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Research Article

NUMERICAL SOLUTIONS FOR THE FOURTH ORDER EXTENDED FISHER-KOLMOGOROV EQUATION WITH HIGH ACCURACY BY DIFFERENTIAL QUADRATURE METHOD

Ali BAŞHAN*[1], Yusuf UÇAR[2], N. Murat YAĞMURLU[3], Alaattin ESEN[4]

[1]*Zonguldak Bulent Ecevit University, Dept. of Mathematics, ZONGULDAK;* ORCID:0000-0001-8500-493X
[2]*Inonu University, Department of Mathematics, MALATYA;* ORCID:0000-0003-1469-5002
[3]*Inonu University, Department of Mathematics, MALATYA;* ORCID:0000-0003-1593-0254
[4]*Inonu University, Department of Mathematics, MALATYA;* ORCID:0000-0002-7927-5941

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ABSTRACT

In this paper, modified cubic B-spline based differential quadrature method (MCB-DQM) has been used to obtain the numerical solutions for the fourth order extended Fisher-Kolmogorov equation (EFK). After using DQM for discretization of the EFK equation, ordinary differential equation systems have been obtained. For time integration, strong stability preserving Runge-Kutta method has been used. Numerical solutions of the three test problems have been investigated. The efficiency and accuracy of the method have been measured by calculating error norms L_2 and L_∞. The present obtained numerical results have been compared with the published numerical results and the comparison has shown that the method is an effective numerical scheme to solve the EFK equation.
Keywords: Partial differential equations, differential quadrature method, EFK equation, modified cubic B-Splines.
AMS Classification: 65M99, 65M12, 65D07, 65L20.

1. INTRODUCTION

In this study, we have investigated numerical solutions of fourth order extended Fisher-Kolmogorov (EFK) equation by using modified cubic B-spline Differential Quadrature Method (MCB-DQM). The EFK equation is given in the following form

$$u_t + \gamma u_{xxxx} - u_{xx} + \psi(u) = 0, \quad x \in [a,b], \ \ t \in [0,T] \tag{1}$$

where $u \in [a,b] \times [0,T]$, $\psi(u) = u^3 - u$ and $\gamma > 0$.

Coullet et al.[1] and van Saarlos [2-4] and Dee and van Saarlos [5] introduced the EFK equation given in the Eq. (1) by adding a stabilizing fourth-order derivative for the value of the $\gamma \neq 0$. For the value of the $\gamma = 0$ the Eq. (1) turns into the standard Fisher-Kolmogorov (FK) equation.

* Corresponding Author: e-mail: alibashan@gmail.com, tel: (372) 291 17 88 / 1788

EFK equ ation has many applications in several scientific field such as spatiotemporal chaos [1] pattern formation in bi-stable systems [5], propagation of domain walls in liquid crystals [6], phase transition near a Lifshitz point [7,8].

EFK equation (1) has been solved with various methods by many researchers. Among others, Danumjaya and Pani [9] used orthogonal cubic spline collocation method, Mittal and Arora [10] used quantic B-spline collocation, Mittal and Dahiya [11] used quintic B-spline differential quadrature method. In this paper, we investigated numerical solutions of EFK equation via modified cubic B-spline differential quadrature method since it has high accurate solutions.

In recent years, DQM which is first introduced by Bellman et al. [12] 1972, has had wide application areas because of its using considerably less number of grid points. DQM is a method in which partial derivative of a function with respect to a coordinate direction is expressed as a linear weighted sum of all the functional values at all mesh points along that direction[13]. Many researchers have developed various types of DQM by using different base functions such as Legendre polynomials and spline functions [12,14-17], Lagrange interpolation polynomials [18-20], Lagrange interpolated trigonometric polynomials [21], Hermite polynomials [22], radial basis functions [23], harmonic functions [24], Sinc functions [25,26], B-spline functions [27-33].

In the present study, MCB-DQM is going to be applied to obtain approximate solutions of the EFK equation. MCB-DQM has been preferred for solving fourth-order EFK equation since it has high accurate solutions, low storage allocation and simple applicability to equation.

2. MODIFIED CUBIC B-SPLINE DQM

We are going to consider the Eq. (1) with the boundary conditions taken from:

$$U(a,t) = g_1(t), \quad U(b,t) = g_2(t), \quad x \in [a,b], \ t \in [0,T] \tag{2}$$

with the initial condition

$$U(x,0) = u_0(x), \ a \le x \le b. \tag{3}$$

Let us take the uniform grid distribution $a = x_1 < x_2 < \cdots < x_N = b$ of a finite interval $[a,b]$ into consideration. Provided that any given function $U(x)$ is smooth enough over the solution domain, its derivatives with respect to x at a grid point x_i can be approximated by a linear summation of all the functional values in the solution domain, namely,

$$\left.\frac{d^{(r)}U}{dx^{(r)}}\right|_{x_i} = \sum_{j=1}^{N} w_{ij}^{(r)}\, U(x_j), \ \ i = 1,2,\dots,N, \ \ r = 1,2,\dots,N-1 \tag{4}$$

where r represents the order of derivative, $w_{ij}^{(r)}$'s denote the weighting coefficients of the r-th order derivative approximation, and N represents the number of nodal points in the given solution domain. Here, the index j indicates the fact that $w_{ij}^{(r)}$ is the corresponding weighting coefficient of the value of the function $U(x_j)$.

In this work, we are going to need the second and fourth order derivatives of the function $U(x)$. So, we are going to begin to find the value of the equation (4) for the $r = 1$.

The main idea behind DQM approximation is to find out the corresponding weighting coefficients $w_{ij}^{(r)}$ by means of a set of base functions spanning the problem domain. While determining the corresponding weighting coefficients, different basis may be used. In the present study, we will try to compute weighting coefficients with modified cubic B-spline basis.

Let $C_m(x)$ be the cubic B-splines with knots at the points x_i where the uniformly distributed N grid points are taken as $a = x_1 < x_2 < \cdots < x_N = b$ on the ordinary real axis. Then, the cubic B-splines $\{C_0, C_1, \dots, C_{N+1}\}$ form a basis for the functions defined over $[a,b]$.

The cubic B-splines $C_m(x)$ are defined by the relationships:

$$C_m(x) = \frac{1}{h^3}\begin{cases}(x-x_{m-2})^3 & ,[x_{m-2},x_{m-1}]\\ (x-x_{m-2})^3-4(x-x_{m-1})^3 & ,[x_{m-1},x_m]\\ (x_{m+2}-x)^3-4(x_{m+1}-x)^3 & ,[x_m,x_{m+1}]\\ (x_{m+2}-x)^3 & ,[x_{m+1},x_{m+2}]\\ 0 & ,otherwise\end{cases}$$

where $h = x_m - x_{m-1}$ for all m [34].

Using the modified cubic B-splines results in a diagonally dominant matrix system of equations. This structur e has great importance for the stability analysis. Modification of cubic B-splines can be carried out differently. Among others, Mittal and Jain [35] have introduced modified cubic B-splines at the grid points as follows

$$\begin{aligned}&\emptyset_1(x) = C_1(x) + 2C_0(x)\\ &\emptyset_2(x) = C_2(x) - C_0(x)\\ &\emptyset_s(x) = C_s(x), \text{ for } s = 3,4,\dots,N-2\\ &\emptyset_{N-1}(x) = C_{N-1}(x) - C_{N+1}(x)\\ &\emptyset_N(x) = C_N(x) + 2C_{N+1}(x)\end{aligned} \quad (5)$$

where $\emptyset_k$, $(k = 1,2,\dots,N)$ forms a basis functions over the $[a,b]$ domain.

2.1. Weighting Coefficients of the First Order Derivative

From Eq. (4) with value of $r = 1$, we have obtained the following equation

$$\emptyset'_k(x_i) = \sum_{j=1}^{N} w_{ij}^{(1)}\, \emptyset_k(x_j) \ \text{ for } i = 1,2,\dots,N;\ k = 1,2,\dots,N \quad (6)$$

For the first grid point x_1, from Eq. (6) we get an equation in the following form

$$\emptyset'_k(x_1) = \sum_{j=1}^{N} w_{1j}^{(1)}\, \emptyset_k(x_j) \ \text{ for } \ k = 1,2,\dots,N \quad (7)$$

and by using the value of modified cubic basis functions

$$\begin{bmatrix}6&1&&&&&\\0&4&1&&&&\\&1&4&1&&&\\&&\ddots&\ddots&\ddots&&\\&&&1&4&1&\\&&&&1&4&0\\&&&&&1&6\end{bmatrix}\begin{bmatrix}w_{1,1}^{(1)}\\w_{1,2}^{(1)}\\w_{1,3}^{(1)}\\\vdots\\w_{1,N-1}^{(1)}\\w_{1,N}^{(1)}\end{bmatrix}=\begin{bmatrix}-6/h\\6/h\\0\\\vdots\\0\\0\end{bmatrix} \quad (8)$$

the above equation system is obtained. Similarly, by using the value of modified cubic basis functions at the grid points x_i, $(2 \le i \le N-1)$, respectively,

$$\begin{bmatrix}6&1&&&&&\\0&4&1&&&&\\&1&4&1&&&\\&&\ddots&\ddots&\ddots&&\\&&&1&4&1&\\&&&&1&4&0\\&&&&&1&6\end{bmatrix}\begin{bmatrix}w_{i,1}^{(1)}\\\vdots\\w_{i,i-1}^{(1)}\\w_{i,i}^{(1)}\\w_{i,i+1}^{(1)}\\\vdots\\w_{i,N}^{(1)}\end{bmatrix}=\begin{bmatrix}0\\\vdots\\0\\-3/h\\0\\3/h\\0\\\vdots\\0\end{bmatrix} \quad (9)$$

the above equation system is obtained. For the last grid point x_N

$$\begin{bmatrix} 6 & 1 & & & & & \\ 0 & 4 & 1 & & & & \\ & 1 & 4 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 1 & 4 & 1 & \\ & & & & 1 & 4 & 0 \\ & & & & & 1 & 6 \end{bmatrix} \begin{bmatrix} w_{N,1}^{(1)} \\ w_{N,2}^{(1)} \\ \vdots \\ w_{N,N-2}^{(1)} \\ w_{N,N-1}^{(1)} \\ w_{N,N}^{(1)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \\ 0 \\ -6/h \\ 6/h \end{bmatrix} \tag{10}$$

the above equation system is obtained. So, weighting coefficients $w_{i,j}^{(1)}$ which are related to the x_i $(i = 1, 2, \dots, N)$ are found quite easily by solving system of Eqs. (8)-(10) with Thomas algorithm.

2.2. Weighting Coefficients of the Second Order Derivative

This method depends on the first order weighting coefficients when obtaining weighting coefficients of the second order derivatives. By the Shu's recurrence formula, the second order weighting coefficients are determined for $i = 1, 2, \dots, N$ and $j = 1, 2, \dots, N$ as below [13]:

$$w_{i,j}^{(2)} = 2w_{i,j}^{(1)} \left(w_{i,i}^{(1)} - \frac{1}{x_i - x_j} \right), \text{ for } i \neq j \tag{11}$$

$$w_{i,i}^{(2)} = -\sum_{j=1, j\neq i}^{N} w_{i,j}^{(2)} \tag{12}$$

2.3. Weighting Coefficients of the Fourth Order Derivative

This method depends on the second order weighting coefficients to obtain the weighting coefficients of the fourth order derivatives. By the matrix multiplication approach, the fourth order weighting coefficients are determined as below [13]:

$$\left[A^{(m)}\right] = \left[A^{(1)}\right]\left[A^{(m-1)}\right] = \left[A^{(m-1)}\right]\left[A^{(1)}\right] \tag{13}$$

where $\left[A^{(m-1)}\right]$ and $\left[A^{(m)}\right]$ are the weighting coefficients matrices of the $(m-1)-th$ and $m-th$ order derivatives, respectively. Although Eq. (13) looks simple, it involves more arithmetic operations as compared to Eqs. (11) and (12). It is noted that the calculation of each weighting coefficient by Eq. (13) involves *N* multiplications and (*N*-1) additions, i.e. a total of (2*N*-1) arithmetic operations. On the other hand, Shu's recurrence relationship (11) involves two multiplications, one division and one subtraction, i.e. a total of four arithmetic operations [13].

3. NUMERICAL DISCRETIZATION

The EFK equation of the form

$$U_t + \gamma U_{xxxx} - U_{xx} + U^3 - U = 0 \tag{14}$$

with the boundary conditions (2) and the initial condition (3) is rewritten as

$$U_t = -\gamma U_{xxxx} + U_{xx} - U^3 + U. \tag{15}$$

Then, the differential quadrature derivative approximations of the second and the fourth orders have been used in Eq. (15)

$$\frac{dU(x_i)}{dt} = -\gamma \sum_{j=1}^{N} w_{ij}^{(4)}\, U(x_j, t) + \sum_{j=1}^{N} w_{ij}^{(2)}\, U(x_j, t) - U^3(x_i, t) + U(x_i, t),\ i = 1, 2, \dots, N \tag{16}$$

and ordinary differential equation (16) is obtained. Then, the ordinary differential equation given by Eq. (16) is integrated with respect to time. Here, we have preferred strong stability-preserving low storage Runge-Kutta43 method[36] due to its advantages such as accuracy, stability and memory allocation properties.

4. NUMERICAL EXAMPLES AND STABILITY

Here, we have obtained the numerical solutions of the EFK by the MCB-DQM. The accuracy of the numerical method is checked using the error norms L_2 and L_∞, respectively:

$$L_2 = \|U-u\|_2 \cong \sqrt{h\sum_{j=1}^{N}\left|U_j-u_j\right|^2},$$

$$L_\infty = \|U-u\|_\infty \cong \max_j\left|U_j-u_j\right|,\ j=1,2,\dots N-1.$$

Since the analytical solution of EFK equation does not exist, newly obtained numerical solution is compared with those solutions obtained when grid number is taken N=160 instead of exact solution.

Stability analysis of a numerical method for a nonlinear differential equation requires the determination of eigenvalues of coefficient matrices. With the numerical discretization of partial differential equation EFK, it turns into an ordinary differential equation.

The stability of a time-dependent problem:

$$\frac{\partial U}{\partial t} = l(U), \tag{17}$$

with the proper initial and boundary conditions, where l is a spatial differential operator. After discretization with DQM, Eq. (17) is reduced into a set of ordinary differential equations in time as follows

$$\frac{d\{U\}}{dt} = [A]\{u\} + \{b\} \tag{18}$$

where $\{u\}$ is an unknown vector of the functional values at the grid points except the left and right boundary points, $\{b\}$ is a vector containing the non-homogenous part and the boundary conditions and A is the coefficient matrix. The stability of a numerical scheme for numerical integration of Eq. (18) depends on the stability of the ordinary differential Eq. (18). If the ordinary differential Eq. (18) is not stable, numerical methods may not generate converged solutions. The stability of Eq. (18) is related to the eigenvalues of the matrix A, since its exact solution is directly determined by the eigenvalues of the matrix A. When all $Re(\lambda_i) \leq 0$ for all it is enough to show the stability of the exact solution of $\{u\}$ as $t \to \infty$ where $Re(\lambda_i)$ denotes the real part of the eigenvalues λ_i of the matrix A. The matrix A in Eq. (18) is determined as $A_{ij} = -\gamma w_{ij}^{(4)} + w_{ij}^{(2)} - \alpha_i^3 + \alpha_i$ where $\alpha_i = U(x_i,t)$ [13]. The eigenvalues of matrix A should be in the stability region as shown in Figure 1 [37].

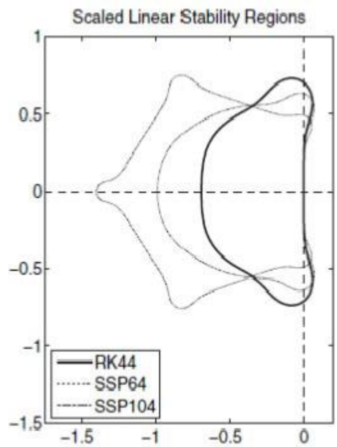

Figure 1. Stability regions of fourth order SSPRK eigenvalues.

4.1 Test Problem 1

The first test problem has initial condition as follows:

$$U(x,0) = -sin(\pi x) \qquad (19)$$

with boundary conditions

$$U(x_0,t) = U(x_N,t) = 0, \qquad (20)$$

at the domain $-4 \le x \le 4$.

We fix the number of grid points N=81 and time increment $\Delta t = 0.0001$ when $\gamma = 0$, $\gamma = 0.0001$ and $\gamma = 0.1$, respectively. Numerical simulations are given in Figures 2-4.

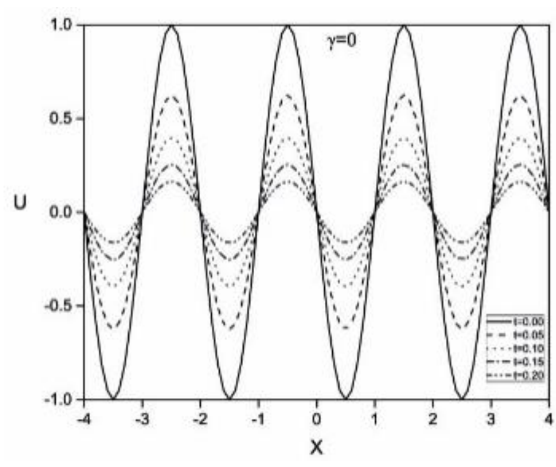

Figure 2. Simulations for $\gamma = 0$.

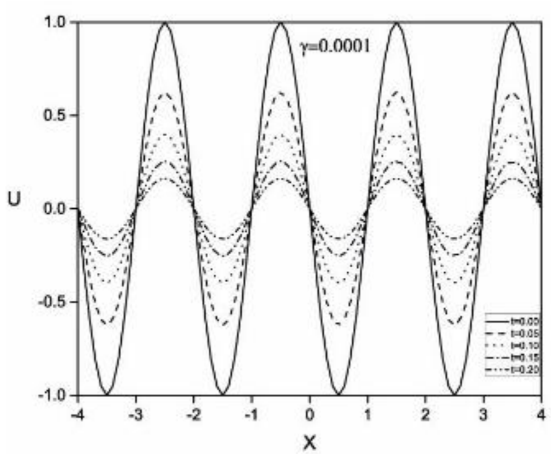

Figure 3. Simulations for $\gamma = 0.0001$.

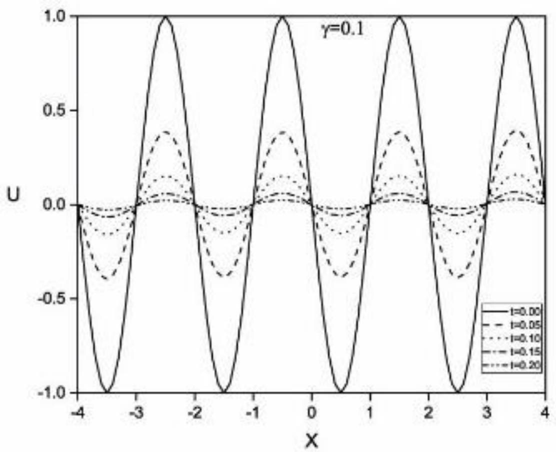

Figure 4. Simulations for $\gamma = 0.1$.

Table 1. L_2 and L_∞ error norms at $t = 0.2$.

	Present (MCB-DQM)		Quin. Coll. [10]		Quin. DQM[11]	
N	L_2	L_∞	L_2	L_∞	L_2	L_∞
20	0.01881	0.01839	0.01158	0.00551	0.02135	0.00116
40	0.00230	0.00222	0.00282	0.00134	0.00222	0.00122
80	0.00024	0.00023	0.00057	0.00028	0.00031	0.00015

As it seen straightforwardly from Figure 2 and Figure 3 that the behaviour of solutions for $\gamma = 0$ and $\gamma = 0.0001$ are similar to each other. Except for $\gamma = 0.1$ simulations given in Figure 4 that solutions decline to 0 very rapidly because of stabilizing behaviour of EFK. The calculated and compared values of the error norms L_2 and L_∞ are given in Table 1. As it is seen straightforward from comparison with earlier works [10, 11] given in Table 1 that the present error norms L_2 and L_∞ are acceptably good.

4.3. Test Problem 2

The second test problem has initial condition as follows:

$$U(x,0) = 10^{-3}exp(-x^2) \tag{21}$$

with boundary conditions

$$U(x_0,t) = U(x_N,t) = 1, \tag{22}$$

at the domain $-4 \leq x \leq 4$.

The behaviours of solutions for time running up from $t = 0.25$ to $t = 4.5$ are given in Figure 5. As it seen from Figure 5 that approximate solution of U declines as time increases and eventually it comes close to the value 1. Numerical results are agreeable to those given in [9].

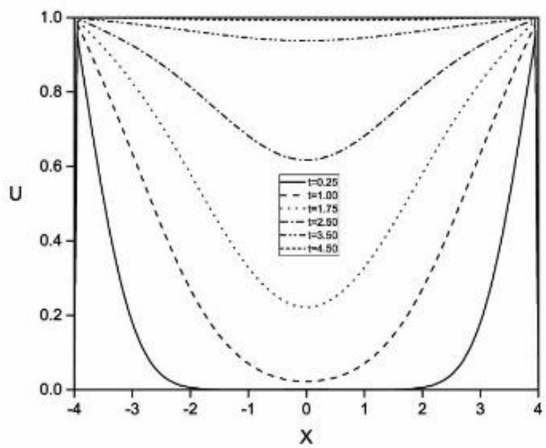

Figure 5. Simulations for $\gamma = 0.0001$.

4.3. Test Problem 3

Our last test problem has initial condition

$$U(x,0) = -10^{-3}exp(-x^2) \tag{23}$$

with boundary conditions

$$U(x_0,t) = U(x_N,t) = -1, \tag{24}$$

at the domain $-4 \leq x \leq 4$.

The behaviours of solutions for time running up from $t = 0.25$ to $t = 4.5$ are given in Figure 6. As it seen from Figure 6 that the approximate solution of U decline by time increases and eventually it come close to the value -1. Numerical results are agreeable to the as given in [9].

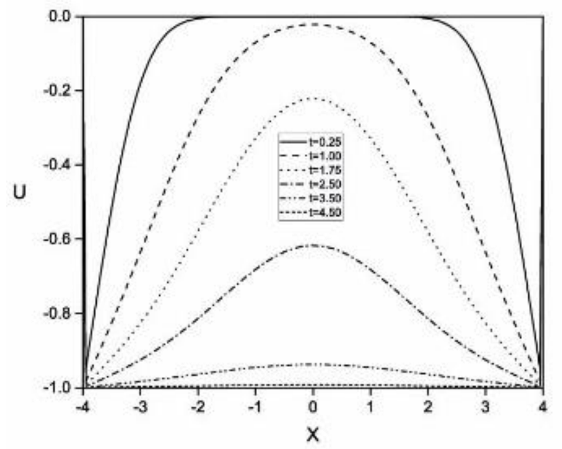

Figure 6. Simulations for $\gamma = 0.0001$.

A matrix stability analysis is also carried out for the MCB-DQM. We have used *matlab r2013b* program to obtain the eigenvalues of the coefficient matrix for the first test problem. Eigenvalues of the suggested method for *N*=21, *N*=41, *N*=81 and *N*=161 are presented in Figure 7. None of them has imaginary parts. All the eigenvalues are in convenience with stability criteria [37].

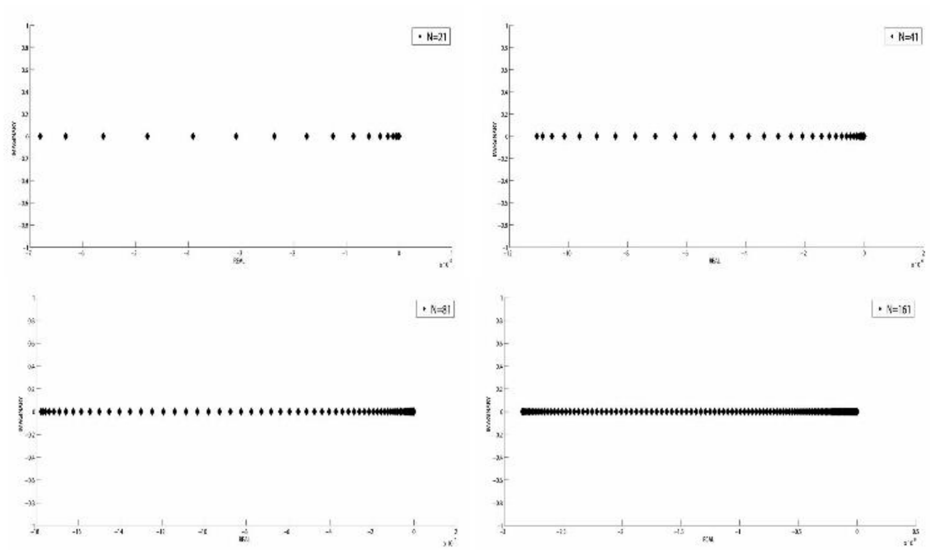

Figure 7. Eigenvalues for various number of grid points.

Also, maximum absolute value of eigenvalues for various value of $\gamma = 0$, $\gamma = 0.0001$ and $\gamma = 0.1$ for different number of grid points *N*=21 , *N*=41, *N*=81 and *N*=161 are tabulated in Table 2. All results are pure real eigenvalues when $\gamma = 0$, $\gamma = 0.0001$ and $\gamma = 0.1$. As it seen from

Table 2 when the number of the grid points are increased, absolute value of eigenvalue grows, so time step should be decrease to obtain the stable solution.

Table 2. Maximum absolute value of eigenvalues at various number of grid points.

		$\gamma = 0$		
Grid Number	21	41	81	161
$Max\|Re(\lambda)\|$	2.6×10^3	1.0×10^4	4.2×10^4	1.6×10^5
$Max\|Im(\lambda)\|$	0	0	0	0
		$\gamma = 0.0001$		
Grid Number	21	41	81	161
$Max\|Re(\lambda)\|$	3.2×10^3	2.1×10^4	2.1×10^5	3.0×10^6
$Max\|Im(\lambda)\|$	0	0	0	0
		$\gamma = 0.1$		
Grid Number	21	41	81	161
$Max\|Re(\lambda)\|$	6.8×10^5	1.1×10^7	1.7×10^8	2.8×10^9
$Max\|Im(\lambda)\|$	0	0	0	0

5. CONCLUSION

In this study, we have implemented MCB-DQM for numerical solution of EFK equation. To obtain the second order weighting coefficients, we used Shu's recurrence formulae and to obtain the fourth order weighting coefficients we used the matrix multiplication approach. The performance and accuracy of the method have been shown by calculating the error norms L_2 and L_∞. As can be observed by the comparison between the obtained values of the error norms of the present method and earlier works, MCB-DQM results are acceptable good. The obtained results show that MCB-DQM can be used to produce reasonable accurate numerical solutions of the EFK equation. As the results of investigation of eigenvalues, the present method is stable. So, MCB-DQM is a reliable one for getting the numerical solutions of some physically important nonlinear problems.

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