_{ij}) where the two subscripts run through the sequence of natural numbers independent of each other [9]. A number $a \in \mathbb{C}$ is called double limit of a double sequence (x_{ij}) if for every $\epsilon > 0$ there exists some $N = N(\epsilon) \in \mathbb{C}$ some such that,

$$|x_{ij} - a| < \epsilon, \text{for all } i, j \geq N \tag{1.1}$$

(see [6]).

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Let $2l\infty$, $2c$ and $2c0$ denote the Banach space bounded, convergent and null double sequences respectively with norm $\|x\| = \sup_{i,j}|xij|$.

Let ν be denote the space of sequences of bounded variation. That is,

$$\nu = \{x = (x_k): \sum_{k=0}^{\infty}|xk - x_{k-1}| < \infty, x_{-1}\} = 0 \tag{1.2}$$

where ν is a Banach space normed by

$$\|x\| = \sum_{k=0}^{\infty}|xk - x_{k-1}|. \qquad \text{(Mursaleen [20])}$$

Let σ be an injective mapping of the set of the positive integers into itself having no finite orbits. A continuous linear functional ϕ on l_∞ is said to be an invariant mean or $\sigma -$ mean if and only if:

1. $\phi(x) \geq 0$ where the sequence $x = (x_k)$ has $x_k \geq 0$ for all k,
2. $\phi(e) = 1$ where $e = \{1,1,1,1, \ldots \ldots . .\}$,
3. $\phi(x_{\sigma(n)}) = \phi(x)$ for all $x \in l_\infty$.

If $x = (x_k)$, write $Tx = (Tx_k) = (x_{\sigma(k)})$. It can be shown that

$$V_\sigma = \{x = (x_k) \in l_\infty: \lim_{m\to\infty} t_{m,k}(x) = L \text{ uniformly in } k, L = \sigma - \lim x\}, \tag{1.3}$$

where $m \geq 0, k > 0$

$$t_{m,k}(x) = \frac{x_k + x_{\sigma(k)} + \cdots + x_{\sigma^m(k)}}{m+1} \; and \; t_{-1,k} = 0, \tag{1.4}$$

where $\sigma^m(k)$ denote the m^{th} - iterate of $\sigma(k)$ at k . In this case σ is the translation mapping, that is, $\sigma(k) = k + 1, \sigma -$mean is called a Banach limit [2] and V_σ , the set of bounded sequences of all whose invariant means are equal, is the set of almost convergent sequences. The special case of (1.4) in which

$\sigma(k) = k + 1$, was given by Lorentz 19], and that the general result can be proved in a similar way. It is familiar that a Banach limit extends the limit functional on c in the sense that

$$\phi(x) = \lim x, \text{ for all } x \in c. \tag{1.5}$$

Remark 1.1: In view of above discussion we have $c \subset V_\sigma$.

Theorem 1.1: A $\sigma -$ mean extends the limit functional on c in the sense that $\phi(x) = \lim x$ for all $x \in c$, if and only if σ has no finite orbits. That is, if and only if for all $k \geq 0, j \geq 1, \sigma^j(k) \neq k$, (see [8]).

Put

$$\phi_{m,k}(x) = t_{m,k}(x) - t_{m-1,k}(x), \tag{1.6}$$

assuming that $t_{-1,k}(x) = 0$.

A straight forward calculation shows that (Mursaleen,[20]),

$$\phi_{m,k}(x) = \begin{cases} \dfrac{1}{m(m+1)}\displaystyle\sum_{j=1}^{m}(J(x_{\sigma^j}(k) - x_{\sigma^{j-1}}(k)), & \text{if } m \geq 1 \\ x_k, & \text{if } m = 0 \end{cases}$$

For any sequence x, y and scalar λ, we have $\phi_{m,k}(x + y) = \phi_{m,k}(x) + \phi_{m,k}(y)$and $\phi_{m,k}(\lambda x) = \lambda\phi_{m,k}(x)$

Definition 1.2: A sequence $x \in l_\infty$ is of $\sigma -$bounded variation if and only if:

1. $\sum|\phi_{m,k}(x)|$ converges uniformly in k,
2. $\lim_m t_{m,k}(x)$, which must exist, should take the same value for all k.

Subsequently invariant means hav e been studied by Ahmad and Mursaleen [1]; Mursaleen ([20],[21]); J.P. King[14]; Raimi[26]; Khan et al. [11] and many others. We denote by BV_σ the space of all sequences of σ −bounded variation (see [20]):

$$BV_\sigma = \left\{x \in l_\infty: \sum_{m=0}^{\infty} \left|\phi_{m,k}(x)\right| < \infty, \text{uniformly in } k\right\}.$$

Theorem 1.2: (see [12]) BV_σ is a Banach space normed by

$$\|x\| = \sup_k \sum_{m=0}^{\infty} \left|\phi_{m,k}(x)\right| \tag{1.7}$$

Definition1.3: A function $f:[0,\infty) \to [0,\infty)$ is called modulus function if it satisfies the following conditions:

(i). $f(t) = 0$, if and only if $t = 0$,
(ii). $f(t+u) \leq f(t) + f(u)$, for all $t, u \geq 0$,
(iii). f is increasing, and
(iv). f is continuous from the right side at zero.

A modulus function f is said to satisfy Δ_2- condition for all values of u if there exists a constant $K > 0$ such that

$f(Lu) \leq KLf(u)$ for all values of $L > 1$. The idea of modulus function was introduced by Nakano in 1953(see [25]). Ruckle

[[27], [25], [29]] used the idea of modulus function f to construct the sequence space

$$X(f) = \{x = (x_k): \sum_{k=1}^{\infty} f(|x_k|) < \infty\}.$$

This space is an FK space and Ruckle [[27], [28], [29]] proved that the intersection of all such $X(f)$ spaces is ϕ, the space of all finite sequences. The space $X(f)$ is closely related to the space $l1$ which is an $X(f)$ space with $f(x) = x$ for all real $x \geq 0$. Thus Ruckle [[27], [28],[29]] proved that, for any modulus function f.

$$X(f) \subset l_1 \text{ and } X(f)^\alpha = l_\infty,$$

Where

$$X(f)^\alpha = \{y = (y_k) \in \omega: \sum_{k=1}^{\infty} f(|y_k x_k|) < \infty\}.$$

This space $X(f)$ is a Banach space with respect to the norm

$$\|x\| = \sum_{k=1}^{\infty} f(|x_k|) < \infty \quad \text{(see[29])}.$$

Spaces of the type $X(f)$ are a special case of the spaces structured by B.Gramsch [5]. From the point of view of local convexity, spaces of the type $X(f)$are quite pathological. Symmetric sequence spaces, which are locally convex have been frequently studied by G.K.Äothe[18], I.J.Maddox [[22],[23],[24]], W.H.Ruckle[[27],[28],[29]] and $\sigma -$convergent sequence space studied by K. Kayaduman and M. Sengönül [15].

Initially, as a generalization of statistical convergence[[3],[4]], the notation of ideal convergence (I −convergence) was introduced and studied by Kostyrko, Mačaj, Šalát and Wilczyňki [[16],[17]]. Later on, it was studied by Šalát, Tripathy and Ziman [[30],[31]], Tripathy and Hazarika [[32],[33],[34]], Hazarika, et,al [7], Khan and Ebadullah [[9], [10]], Khan et al. [11] and many others.

Definition1.4: A double sequence $x = (x_{ij}) \in {}_2\omega$ is said to be I −convergent to a number L if for every $\epsilon > 0$, we have

$\{(i;\ j) : |x_{ij} - L| \geq \epsilon\} \in I.$ (1.8)

In this case, we write $I - \lim x_{ij} = L$.

Definition.1.5: Let X be a non-empty set. Then, a family of sets $I \subseteq 2^X$ is said to be an Ideal in X if

(i). $\phi \in I$;
(ii). I is additive; that is, $A, B \in I \Rightarrow A \cup B \in I$;
(iii). I is hereditary that is, $A \in I, B \subseteq A \Rightarrow B \in I$.

An Ideal $I \subseteq 2^X$ is called non trivial if $I \neq 2^X$.
A non-trivial ideal $I \subseteq 2^X$ is called admissible if

$$\{\{x\} : x \in X\} \subseteq I.$$

A non-trivial ideal I is maximal if there cannot exist any non-trivial ideal $J \neq I$ containing I as a subset.

Definition.1.6: A non-empty family of sets $\mathcal{F} \subseteq 2^X$ is said to be filter on X if and only if

(i). $\phi \notin \mathcal{F}$;
(ii). for, $A, B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$;
(iii). for each $A \in \mathcal{F}$ and $A \subseteq B$ implies $B \in \mathcal{F}$.

For each ideal I, there is a filter $\mathcal{F}(I)$ corresponding to I. That is,

$\mathcal{F}(I) = \{K \subseteq N : K^C \in I,\ \text{where } K^C = N - K\}.$ (1.9)

Definition.1.7: A double sequence $(x_{ij}) \in 2\omega$ is said to be I- null if $L = 0$. In this case, we write.

$I - \lim x_{ij} = 0.$ (1.10)

Definition.1.8: A double sequence $(x_{ij}) \in 2\omega$ is said to be I −cauchy if for every $\epsilon > 0$ there exists numbers $m = m(\epsilon),\ n = n(\epsilon)$ such that

$\{(i,j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - x_{mn}| \geq \epsilon\} \in I.$ (1.11)

Definition.1.9: A double sequence $(x_{ij}) \in 2\omega$ is said to be I-bounded if there exists $M > 0$ such that

$\{(i,j) \in \mathbb{N} \times \mathbb{N} : |x_{ij}| > M\} \in I.$ (1.12)

Definition.1.10: A double sequence space E is said to be solid or normal if $(x_{ij}) \in E$ implies that $(\alpha_{ij} x_{ij}) \in E$ for all sequence of scalars (α_{ij}) with $|\alpha_{ij}| < 1$ for all $(i,j) \in \mathbb{N} \times \mathbb{N}$.

Definition.1.11: A double sequence space E is said to be symmetric if $(x_{\pi(i,j)}) \in E$ whenever $(x_{ij}) \in E$, where $\pi(i,j)$ is a permutation on $\mathbb{N} \times \mathbb{N}$.

Definition.1.12: A double sequence space E is said to be sequence algebra if $(x_{ij}).(y_{ij}) \in E$ whenever

$$(x_{ij}), (y_{ij}) \in E.$$

Definition.1.13: A double sequence space E is said to be convergence free if $(y_{ij}) \in E$ whenever $(x_{ij}) \in E$ and $x_{ij} = 0$ implies $y_{ij} = 0$, for all $(i,j) \in \mathbb{N} \times \mathbb{N}$.

Definition.1.14: Let $K = \{(n_i, k_j) : (i,j) : n_1 < n_2 < n_3 < \ldots.$ and $k_1 < k_2 < k_3 < \ldots.\} \subseteq \mathbb{N} \times \mathbb{N}$ and E be a double sequence space. A K-step space of E is a sequence spaces

$$\lambda_K^E = \{(\alpha_{ij} x_{ij}) : (x_{ij}) \in E\}.$$

Definition.1.15: A cannonical preimage of a sequence $(a_{n_i k_j}) \in \lambda_K^E$ is a sequence $(b_{nk}) \in E$ defined as follows

$$b_{nk=}\begin{cases} a_{nk}, & n,k \in K \\ 0, & \text{otherwise.} \end{cases}$$

Definition.1.16: A sequence space E is said to be monotone if it contains the cannonical preimages of all its stepspaces.

Remark: If $= I_f$, the class of all finite subsets of $\mathbb{N}$. Then I is an admissible ideal in $\mathbb{N}$ and If convergence coincides with the usual convergence.

Definition1.17: If $I = I_\delta = \{A \subseteq \mathbb{N} : \delta(A) = 0\}$. Then I is an admissible ideal in $\mathbb{N}$ and we call the I_δ -convergence as the logarithmic statistical convergence.

Definition1.18: If $I = I_d = \{A \subseteq \mathbb{N} : d(A) = 0\}$. Then I is an admissible ideal in $\mathbb{N}$ and we call the Id $-$convergence as asymptotic statistical convergence.

We used the following lemmas for establishing some results of this article.

Lemma.1.1:[34] Every solid space is monotone.

Lemma.1.2: Let $K \in \mathcal{F}(I)$ and $M \subseteq N$. If $M \notin I$, then $M \cap K \notin I$.

Lemma.1.3: If $I \subseteq 2^N$and $M \subseteq N$. If $M \notin I$, then $M \cap N \notin I$.

For $m;\ n > 0$

$$\mathbf{2}\boldsymbol{BV_\sigma^I} = \{\boldsymbol{x} = (\boldsymbol{x_{ij}}) \in \mathbf{2}\boldsymbol{\omega}: \{(\boldsymbol{i},\boldsymbol{j}): |\emptyset_{\boldsymbol{mnij}}(\boldsymbol{x}) - \boldsymbol{L}| \geq \boldsymbol{\epsilon}\} \in \boldsymbol{I};\ \ \textbf{for some } \boldsymbol{L} \in \mathbb{C}\} \tag{1.13}$$

(See [11]).

2. MAIN RESULTS

In this article, we introduce the following double sequence spaces:

$$2\boldsymbol{B}V_\sigma^I(f) = \{x = (x_{ij})\ 2\omega: \{(i,j): \textstyle\sum_{m,n=0}^{\infty} f(|\emptyset_{mnij}(x) - L|) \geq \epsilon\ \} \in I;\ \text{for some } L \in \mathbb{C}\}; \tag{2.1}$$

$$2(0BV_\sigma^I(f)) = \{x = (x_{ij}) \in 2\omega: \{(i,j): \textstyle\sum_{m,n=0}^{\infty} f(|\emptyset_{mnij}(x)|) \geq \epsilon\} \in I\}; \tag{2.2}$$

$$2(\infty BV_\sigma^I(f)) = \{x = (x_{ij}) \in 2\omega:\ \{(i,j): \exists\, K > 0 : \textstyle\sum_{m,n=0}^{\infty} f(|\emptyset_{mnij}(x)|) \geq K\} \in I\}; \tag{2.3}$$

$$2(\infty BV_\sigma(f)) = \{x = (x_{ij}) \in 2\omega: \sup_{i,j} \textstyle\sum_{m,n=0}^{\infty} f(|\emptyset_{mnij}(x)|) < \infty\}. \tag{2.4}$$

We also denote

$$2(M_{BV_\sigma^I}(f)) = \ 2BV_\sigma^I(f)\ \cap\ 2(\infty BV\sigma(f)).$$

and

$$2(0M_{BV_\sigma^I}(f)) = 2(0BV_\sigma^I(f))\ \cap\ 2(\infty BV\sigma(f)).$$

Theorem.2.1: For any modulus function f, the classes of double sequence $2BV_\sigma^I(f)$, $2(0BV_\sigma^I(f))$, $2(0M_{BV_\sigma^I}(f))$and $2(M_{BV_\sigma^I}(f))$ are linear spaces.

Proof: Let $x = (x_{ij}),\ y = (y_{ij}) \in 2BV_\sigma^I(f)$ be any two arbitrary elements, and let α, β are scalars.

Now, since $x = (x_{ij}),\ y = (y_{ij}) \in 2BV_\sigma^I(f)$. Then this implies that there exists some positive numbers $L1,\ L2\ \in \mathbb{C}$ and such that the sets

$$A_1 = \{(i,j): \textstyle\sum_{m,n=0}^{\infty} f(|\emptyset_{mnij}(x) - L_1|) \geq \frac{\epsilon}{2}\} \in I, \tag{2.5}$$

$$A_2 = \{(i,j): \textstyle\sum_{m,n=0}^{\infty} f(|\emptyset_{mnij}(x) - L_2|) \geq \frac{\epsilon}{2}\} \in I\ . \tag{2.6}$$

Now let

$B_1 = \{(i,j): \sum_{m,n=0}^{\infty} f\left(\left|\emptyset_{mnij}(x) - L_1\right|\right) < \frac{\epsilon}{2}\} \in \mathcal{F}(I)$ (2.7)

$B_2 = \{(i,j): \sum_{m,n=0}^{\infty} f\left(\left|\emptyset_{mnij}(x) - L_2\right|\right) < \frac{\epsilon}{2}\} \in \mathcal{F}(I)$ (2.8)

be such that $B_1^c, B_1^c \in I$. Since f is a modulus function, we have

$$\sum_{m,n=0}^{\infty} f\left(\left|\phi_{mnij}(\alpha x + \beta y) - (\alpha L_1 + \beta L_2)\right|\right)$$
$$= \sum_{m,n=0}^{\infty} f\left(\left|(\alpha\phi_{mnij}(x) + \beta\phi_{mnij}(y)) - (\alpha L_1 + \beta L_2)\right|\right)$$
$$= \sum_{m,n=0}^{\infty} f\left(\left|\alpha\left(\phi_{mnij}(x) - L_1\right) + \beta\left(\phi_{mnij}(y) - L_2\right)\right|\right)$$
$$\leq \sum_{m,n=0}^{\infty} f\left(|\alpha|\left|\emptyset_{mnij}(x) - L_1\right|\right) + \sum_{m,n=0}^{\infty} f\left(|\beta|\left|\emptyset_{mnij}(y) - L_2\right|\right)$$
$$\leq \sum_{m,n=0}^{\infty} f\left(\left|\phi_{mnij}(x) - L_1\right|\right) + \sum_{m,n=0}^{\infty} f\left(\left|\phi_{mnij}(y) - L_2\right|\right)$$
$$\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

this implies that

$$\{(i,j): \sum_{m,n=0}^{\infty} f\left(\left|\phi_{mnij}(\alpha x + \beta y) - (\alpha L_1 + \beta L_2)\right|\right) \geq \epsilon\} \in I.$$

Thus $\alpha(x_{ij}) + \beta(y_{ij}) \in 2BV_\sigma^I(f)$. As (x_{ij}) and $\left(y_{ij}\right)$ are two arbitrary elements, then $\alpha(x_{ij}) + \beta(y_{ij}) \in 2BV_\sigma^I(f)$ for all (x_{ij}),$(y_{ij}) \in 2BV_\sigma^I(f)$ and for all scalars α, β.

Hence $2BV_\sigma^I(f)$ is linear space. The proof for other spaces will follow similarly.

Theorem.2.2. A sequence $x = \left(x_{ij}\right) \in 2(M_{BV_\sigma^I}(f))$ I-convergent if and only if for every $\epsilon > 0$, there exists $M\epsilon,\ N\epsilon \in \mathbb{N}$ such that

$$\{(i,j): \sum_{m,n=0}^{\infty} f\left(\left|\phi_{mnij}\left(x_{ij}\right) - \emptyset_{mnij}(x_{M\epsilon N\epsilon})\right|\right) < \epsilon\} \in \mathcal{F}(I).$$

Proof: Let $x = \left(x_{ij}\right), \in 2(M_{BV_\sigma^I}(f))$, Suppose $I - \lim x = L$. Then, the set

$$B_\epsilon = \left\{(i,j): \sum_{m,n=0}^{\infty} f\left(\left|\phi_{mnij}\left(x_{ij}\right) - L\right|\right) < \frac{\epsilon}{2}\right\} \in \mathcal{F}(I), \qquad \text{for all } \epsilon > 0$$

Fix $M_\epsilon, N_\epsilon \in B_\epsilon$. Then we have

$$\sum_{m,n=0}^{\infty} f\left(\left|\phi_{mnij}\left(x_{ij}\right) - \phi_{mnij}(x_{M\epsilon N\epsilon})\right|\right) \leq \sum_{m,n=0}^{\infty} f\left(\left|\phi_{mnij}(x_{M\epsilon N\epsilon}) - L\right|\right) + \sum_{m,n=0}^{\infty} f\left(\left|L - \phi_{mnij}\left(x_{ij}\right)\right|\right) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

which holds for all $(i,j) \in B_\epsilon$.

Hence

$$\{(i,j): \sum_{m,n=0}^{\infty} f\left(\left|\phi_{mnij}\left(x_{ij}\right) - \phi_{mnij}(x_{M\epsilon N\epsilon})\right|\right) < \epsilon\} \in \mathcal{F}(I).$$

Conversely, suppose that

$$\{(i,j): \sum_{m,n=0}^{\infty} f\left(\left|\phi_{mnij}\left(x_{ij}\right) - \phi_{mnij}(x_{M\epsilon N\epsilon})\right|\right) < \epsilon\} \in \mathcal{F}(I).$$

Then, being f a modulus function and by using basic triangular inequality, we have

$\{(i,j): \left| \sum_{m,n=0}^{\infty} f\left(\left|\emptyset_{mnij}(x_{ij})\right|\right) - \sum_{m,n=0}^{\infty} f\left(\left|\emptyset_{mnij}(x_{M\epsilon N\epsilon})\right|\right)\right| < \epsilon\} \in \mathcal{F}(I)$, for all $\epsilon > 0$.

Then, the set

$$C_\epsilon = \left\{(i,j): \sum_{m,n=0}^{\infty} f(\left|\emptyset_{mnij}(x_{ij})\right| \in \left[\sum_{m,n=0}^{\infty} f\left(\left|\emptyset_{mnij}(x_{M\epsilon N\epsilon})\right|\right) - \epsilon, \sum_{m,n=0}^{\infty} f\left(\left|\emptyset_{mnij}(x_{M\epsilon N\epsilon})\right|\right) + \epsilon\right]\right\} \in \mathcal{F}(I)$$

Let

$$J_\epsilon = \left[\sum_{m,n=0}^{\infty} f\left(\left|\emptyset_{mnij}(x_{M\epsilon N\epsilon})\right|\right) - \epsilon, \sum_{m,n=0}^{\infty} f\left(\left|\emptyset_{mnij}(x_{M\epsilon N\epsilon})\right|\right) + \epsilon\right].$$

If we fix $\epsilon > 0$ then, we have $C\epsilon \in \mathcal{F}(I)$ as well as $C_{\frac{\epsilon}{2}} \in \mathcal{F}(I)$.

Hence $C\epsilon \cap C_{\frac{\epsilon}{2}} \in \mathcal{F}(I)$. This implies that $J = J_\epsilon \cap J_{\frac{\epsilon}{2}} \neq \emptyset$.

That is

$$\{(i,j): \sum_{m,n=0}^{\infty} f\left(\left|\emptyset_{mnij} x_{i,j}\right|\right) \in J\} \in \mathcal{F}(I).$$

This shows that

$$diam\, J \leq diam\, J_\epsilon$$

where the $diam\ J$ denotes the length of interval J. In this way, by induction we get the sequence of closed intervalsType equation here.

$$J_\epsilon = I0 \supseteq I1 \supseteq I2 \supseteq \ldots\ldots \supseteq Ik \supseteq \ldots..$$

with the property that $diam\, I_k \leq \frac{1}{2} diam\, I_{k-1}$ for $(k = 2,3,4, \ldots.)$ and

$$\left\{(i,j): \sum_{m,n=0}^{\infty} f\left(\left|\emptyset_{mnij}(x_{i,j})\right|\right) \in I_k\right\} \in \mathcal{F}(I) \text{ for } (k = 2,3,4, \ldots.).$$

Then there exists a $\xi \in I_k$ where $k \in \mathbb{N}$ such that

$$\xi = I - \lim_{i,j} \sum_{m,n=0}^{\infty} f\left(\left|\emptyset_{mnij} x_{i,j}\right|\right),$$

showing that $x = (x_{ij}) \in 2(M_{BV_\sigma^I}(f))$ is I-convergent. Hence the result holds.

Theorem.2.3: Let $f1$ and $f2$ be two modulus functions and satisfying $\Delta 2 -$ condition, then

(a). $\chi(f2) \subseteq \chi(f1f2)$
(b). $\chi(f1) \cap \chi(f2) \subseteq \chi(f1 + f2)$

where $\chi = 2\ (0BV_\sigma^I), 2BV_\sigma^I, 2M_{BV_\sigma^I},\ 2\left(0M_{BV_\sigma^I}\right)$.

Proof: (a). Let $x = (x_{ij}) \in 2(0BV_\sigma^I(f))$ be an arbitrary element. Then the set

$$\{(\mathrm{i},\mathrm{j}): \sum_{m,n=0}^{\infty} f_2\left(\left|\phi_{mnij}(x)\right|\right) \geq \epsilon\} \in I. \tag{2.9}$$

Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f_1(t) < \epsilon$

for $0 < t \leq \delta$.

Let us write $y_{ij} = f_2(|\phi_{mnij}(x)|)$ and consider,

$$\lim_{ij} f_1(y_{ij}) = \lim_{y_{ij}\le\delta,i,j\in\mathbb{N}} f_1(y_{ij}) + \lim_{y_{ij}>\delta,i,j\in\mathbb{N}} f_1(y_{ij}). \quad (2.10)$$

Now, since f_1 is modulus function.
Therefore, we have

$$\lim_{y_{ij}\le\delta,i,j\in\mathbb{N}} f_1(y_{ij}) \le f_1(2) \lim_{y_{ij}\le\delta,ij\in\mathbb{N}}(y_{ij}). \quad (2.11)$$

For $y_{ij} > \delta$, we have $y_{ij} < \frac{y_{ij}}{\delta} < 1 + \frac{y_{ij}}{\delta}$. Now, since f_1 is non-decreasing, it follows that

$$f_1(y_{ij}) < f_1\left(1 + \frac{y_{ij}}{\delta}\right) < \frac{1}{2} f_1(2) + \frac{1}{2} f_1\left(\frac{2y_{ij}}{\delta}\right). \quad (2.12)$$

Again, since f_1 satisfies the $\Delta 2 -$ condition, we have,

$$\begin{aligned} f_1(y_{ij}) &< \frac{1}{2} K \frac{y_{ij}}{\delta} f_1(2) + \frac{1}{2} K f_1\left(\frac{2y_{ij}}{\delta}\right) \\ &< \frac{1}{2} K \frac{y_{ij}}{\delta} f_1(2) + \frac{1}{2} K \frac{y_{ij}}{\delta} f_1(2) \\ &= K_1 \frac{y_{ij}}{\delta} f_1(2) \end{aligned} \quad (2.13)$$

This implies that,

$$f_1(y_{ij}) < K \frac{y_{ij}}{\delta} f_1(2). \quad (2.14)$$

Hence, we have

$$\lim_{y_{ij}>\delta,i,j\in\mathbb{N}} f_1(y_{ij}) \le \max\{1, K\delta^{-1} f_1(2) \lim_{y_{ij}>\delta,i,j\in\mathbb{N}}(y_{ij})\}. \quad (2.15)$$

Therefore from (2.9), (2.11) and (2.15) , it follows that

$$\{(\mathrm{i,j}): \sum_{m,n=0}^{\infty} f_1(y_{ij}) \ge \epsilon\} \in I,$$

i.e

$$\{(\mathrm{i,j}): \sum_{m,n=0}^{\infty} f_1 f_2(|\phi_{mnij}(x)|) \ge \epsilon\} \in I,$$

this implies that $x = (x_{ij}) \in 2\,(0BV_\sigma^I(f_1 f_2))$. Hence $\chi(f2) \subseteq \chi(f_1 f_2)$ for $\chi = 2\,(0BV_\sigma^I)$. The other cases can be proved in similar way.

(b). Let $x = (x_{ij}) \in 2\,(0BV_\sigma^I(f_1)) \cap 2\,(0BV_\sigma^I(f_2))$. Let $\epsilon > 0$ be given. Then, the sets

$$\{(\mathrm{i,j}): \sum_{m,n=0}^{\infty} f_1(|\phi_{mnij}(x)|) \ge \frac{\epsilon}{2}\} \in I \quad (2.16)$$

$$\{(\mathrm{i,j}): \sum_{m,n=0}^{\infty} f_2(|\phi_{mnij}(x)|) \ge \frac{\epsilon}{2}\} \in I. \quad (2.17)$$

Therefore, the inclusion

$$\begin{aligned} &\{(\mathrm{i,j}): \sum_{m,n=0}^{\infty} (f_1 + f_2)(|\phi_{mnij}(x)|) \ge \epsilon\} \\ &\subseteq \left[\left\{(\mathrm{i,j}): \sum_{m,n=0}^{\infty} f_1(|\phi_{mnij}(x)|) \ge \epsilon\right\}\right. \\ &\cup \left.\left\{(\mathrm{i,j}): \sum_{m,n=0}^{\infty} f_2(|\phi_{mnij}(x)|) \ge \epsilon\right\}\right] \end{aligned}$$

implies that

$$\{(\text{i},\text{j}): \sum_{m,n=0}^{\infty} (f_1+f_2)(|\phi_{mnij}(x)|) \geq \epsilon\} \in I.$$

Hence, we get $2\,(0BV_\sigma^I\,(f_1)) \cap 2\,(0BV_\sigma^I\,(f_2)) \subseteq 2\,(0BV_\sigma^I\,(f_1+f_2))$.

For $\chi = 2BV_\sigma^I$; $2(M_{BV_\sigma^I})$; $2\,(0\,M_{BV_\sigma^I})$ the inclusion are similar.

For $f_2(x) = x \; and \; f_1(x) = f(x) \; for \; all \; x \in [0,1)$ we have the following corollary.

Corollary.2.4: $\chi \subseteq \chi(f)$ for $\chi = 2\,(0BV_\sigma^I\,), 2BV_\sigma^I\,; 2(M_{BV_\sigma^I}); \; 2\,(0\,M_{BV_\sigma^I}\,)$.

Theorem.2.5: For any modulus function f, the spaces $2\,(0BV_\sigma^I\,(f))$ and $2\,(0\,M_{BV_\sigma^I}\,(f))$ are solid and monotone.

Proof: Here we consider $2\,(0BV_\sigma^I\,(f))$ and for $2\,(0\,M_{BV_\sigma^I}\,(f))$ the proof shall be similar.

Let $x = (x_{ij}) \in 2\,(0BV_\sigma^I\,(f))$ be an arbitrary element, then the set

$$\{(\text{i},\text{j}): \textstyle\sum_{m,n=0}^{\infty} f(|\phi_{mnij}(x)|) \geq \epsilon\} \in I. \tag{2.18}$$

Let (α_{ij}) be a sequence of scalars with $|\alpha_{ij}| \leq 1$ for all $i,j \in \mathbb{N}$.

Now, since f is a modulus function. Then the result follows from (2.18) and the inequality

$$f(|\alpha_{ij}\phi_{mnij}(x)|) \leq |\alpha_{ij}| f(|\phi_{mnij}(x)|) \leq f(|\phi_{mnij}(x)|).$$

Therefore,

$$\left\{(\text{i},\text{j}): \sum_{m,n=0}^{\infty} f(|\alpha_{ij}\phi_{mnij}(x)|) \geq \epsilon\right\} \subseteq \left\{(\text{i},\text{j}): \sum_{m,n=0}^{\infty} f(|\phi_{mnij}(x)|) \geq \epsilon\right\} \in I$$

implies that

$$\left\{(i,j): \sum_{m,n=0}^{\infty} f(|\alpha_{ij}\phi_{mnij}(x)|) \geq \epsilon\right\} \in I.$$

Thus we have $(\alpha_{ij}x_{ij}) \in 2\,(0BV_\sigma^I\,(f))$. Hence $2\,(0BV_\sigma^I\,(f))$ is solid. Therefore $2\,(0BV_\sigma^I\,(f))$ is monotone. Since every solid sequence space is monotone.

Theorem.2.6: For any modulus function f, the space $2BV_\sigma^I\,(f)$ and $2(M_{BV_\sigma^I}(f))$ are neither solid nor monotone in general.

Proof: Here we give counter example for establishment of this result. $\chi = 2BV_\sigma^I$ and $2(M_{BV_\sigma^I})$. Let us consider $I = I_f$ and $f(x) = x$ for all $x = (x_{ij})$ and $x \in [0,\infty)$. Consider, the K-step space $\chi_K(f)$ of $\chi(f)$ defined as follows:

Let $x = (x_{ij}) \in \chi(f)$ and $y = (y_{ij}) \in \chi_K(f)$ be such that

$$y_{ij} = \begin{cases} x_{ij}\,; & \text{if i, j are even} \\ 0; & \text{otherwise.} \end{cases}$$

Consider the sequence (x_{ij}) defined by $x_{ij} = 1$ for all $i,j \in \mathbb{N}$. Then $x = (x_{ij}) \in 2BV_\sigma^I\,(f)$ and $2(M_{BV_\sigma^I}(f))$, but K-step space preimage does not belong to $BV_\sigma^I\,(f)$ and $2(M_{BV_\sigma^I}(f))$,

Thus $2BV_\sigma^I\,(f)$ and $2(M_{BV_\sigma^I}(f))$ are not monotone and hence they are not solid.

Theorem.2.7: For any modulus function f, the spaces $2BV_\sigma^I\,(f)$ and $2\,(0BV_\sigma^I\,(f))$ are sequence algebra.

Proof: Let $x = (x_{ij}),\; y = (y_{ij}) \in 2\,(0BV_\sigma^I\,(f))$ be any two arbitrary elements.

Then, the sets

$$\left\{(\mathrm{i,j}): \sum_{m,n=0}^{\infty} f\left(\left|\phi_{mnij}(x)\right|\right) \geq \epsilon\right\} \in I,$$

And

$$\left\{(\mathrm{i,j}): \sum_{m,n=0}^{\infty} f\left(\left|\phi_{mnij}(y)\right|\right) \geq \epsilon\right\} \in I,$$

Therefore,

$$\left\{(\mathrm{i,j}): \sum_{m,n=0}^{\infty} f\left(\left|\phi_{mnij}(x).\phi_{mnij}(y)\right|\right) \geq \epsilon\right\} \in I.$$

Thus, we have $(x_{ij}).(y_{ij}) \in 2\left(0BV_\sigma^I(f)\right)$. Hence $2\left(0BV_\sigma^I(f)\right)$ is sequence algebra and for $2BV_\sigma^I(f)$ the result can be proved similarly.

Theorem.2.8: If I is not maximal and $I \neq If$, then the spaces $2BV_\sigma^I(f)$and $2\left(0BV_\sigma^I(f)\right)$ are not symmetric.

Proof: Let $A \in I$ be an infinite set and $f(x) = x$ for all $x = (x_{ij})$ and $x_{ij} \in [0,\infty)$. If

$$x_{ij} = \begin{cases} 1; & \text{if } (i,j) \in A \\ 0; & \text{otherwise.} \end{cases}$$

Then, it is clearly seen that $(x_{ij}) \in 2\,(0BV_\sigma^I(f)) \subset 2BV_\sigma^I(f)$

Let $K \subseteq \mathbb{N} \times \mathbb{N}$ be such that $K \notin I$ and $K^c \notin I$. Let $\phi: K \to A$ and $\psi: K^c \to A^C$ be a bijective maps (as all four sets are infinite). Then, the mapping $\pi: \mathbb{N}\times\mathbb{N} \to \mathbb{N}\times\mathbb{N}$ defined by

$$\pi(i,j) = \begin{cases} \phi(i,j); & \text{if } (i,j) \in K \\ \psi(i,j); & \text{otherwise} \end{cases}$$

is a permutation on $\mathbb{N}\times\mathbb{N}$.

But $(x_{\pi(i,j)}) \notin 2BV_\sigma^I(f)$ and hence $(x_{\pi(i,j)}) \notin 2\left(0BV_\sigma^I(f)\right)$ showing that $2BV_\sigma^I(f)$ and $2\left(0BV_\sigma^I(f)\right)$ are not symmetric double sequence spaces.

Theorem.2.9: Let f be any modulus function. Then

$$2\left(0BV_\sigma^I(f)\right) \subset 2BV_\sigma^I(f) \subset 2\left(\infty BV_\sigma^I(f)\right).$$

Proof: The inclusion $2\left(0BV_\sigma^I(f)\right) \subset 2BV_\sigma^I(f)$ is obvious.

Next, let us consider $x = (x_{ij}) \in 2BV_\sigma^I(f)$. Then there exists $L \in \mathbb{C}$ such that

$$\left\{(\mathrm{i,j}): \sum_{m,n=0}^{\infty} f\left(\left|\phi_{mnij}(x) - L\right|\right) \geq \epsilon\right\} \in I.$$

We have

$$f\left(\left|\phi_{mnij}(x)\right|\right) \leq \frac{1}{2} f\left(\left|\phi_{mnij}(x) - L\right|\right) + f\left(\frac{1}{2}|L|\right).$$

Now taking supremum over i,j on both sides, we get $x = (x_{ij}) \in 2BV_\sigma^I(f)$.

Hence $2\,(0BV_\sigma^I(f)) \subset 2BV_\sigma^I(f) \subset 2\,(\infty BV_\sigma^I(f))$.

Next, we show that the inclusions are proper.

For this, let us consider $I = I_d$, $f(x) = x^2$ for all $x \in [0,\infty)$. Consider the sequence (x_{ij}) defined by $x_{ij} = 1 \;\forall\, i,j$. Then $(x_{ij}) \in 2BV_\sigma^I(f)$ but $(x_{ij}) \notin 2\,(0BV_\sigma^I(f))$.

Again, consider the sequence (y_{ij}) defined by

$$y_{ij} = \begin{cases} 2, & \text{if } i,j \text{ even} \\ 0, & \text{otherwise.} \end{cases}$$

Then $(y_{ij}) \in 2\left({}_\infty BV_\sigma^I(f)\right)$ but $(y_{ij}) \notin 2BV_\sigma^I(f)$.

3. CONCLUSIONS

In this paper we have studied the concept of I- convergent double sequence space of invariant mean which is defined by modulus function. Recently V. A. Khan, Ayhan Esi and Mohd Shafiq [13] studied the notion of BV_σ -ideal convergent sequence spaces defined by modulus function and with the help of this we defined different spaces such as $2\left({}_0BV_\sigma^I(f)\right)$, $2BV_\sigma^I(f)$ and $2\left({}_\infty BV_\sigma^I(f)\right)$ for double sequence by using modulus function. These results provide new tools to deal with the I-convergence in double sequence and problems of sequences occurring in many branches of science and engineering.

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