Research Article

SOME INTEGRAL INEQUALITIES FOR THE NEW CONVEX FUNCTIONS

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ABSTRACT

In this study, we obtained the Hermite-Hadamard integral inequality for $M_{\phi}A$-function. Then we gave a new identity for $M_{\phi}A$-function and using these identity, we obtained the theorems and the results.

Keywords: $M_{\phi}A$-function, Hermite-Hadamard type inequality.

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1. INTRODUCTION

Let $f: I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function defined on the interval $I$ of real numbers and $a < b$. The following inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

holds. This double inequality is known in the literature as Hermite-Hadamard integral inequality for convex functions. Note that some of the classical inequalities for means can be derived from (1.1) for appropriate particular selections of the mapping $f$. Both inequalities hold in the reversed direction if $f$ is concave. For some results which generalize, improve and extend the inequalities (1.1) we refer the reader to the recent papers (see [1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13]).

In [7], Varosanec got the new convex class as follow:

**Definition 1** [7] Let $f: J \subseteq [0, \infty) \rightarrow \mathbb{R}$, be a non-negative function, $h \neq 0$. We say that $f: J \subseteq [0, \infty) \rightarrow \mathbb{R}$ is an $h$-convex function, or that $f$ belongs to the class $S_X(h,I)$, if $f$ is non-negative and for all $x, y \in J$, $\alpha \in (0,1)$ we have

$$f(\alpha x + (1-\alpha)y) \leq h(\alpha)f(x) + h(1-\alpha)f(y).$$

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If inequality (1.2) is reversed, then \( f \) is said to be \( h \)-concave, i.e. \( f \in SV(h, I) \).

**Theorem 1** [7] Assume that the function \( f : C \subseteq X \to [0, \infty) \) is an \( h \)-convex function with \( h \in L[0,1] \). Let \( y, x \in C \) with \( y \neq x \) and assume that the mapping \( t \to f[(1-t)x+ty], \ t \in [0,1] \) is Lebesgue integrable on \([0,1] \). Then

\[
\frac{1}{2h(\frac{1}{2})} f(\frac{x+y}{2}) \leq \int_0^1 f[(1-t)x+ty] dt \leq \int_0^1 [f(x)+f(y)] \frac{1}{h(t)} dt .
\] (1.3)

In [5], Dragomir et.al. gave the new theorem for the Hermite-Hadamard inequality via \( P \)-function as follow:

**Definition 2** [5] A function \( f : I \subseteq [0, \infty) \to \mathbb{I} \) is said to be \( P \)-function, if

\[
f(x+(1-t)y) \leq f(x) + f(y)
\] (1.4)

for \( \forall x, y \in I, \ t \in [0,1] \).

**Theorem 2** Let \( f \in P(I), \ a, b \in I \) with \( a < b \) and \( f \in L_1[a,b] \). Then

\[
f\left(\frac{a+b}{2}\right) \leq \frac{2}{b-a} \int_a^b f(x)dx \leq 2(f(a)+f(b)) .
\] (1.5)

Both inequalities are the best possible.

In [14], Ion, D. A. revealed the new identity for quasi-convex function as follow:

**Lemma 1** Assume \( a, b \in \mathbb{I} \) with \( a < b \) and \( f : [a,b] \to \mathbb{I} \) is a differentiable function on \((a,b)\).

If \( f' \in L^1(a,b) \) then the following equality holds

\[
\frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx = \frac{b-a}{2} \int_0^1 (1-2t)f'(ta+(1-t)b)dt .
\] (1.6)

In this study, we have gotten the generalization of the (1.6) equation for \( M_\varphi A - p \)-function. We use the identity the theorems and corollary that is descent from previous study.

2. MAIN RESULTS

**Definition 3** Let \( I \) be a interval, \( \varphi : I \to \mathbb{I} \) be a continuous and strictly monotonic function.

\( f : I \to \mathbb{I} \) is said to be \( M_\varphi A - p \)-function, if

\[
f(\varphi^{-1}(t\varphi(x)+(1-t)\varphi(y))) \leq f(a)+f(b)
\] (2.1)

for all \( x, y \in I \) ve \( t \in [0,1] \).
Lemma 2 Let $f : I \subseteq [0, \infty) \rightarrow \mathbb{R}$ be a differentiable function on $I$ and $\phi : I \rightarrow \mathbb{R}$ be a continuous and strictly monotonic function and $a, b \in I$ with $0 < a < b$. If $f' \in L([a, b])$, then we get

$$\frac{f(a)+f(b)}{2} - \frac{1}{\phi(b)-\phi(a)} \int_a^b f(x)\phi'(x)dx$$

(2.2)

Proof. Firstly we use partial integration method on the right of (2.2) equality as follow

$$\int_0^1 (1-2t)\left((\phi(a)+(1-t)\phi(b))f'(\phi(a)+(1-t)\phi(b))\right)dt$$

$$= \frac{(1-2t)}{\phi(b)-\phi(a)} f\left((\phi(a)+(1-t)\phi(b))\right)\bigg|_0^1 + \frac{2}{\phi(b)-\phi(a)} \int_0^1 f\left((\phi(a)+(1-t)\phi(b))\right)\bigg|_0^b$$

$$= \frac{f(a)+f(b)}{\phi(b)-\phi(a)} - \frac{2}{\phi(b)-\phi(a)} \int_a^b f(x)\phi'(x)dx$$

If we compare both sides of the last equality with $\frac{\phi(b)-\phi(a)}{2}$, the proof is completed.

Theorem 3 Let $f : I \subseteq [0, \infty) \rightarrow \mathbb{R}$ be a differentiable function on $I$ and $a, b \in I$ with $a < b$, $\phi : I \rightarrow \mathbb{R}$ be a continuous and strictly monotonic function such that $\phi^{-1} : \phi^{-1}(f') \rightarrow (f')$ is continuously differentiable, $f' \in L([a, b])$ and $f'$ is $M_{\phi}A - p$ - function, we have

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{\phi(b)-\phi(a)} \int_a^b f(x)\phi'(x)dx\right|$$

(2.3)

$$\leq \frac{\phi(b)-\phi(a)}{2} \left[A_1(t)+A_2(t)\right]$$

where

$$A_1(t) = \int_0^{1/2} (1-2t)\left((\phi(a)+(1-t)\phi(b))\right)dt$$

(2.4)

$$A_2(t) = \int_{1/2}^1 (2t-1)\left((\phi(a)+(1-t)\phi(b))\right)dt$$

(2.5)

Proof. Firstly we take absolute value on both sides of the equality and then use the $f'$ is $M_{\phi}A - p$ - function, we get

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{\phi(b)-\phi(a)} \int_a^b f(x)\phi'(x)dx\right|$$

(2.6)
\[
\left| f(a) - f(b) \right| \leq \frac{|\varphi(b) - \varphi(a)|}{2} \left[ \int_{0}^{b-2t} \left[ \varphi^{-1}' \right]' (t\varphi(a) + (1-t)\varphi(b)) \left| f' \left( \varphi^{-1}(t\varphi(a) + (1-t)\varphi(b)) \right) \right| dt \right]
\]

\[
= \frac{|\varphi(b) - \varphi(a)|}{2} \left[ \int_{0}^{b-2t} \left[ \varphi^{-1}' \right]' (t\varphi(a) + (1-t)\varphi(b)) \left| f' \left( \varphi^{-1}(t\varphi(a) + (1-t)\varphi(b)) \right) \right| dt \right]
\]

\[
+ \int_{0}^{(1-t)a'} \left[ \varphi^{-1}' \right]' (t\varphi(a) + (1-t)\varphi(b)) \left| f' \left( \varphi^{-1}(t\varphi(a) + (1-t)\varphi(b)) \right) \right| dt
\]

\[
\leq \frac{|\varphi(b) - \varphi(a)|}{2} \left[ \int_{0}^{b-2t} \left[ \varphi^{-1}' \right]' (t\varphi(a) + (1-t)\varphi(b)) \left| f' \left( \varphi^{-1}(t\varphi(a) + (1-t)\varphi(b)) \right) \right| dt \right] \left| f'(a) \right| + \left| f'(b) \right|
\]

This proof is completed.

**Corollary 1**

**i.** If we take \( \varphi(x) = mx + n \) to (2.3), we get

\[
\left| f(a) - f(b) \right| \leq \frac{b-a}{2} \left[ \left| f'(a) \right| + \left| f'(b) \right| \right].
\]

(2.7)

**ii.** If we take \( \varphi(x) = \ln x \) to (2.3), we get

\[
\left| f\left( \sqrt{ab} \right) - \frac{1}{\ln b - \ln a} \int_{a}^{b} f(x) dx \right| \leq \frac{\ln b - \ln a}{2} \left[ B_1(t) + B_2(t) \right] \left[ \left| f'(a) \right| + \left| f'(b) \right| \right]
\]

where

\[
B_1(t) = \int_{0}^{\frac{1}{2}} (1-2t)a'b^{-t} dt,
\]

\[
B_2(t) = \int_{\frac{1}{2}}^{1} (2t-1)a'b^{-t} dt.
\]

**iii.** If we take \( \varphi(x) = x^{-1} \) to (2.3), we get

\[
\left| f\left( \frac{a}{a+b} \right) - \frac{1}{b-a} \int_{a}^{b} \frac{f(x)}{x^2} dx \right| \leq \frac{b-a}{2ab} \left[ C_1(t) + C_2(t) \right] \left[ \left| f'(a) \right| + \left| f'(b) \right| \right]
\]

where

\[
C_1(t) = \int_{0}^{\frac{1}{2}} (1-2t) \frac{(ab)^2}{(tb+(1-t)a)^2} dt,
\]

\[
C_2(t) = \int_{\frac{1}{2}}^{1} (2t-1) \frac{(ab)^2}{(tb+(1-t)a)^2} dt.
\]
Theorem 4 Let $f : I \subseteq [0, \infty) \to \mathbb{R}$ be differentiable on $I^0$ and $a, b \in I^0$ with $a < b$, \( \varphi : I \to \mathbb{R} \) be a continuous and strictly monotonic function such that \( \varphi^{-1} : \varphi(I) \to \varphi(I) \) is continuously differentiable functions. If \( |f'|^q, q > 1, \frac{1}{p} + \frac{1}{q} = 1 \) is $M_{\varphi}A - p$-function on \([a, b]\) then we get
\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{\varphi(b) - \varphi(a)} \int_a^b f(x) \varphi'(x) dx \right| \leq \frac{|\varphi(b) - \varphi(a)|}{2} \left[ D_1^{\frac{1}{p}}(t) + D_2^{\frac{1}{q}}(t) \right] \left[ \left| f'(\varphi(a) + (1-t)\varphi(b)) \right|^p \right]^{\frac{1}{q}}
\]
where
\[
D_1 = \int_0^{\frac{1}{2}} \left| \varphi^{-1}(t\varphi(a) + (1-t)\varphi(b)) \right|^q dt,
\]
\[
D_2 = \int_{\frac{1}{2}}^1 \left| \varphi^{-1}(t\varphi(a) + (1-t)\varphi(b)) \right|^q dt.
\]

Proof. By using Hölder inequality on (2.6) inequality, we get
\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{\varphi(b) - \varphi(a)} \int_a^b f(x) \varphi'(x) dx \right| \leq \frac{|\varphi(b) - \varphi(a)|}{2} \left[ \left( \int_0^{\frac{1}{2}} \left| \varphi^{-1}(t\varphi(a) + (1-t)\varphi(b)) \right|^p dt \right)^{\frac{1}{p}} + \left( \int_{\frac{1}{2}}^1 \left| \varphi^{-1}(t\varphi(a) + (1-t)\varphi(b)) \right|^q dt \right)^{\frac{1}{q}} \right].
\]

Since \( |f'|^q, q > 1 \), is $M_{\varphi}A - p$-function, we have
\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{\varphi(b) - \varphi(a)} \int_a^b f(x) \varphi'(x) dx \right| \leq \frac{|\varphi(b) - \varphi(a)|}{2^{\frac{1}{p}}(p+1)^{\frac{1}{q}}} \left[ \left( \int_0^{\frac{1}{2}} \left| \varphi^{-1}(t\varphi(a) + (1-t)\varphi(b)) \right|^p dt \right)^{\frac{1}{p}} + \left( \int_{\frac{1}{2}}^1 \left| \varphi^{-1}(t\varphi(a) + (1-t)\varphi(b)) \right|^q dt \right)^{\frac{1}{q}} \right] \left( \left| f'(\varphi(a) + (1-t)\varphi(b)) \right|^p \right)^{\frac{1}{q}}.
\]

This completed is proof.

Corollary 2 i. If we take $\varphi(x) = mx + n$ to (2.8), we obtain
\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{|b-a|}{4(p+1)^{\frac{1}{q}}} \left( \left| f'(a) \right|^p + \left| f'(b) \right|^q \right)^{\frac{1}{q}}.
\]
\(ii.\) If we take \( \phi(x) = \ln x \) to (2.8), we obtain

\[
\left| f\left(\sqrt{ab}\right) \frac{1}{\ln b - \ln a} \int_a^b f(x) \, dx \right| \leq \frac{\ln b - \ln a}{(p + 1)\frac{1}{2}} \left[ \frac{1}{B_1^q(t) + B_2^q(t)} \right] \left[ \left\| f'(a) \right\| + \left\| f'(b) \right\| \right]^{\frac{1}{q}}
\]

where

\[
E_1 = \int_0^{\frac{1}{2}} a^{q\phi}(b^{(1-t)}) \, dt,
\]

\[
E_2 = \int_{\frac{1}{2}}^1 a^{q\phi}(b^{(1-t)}) \, dt.
\]

\(iii.\) If we take \( \phi(x) = x^{-1} \), to (2.8), we obtain

\[
\left| f\left(\frac{2ab}{a+b}\right) \frac{ab}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{b-a}{2^{\frac{1}{2}(p+1)}\frac{1}{2}} \left[ \frac{1}{F_1^q(t) + F_2^q(t)} \right] \left[ \left\| f'(a) \right\| + \left\| f'(b) \right\| \right]^{\frac{1}{q}}
\]

where

\[
F_1(t) = \int_0^{\frac{1}{2}} \left( \frac{ab}{(tb+(1-t)a)^{2q}} \right) \, dt,
\]

\[
F_2(t) = \int_{\frac{1}{2}}^1 \left( \frac{ab}{(tb+(1-t)a)^{2q}} \right) \, dt.
\]

**Theorem 5** Let \( f : I \subseteq \left[0,\infty\right) \rightarrow \mathbb{R} \) be differentiable on \( I^0 \) and \( a, b \in I^0 \) with \( a < b \), \( \phi : I \rightarrow \mathbb{R} \) be a continuous and strictly monotonic function such that \( \phi^{-1} : \phi(I) \rightarrow (\phi' \text{ is continuously differentiable functions. If } \left\| f' \right\|, q \geq 1 \) is a \( M_{\phi}^A - p \)-function on \( [a,b] \) then we get

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{\phi(b) - \phi(a)} \int_a^b f(x) \phi(x) \, dx \right| \leq \frac{\left\| \phi(b) - \phi(a) \right\|}{2} \left[ \frac{1}{G_1^q(t) + G_2^q(t)} \right] \left[ \left\| f'(a) \right\| + \left\| f'(b) \right\| \right]^{\frac{1}{q}}
\]

where

\[
G_1 = \int_0^{\frac{1}{2}} \left( (\phi^{-1})' \left( (t\phi(a) + (1-t)\phi(b)) \right) \right) \, dt.
\]
\[ G_2 = \frac{1}{2} \left( 2t-1 \right) \left( \varphi^{-1} \right) \left( t\varphi(a)+(1-t)\varphi(b) \right) \int_0^t dt. \]

Proof. We use with the power mean inequality on (2.6) and the \(|f|^q, q \geq 1\), is \( M_p - \varphi \)-function then we get

\[
\left| \frac{f(a)+f(b)}{2} - \frac{1}{\varphi(b)-\varphi(a)} \int_a^b f(x)\varphi'(x)dx \right|
\leq \frac{\varphi(b)-\varphi(a)}{2} \left[ \left( 1 \right)^{\frac{1}{q}} \left( \varphi^{-1} \right) \left( (1-t)\varphi(a)+(1-t)\varphi(b) \right) \int_0^t \left| f'(\varphi^{-1}(t\varphi(a)+(1-t)\varphi(b))) \right|^q dt \right]
\]

\[
+ \left( \frac{1}{2} \right)^{\frac{1}{q}} \left( \varphi^{-1} \right) \left( (2t-1)\varphi(b) \right) \int_0^t \left| f'(\varphi^{-1}(t\varphi(a)+(1-t)\varphi(b))) \right|^q dt \right] \left[ \left| f'(a) \right|^q + \left| f'(b) \right|^q \right]^{\frac{1}{q}}
\]

**Corollary 3** 

**i.** If we take \( \varphi(x) = mx + n \) to (2.11), we obtain

\[
\left| \frac{a+b}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{2} \left[ \left| f'(a) \right|^q + \left| f'(b) \right|^q \right]^{\frac{1}{q}} \tag{2.12}
\]

**ii.** If we take \( \varphi(x) = \ln x \) to (2.11), we obtain

\[
\left| \frac{e^{ad}}{\ln b - \ln a} \int_a^b f(x)dx \right| \leq \frac{\ln b - \ln a}{2} \left[ \frac{1}{2} H_1^q(t) + \frac{1}{2} H_2^q(t) \right] \left[ \left| f'(a) \right|^q + \left| f'(b) \right|^q \right]^{\frac{1}{q}}
\]

where

\[
H_1 = \int_0^1 a^{\varphi(t)} b^{\varphi(1-t)} dt
\]

\[
H_2 = \int_{\frac{1}{2}}^1 a^{\varphi(t)} b^{\varphi(1-t)} dt
\]

**iii.** If we take \( \varphi(x) = x^{-1} \), to (2.11), we obtain
\[
\left( \frac{f(2ab)}{(a+b)} \right) \int_{a}^{b} \frac{f(x)}{x^2} \, dx \leq \frac{b-a}{2} \left[ K_1(t) + K_2(t) \right] \left[ \left| f'(a) \right|^q + \left| f'(b) \right|^q \right]^\frac{1}{q}
\]

where

\[
F_1(t) = \int_{0}^{1} \frac{(ab)^2}{(tb+(1-t)a)^{2q}} \, dt
\]

\[
F_2(t) = \int_{0}^{1} \frac{(ab)^2}{(tb+(1-t)a)^{2q}} \, dt
\]

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