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Review Paper / Derleme Makalesi

A COMMON EVALUATION OF THE MULTILAYERED COMPOSITE PLATE AND SHELLS ANALYSIS

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ABSTRACT

This review article is devoted to the developments and ideas for the analysis of multilayered composite plate and shells. In the first part; the paper presents a through review of the literature involving the use in the modeling of multilayered plates and shells. A second part reviews relevant key points that should be considered for an accurate stress and strain fields, herein referred to as C^0 requirements (zig-zag form of the displacement field in the thickness direction and continuity of transverse normal and shear stresses at each layer interface). In the third part, the paper explains one of the mixed variation principles Hellinger-Reissner (HR) in view of C^0 requirements. It is then shown that Reissner's mixed variational theorem (RMVT) can be simply constructed by adding the constraint equations (Lagrange multiplier) for the transverse stresses to Hellinger Reissner Principle (HR). The mixed form of Hooke's law has also been derived in this section. The final part of the paper is devoted to giving an overview with selected results of numerical performance that can be acquired by RMVT applications; comparison to elasticity solutions and other significant analyses, based on classical and refined approaches are given. It is concluded that RMVT leads to a better description than classical analysis formulated with only displacement variables.
Keywords: FSDT, HSDT, layerwise, laminated composite, RMVT, C^0 requirements.

TABAKALI KOMPOZİT PLAK VE KABUKLARIN ANALİZİ ÜZERİNE GENEL DEĞERLENDİRME

ÖZET

Bu derleme makalesi ile tabakalı kompozit plak ve kabukların analizinin geçmişten günümüze gelişimi ve varılan sonuçların anlatılması amaçlanmıştır. İlk bölümde; kompozit plak ve kabukların analizi üzerine yapılan çalışmalar gözden geçirilmiştir. İkinci bölüm; analizde doğru gerilme ve deplasman alanlarının dikkate alınması için gerekli anahtar noktaların neler olduğuna adanmış; C^0 gereksinimleri adı verilen bu anahtar noktaların(kalınlık doğrultusundaki deplasman alanının zig-zag biçiminde olması ile enine normal ve kayma gerilmelerinin tabaka arayüzlerinde sürekliliği) sağlanmasının gerekleri açıklanmıştır. Üçüncü bölüm, C^0 gereksinimleri ışığı altında karışık varyasyon prensiplerinden Hellinger-Reissner (HR) açıklanmasına ayrılmıştır. Sonrasında, Reissner'ın Karışık Varyasyon Teoreminin (RMVT); HR prensibindeki kısıtların Lagrange çarpanı yöntemiyle sadece kayma gerilmelerine indirgenmesiyle kolaylıkla elde edilebileceği gösterilmiştir. Ayrıca; Hooke kanunun karışık varyasyon teoremine göre düzenlenişi de bu bölümde verilmiştir. Makalenin son bölümünde RMVT ile diğer teorilerin karşılaştırmalı sonuçlarına yer verilmiştir.
Anahtar Sözcükler: FSDT, HSDT, tabaka duyarlı, tabakalı kompozit, RMVT, C^0 gereksinimleri.

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1. INTRODUCTION

Composite materials consist of two or more materials which together produce desirable properties such as stiffness, strength, corrosion resistance, thermal properties and fatique life that can not be achieved with any of the constituents alone. Some of the properties that can be improved by forming a composite material are stiffness, strength, weight reduction, corrosion resistance, thermal properties and fatique life. Due to its enhanced properties; there has been a major effort to develop composite material systems in all types of engineering structures. (e.g., aerospace,automotive as well as in bridge and building construction) in the last two decades. Examples of multilayered, anisotropic structures are sandwich constructions and composite structures made of orthotropic laminae. In most of the applications, these structures mostly appear as flat (plates) or curved panels (shells). In this section; the plate and shell theories are explained from single layer structures through multilayered ones in the literature.

First studies in the plate and shell literature are grouped as Love First Approximation Theory (LFAT) by Kirchhoff [1] and Love [2] with an assumption that normals to the reference surface Ω remain normal in the deformed states and do not change in length.Likewise, Cauchy [3] and Poisson [4] have studied in thin shell assumptions which can be assigned to the first grouping.

Reissner [5] and Mindlin [6] considered not only the work done by in-plane stresses but also the work done by transverse shear stresses in their studies and they are grouped as Love Second Approximation Theory (LSAT). Koiter [7] recommended that a refinement of Love's first approximation theory (LSAT) is indeed meaningless, unless the effects of transverse shear and normal stresses are taken into account at the same time.

Extensions of Kirchoff-Love First Approximation theory to layered structures are known as Classsical Lamination Theory.[8] Applications of LSAT theories to multilayered structures are referred as the First Order Shear Deformation Theory (FSDT) by Whitney [9]. However, the drawback of FSDT comes from the representation of the constant transverse shear strains through laminate thickness and this discrepancy between the actual quadratic stress state and the constant stress state predicted by the first order theory is often corrected in computing transverse shear force resultants by multiplying the transverse stress integrals with a shear correction coefficient parameter.

Due to the need for shear correction coefficients used in the first order theory, higher order theories are developed to have quadratic variation of the transverse shear strains and transverse shear stresses through each layer by expanding the displacement field in terms of the thickness coordinate up to any desired degree but especially third degree and this theories are referred as higher order shear deformation theory (HOT) or third order shear deformation theory (TSDT). [10]

Following Reddy [11] these types of theories such as CLT, FSDT or HSDT are grouped as Equivalent Single Layer Theories (ESLM) which have a number of unknown variables that are independent of the number of constitutive layers $\mathrm{N_L}$. In addition to their inherent simplicity and low computational cost, the ESL models often provide sufficiently accurate description of global response for thin to moderately thick laminates, e.g., gross deflections, critical buckling loads. However, the comparison of ESL models with 3-D elasticity exact solutions by Pagano [12,13], Pagano and Hatfield [14], Srinivas and Rao [15,16], Noor [17,18] shows that ESL models are often incapable of accurately describing the state of stress and strain at the ply level near geometric and material discontinuities. It is then realized that this dicrepancy is a result of the assumption taken in all equivalent single layer laminate theories that the displacements are continous and differentiable (C^1) functions of the thickness coordinate contrary to the actual zig-zag form of the laminated plates. Therefore; a possible, natural manner of including the zig-zag effect could be implemented by applying CLT, FSDT of TSDT at a layer level, that is, each layer is seen as an independent plate which is known as layerwise theory in the literature. Relevant examples of these types of theories are found in the articles by Srinivas [19], who applied CLT in each layer, and by Cho,Bert and Striz [20], who implemented the HOT by Lo,Christensen, and Wu [21] in each layer. Generalization of layerwise theory were given by Reddy [11] who expressed the diplacement variables in the thickness direction in terms of Lagrange polynomials and stated that it has to be avoided the use of Hermite interpolation functions which is kinematically incorrect for general laminates since the transverse strains are forced to be continuous through the thickness.

Providing the displacement fields zig-zag form by choosing such as not to be differentiable functions in the layer acrosses and the equality conditions of the inplane stresses (τ_{xz}, τ_{yz}) between the layers are both called as C^0 requirements and this requirements will be discussed in the second section of the article.

Although layerwise theory(LWT) has more computational effort than equivalent single layer theory (ESL); results in the literature show that LWT is more aggreable with 3-D elasticity solutions. However; neither LWT nor ESL are sufficient with providing the second C^0 requirement (interlaminar stress continuity of transverse shear and normal stresses) if the displacement field is only taken as unknowns. This situation directed the researchers studies through the mixed layerwise methods which have interlaminar transverse shear stresses as unknowns in addition to the displacement field variables.

The third part of the paper is devoted to explaining the Reissner's mixed variation theorem (RMVT) which is a special case of Hellinger-Reissner (HR) variational principle, in view of C^0 requirements.

Murakami[22-23] was the first to apply RMVT to multilayered structures by assuming two independent fields for displacement and transverse stress variables. Murakami [24-25] showed that RMVT does not experience any particular difficulty when including transverse normal stresses which is recommended by Koiter [7]. Carrera[37] made a full Equivalent Single Layer Mixed (ESLM) description in which Murakami's [23] zig-zag functions is used for displacement field through total thickness and expressed the stress variables in terms of the displacement variables by using a weak form of Hooke's law. Carrera [24-25] has also used Layerwise Mixed (LM) description which does not require any zig-zag function for the simulation of the zig-zag effects by RMVT. Large deflection of post-buckling was also analysed by Carrera and Kröplin [26]. Nonlinear dynamic problems were solved by Carrera and Krause [27]. Molerio and others. [28] studied a layerwise mixed least-squares finite element model for static analysis of multilayered composite plates.

Lekhnitskii [29] developed an approach that was originally for beams, and which describes interlaminar continuous transverse shear stress as well as zigzag effects, was extented to plates by Ren [30]. Di Sciuva [31] proposed a general quadrilateral multilayered plate element formulated on the basis of a refined third order shear deformation theory and makes use of a displacement field that fulfils a priori the geometric and stress continuity conditions at the interfaces between the layers. Idlbi, Karama and Touratier [32] compared the accuracy of three shear deformation laminated plate theories proposed by Reissner [33], Reddy [34] and Touratier's [35] sinus approach with the elasticity solution given by Pagano[12]. The interlayer continuity conditions, given in Beakou and Touratier [36] were then introduced to Touratier's [35] sinus approach that was the best model among those compared.

The first discussion on the application of Reissner's Mixed Variation Theorem to shells was made by Reissner [38]. Toledano and Murakamasi's [39-40] theory was extended to cylindrical shells by Bhaskar and Varadan [41] in which a cubic term was added to the in-plane displacement representation and the transverse shear stress field was taken to the fourth order in each layer. Jing and Tzeng [42] proposed a mixed principle which was obtained by RMVT by discarding the transverse normal stress (σ_{zz}) contribution which contrasts with Koiter's [7] recommendation. Carrera [43,44] has also proposed an extension of RMVT to shells which shows good agreement with 3-d elasticity solutions for curved panels and shells.

2. C^0 REQUIREMENTS

As can be seen from Fig. 1; multilayered composite plates could have different mechanical properties and fiber orientations in each layer.

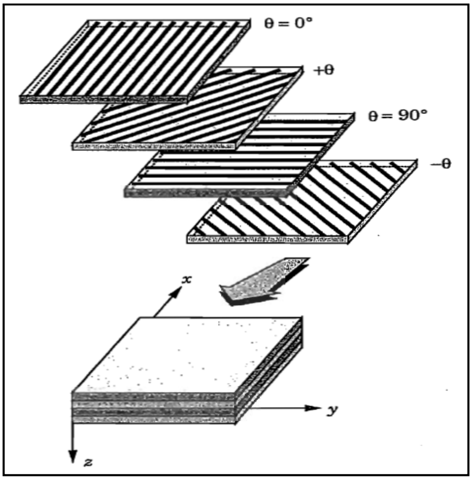

Figure 1. A laminate made up of laminae with different fiber orientations

Transverse discontinuous mechanical properties cause displacement fields **u**=(u_1, u_2, u_3) in the thickness direction which can exhibit a rapid change of their slopes in correspondence to each layer interface. Fig.2 shows how the scenarios of displacement **u** distributions in a laminated plate could appear in the exact solution or experiments. This displacement distribution is known as zigzag effect in the literature. Although in-plane stresses $\boldsymbol{\sigma}_{\mathbf{p}} = (\sigma_{xx}, \sigma_{yy}, \sigma_{xy})$ can in general be discontinuous as a result of equilibrium reasons; transverse stresses $\boldsymbol{\sigma}_{\mathbf{n}} = (\sigma_{xz}, \sigma_{yz}, \sigma_{zz})$ have to be continuous at each layer interface. This is often referred to in the literature as interlaminar continuity (IC) of transverse shear and normal stresses.

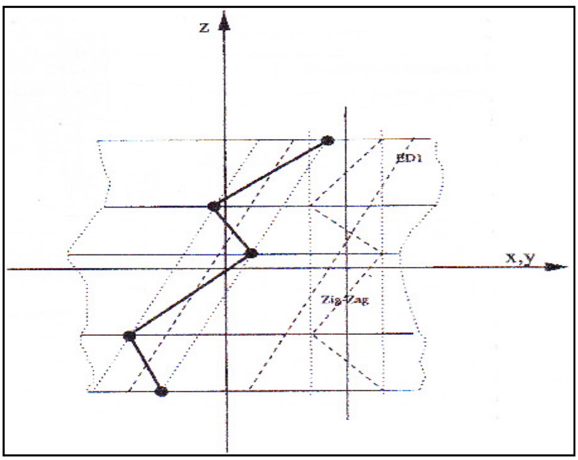

Figure 2. Zigzag form of the displacement field in multilayered plates

Zigzag effect and interlaminar continuity (IC) of transverse shear and normal stresses are both called as C^0 requirements. Fig. 3 shows that both displacement and transverse stresses, due to compatibiliy and equilibrium reasons, are C^0 continuous functions in the thickness z direction. **u** and $\boldsymbol{\sigma_n}$ have discontinuous first derivatives with correspondance to each interface where the mechanical properties change.

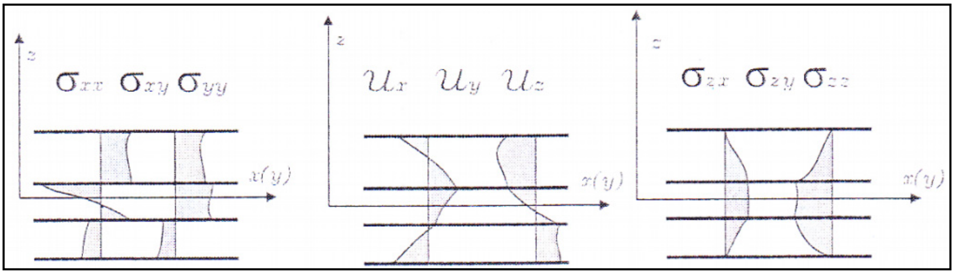

Figure 3. $\mathbf{C^0}$ requirements. Displacement and stress ($\boldsymbol{\sigma_p}$, u and $\boldsymbol{\sigma_n}$) fields in the thickness plate direction in three layered plate.

Since the zigzag effect described above can not be taken account in equivalent single layer theory (ESL); researchers directed their studies through layerwise theory (LWT). However; layerwise theory based on just displacements can not be sufficiently enough to describe the actual stress and strain state in multilayered plates and shells; therefore Reissner's mixed variational theorem (RMVT) has been the most suitable theory which takes the transverse stresses in multilayered plate and shell interfaces in addition to the displacement variables.

3. REISSNER'S MIXED VARIATION THEOREM (RMVT)

Since Reissner's mixed variation theorem (RMVT) has become the most suitable one in the recent researches in multilayered plate and shells; Hellinger-Reissner principle (HR) which is the basis of RMVT, has to be realised.

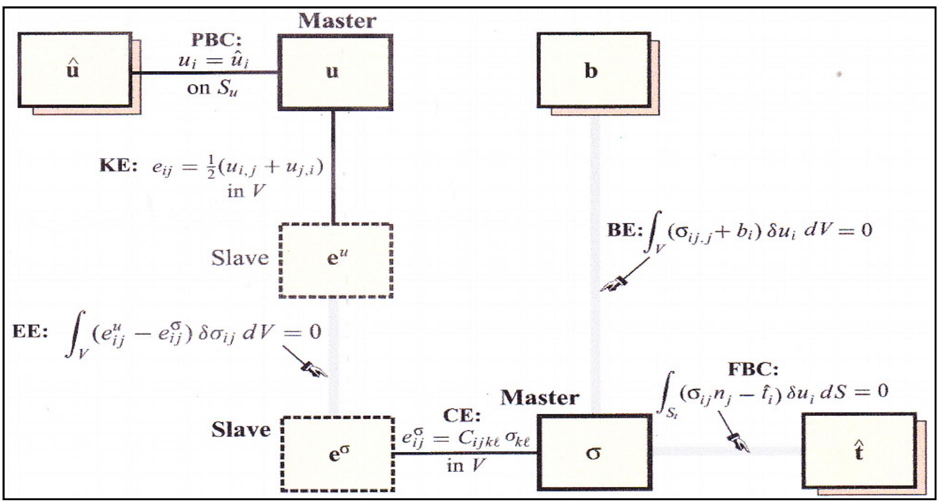

Figure 4. Tonti diagram for generalized Hellinger-Reissner (HR) principle

As can be seen from Fig. 4; displacement field **u** and the stress field **σ** are master(primary) unknowns, that could be subjected to the δ process of variational calculus, in HR principle. $\mathbf{e}^{\mathbf{u}}$ and $\mathbf{e}^{\boldsymbol{\sigma}}$ are respectively strains derived from displacement and stress masters and they are called as slaves in Tonti diagram. $\mathbf{e}^{\mathbf{u}}$ derives from its master **(u)** displacement field by using the kinematic- strain-displacement equations (KE). $\mathbf{e}^{\boldsymbol{\sigma}}$ is derived from its master **(σ)** stress variable by using the constitutive-stress-strain equations (CE). Two slave fields ($\mathbf{e}^{\mathbf{u}}$ and $\mathbf{e}^{\boldsymbol{\sigma}}$) are linked together called as weakened link since multiplied by Lagrange constraint $(\delta\sigma_{ij})$. The other weak connections are the balance equations BE (in elasticity called as the stress equilibrium equations), and the flux boundary conditions FBC (in elasticity called as the traction boundary conditions) and these appear in Fig 4. as shaded lines.

Adding the weak link contribution gives;

$$\int (e^{u}_{ij} - e^{\sigma}_{ij})\delta\sigma_{ij}dV - \int (\sigma_{ij,j} + b_i)\delta u_i dV + \int (\sigma_{ij}n_j - \hat{t}_i)\delta u_i dS_t = 0 \quad (1)$$

Next, by integrating the $\sigma_{ij,j}\delta u_i$ term by parts to eliminate the stress derivatives, splitting the surface integral dS into dS_u U dS_t ;

$$-\int \sigma_{ij,j}\delta u_i dV = \int \sigma_{ij}\delta e^{u}_{ij}dV - \int \sigma_{ij}n_j\delta u_i dS = \int \sigma_{ij}\delta e^{u}_{ij}dV - \int \sigma_{ij}n_j\delta u_i dS_u - \int \sigma_{ij}n_j\delta u_i dS_t \quad (2.a)$$

Then enforcing the boundary condition $u_i = \hat{u}_i$ and $\delta\hat{u}_i = 0$ over dS_u;

$$-\int \sigma_{ij,j}\delta u_i dV = \int \sigma_{ij}\delta e^{u}_{ij}dV - \int \sigma_{ij}n_j\delta u_i dS_t \quad (2.b)$$

Substituting Eq.(2.b) into Eq.(1) and simplification of the cancelling terms $\int \sigma_{ij}n_j\delta u_i dS_t$; the following Hellinger-Reissner variational statement is obtained;

$$\delta\Pi_{HR} = \int [\ (e^{u}_{ij} - e^{\sigma}_{ij})\delta\sigma_{ij} + \sigma_{ij}\delta e^{u}_{ij} - b_i\delta u_i\ ]\, dV - \int \hat{t}_i\,\delta u_i dS_t \quad (3)$$

As can be seen from Eq. (3); variation process (δ) is over stresses $(\delta\sigma_{ij})$ and strains (δe^{u}_{ij}) derived from (u) displacements. The *variational index* of its primary variable is the highest derivative (m) of that field that appears in the variational principle and displacement shape functions must be C^{m-1} continuous between finite elements and C^{m} inside elements in finite element method.

As can be seen from Eq. (4) that displacement variation index $m_u = 1$; since first order derivatives appear.

$$e^{u}_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad (4)$$

The variational index m_σ of the stresses is 0 because no stress derivatives appear in Eq. (3). This results in choosing displacement shape functions continous and differentiable (C^1) inside elements and only continuous (C^0) in the element boundaries. Similarly; stress shape functions must be only continuous (C^0) inside elements and (C^{-1}) in the boundaries which cause jumps as expected for inplane stresses $(\sigma_{xx}, \sigma_{yy}, \sigma_{xy})$.

Reissner's Mixed Variation Theorem (RMVT) [35]; is a special case of HR principle and as can be seen from Eq.(5) and Eq.(3); the difference between RMVT and HR principle comes from chosen Lagrange multipliers which are only out of plan stresses $(\delta\sigma_n)$ for RMVT.

$$\delta\Pi_{RMVT} = \int [\ (e^{u}_{n} - e^{\sigma}_{n})\delta\sigma_{n} + \sigma_{ij}\delta e^{u}_{ij} - b_i\delta u_i\ ]\, dV - \int \hat{t}_i\,\delta u_i dS_t \quad (5)$$

Splitting σ_{ij} term into σ_p(in-plan) and σ_n(out of plan-normal) stresses and δe^{u}_{ij} term into δe^{u}_{p} and δe^{u}_{n}; Eq. (5) becomes;

$$\delta\Pi_{RMVT} = \int [\ (e^{u}_{n} - e^{\sigma}_{n})\delta\sigma_{n} + \sigma_{p}\delta e^{u}_{p} + \sigma_{n}\delta e^{u}_{n} - b_i\delta u_i\ ]\, dV - \int \hat{t}_i\,\delta u_i dS_t \quad (6)$$

The transformed stress-strain relations of an orthotropic lamina in a plane state of stress are

$$\sigma_P = \begin{bmatrix} \sigma_{xx}^k \\ \sigma_{yy}^k \\ \sigma_{xy}^k \end{bmatrix} = \begin{bmatrix} \overline{C_{11}^k} & \overline{C_{12}^k} & \overline{C_{16}^k} \\ \overline{C_{12}^k} & \overline{C_{22}^k} & \overline{C_{26}^k} \\ \overline{C_{16}^k} & \overline{C_{26}^k} & \overline{C_{66}^k} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx}^k \\ \varepsilon_{yy}^k \\ \gamma_{xy}^k \end{bmatrix} + \begin{bmatrix} 0 & 0 & \overline{C_{13}^k} \\ 0 & 0 & \overline{C_{23}^k} \\ 0 & 0 & \overline{C_{36}^k} \end{bmatrix} \begin{bmatrix} \gamma_{xz}^k \\ \gamma_{yz}^k \\ \epsilon_{zz}^k \end{bmatrix} \tag{7}$$

where the $\overline{C_{ij}^k}$ are the transformed elastic coefficients related to the (x,y,z) coordinate system, which are related to the elastic coefficients in the material coordinates C_{ij}^k. In order to reduce the number of terms which needs differentiation; by rearranging Eq.(7) using transverse shear stresses instead of shear strains, takes the form:

$$\sigma_P = \begin{bmatrix} \sigma_{xx}^k \\ \sigma_{yy}^k \\ \sigma_{xy}^k \end{bmatrix} = \begin{bmatrix} \widehat{C_{11}^k} & \widehat{C_{12}^k} & \widehat{C_{16}^k} \\ \widehat{C_{12}^k} & \widehat{C_{22}^k} & \widehat{C_{26}^k} \\ \widehat{C_{16}^k} & \widehat{C_{26}^k} & \widehat{C_{66}^k} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx}^k \\ \varepsilon_{yy}^k \\ \gamma_{xy}^k \end{bmatrix} + \begin{bmatrix} 0 & 0 & \widehat{C_{13}^k} \\ 0 & 0 & \widehat{C_{23}^k} \\ 0 & 0 & \widehat{C_{36}^k} \end{bmatrix} \begin{bmatrix} \tau_{xz}^k \\ \tau_{yz}^k \\ \sigma_{zz}^k \end{bmatrix} \tag{8}$$

Whitney and Pagano [36] gives the relation between $\widehat{C_{ij}^k}$ in Eq.(8) and $\overline{C_{ij}^k}$ in Eq.(7) where $\sigma_1 = \sigma_x$; $\sigma_2 = \sigma_y$; $\sigma_3 = \sigma_z$; $\sigma_6 = \tau_{xy}$;

$$\sigma_i = \overline{C_{ij}}\epsilon_j \qquad (i, j = 1,2,3,6) \tag{9}$$

For i=3 in Eq.(3) takes;

$$\sigma_3 = \overline{C_{3j}}\,\epsilon_j = \overline{C_{31}}\,\epsilon_1 + \overline{C_{32}}\,\epsilon_2 + \overline{C_{33}}\,\epsilon_3 + \overline{C_{36}}\,\epsilon_6 \tag{10}$$

By solving Eq.(10) for ϵ_3;

$$\epsilon_3 = \frac{\sigma_3 - (\overline{C_{31}}\,\epsilon_1 + \overline{C_{32}}\,\epsilon_2 + \overline{C_{36}}\,\epsilon_6)}{\overline{C_{33}}} \tag{11}$$

By substituting Eq.(11) into Eq.(9);

$$\sigma_i = \left(\overline{C_{i\alpha}} - \frac{\overline{C_{i3}}\ \overline{C_{3\alpha}}}{\overline{C_{33}}}\right)\epsilon_\alpha + \frac{\overline{C_{i3}}}{\overline{C_{33}}}\sigma_3 = \widehat{C_{i\alpha}^k}\epsilon_\alpha + \widehat{C_{i3}^k}\sigma_3 \quad (i = 1,2,3,6, \alpha = 1,2,6) \tag{12}$$

is obtained.

As can be seen from Eq.(12) or in matrix form Eq.(8); in plan stresses (σ_P) are written in terms of in plan strains and out of plan shear stresses and this change reduces the differentiation needs compared with Eq.(7) which has transverse shear strains.

Similarly; Whitney and Pagano [36] writes e_n^σ matrix in Eq.(6) in terms of inplane strains (ε_p) and transverse stresses as;

$$e_n^\sigma = \begin{bmatrix} \gamma_{xz}^k \\ \gamma_{yz}^k \\ \varepsilon_{zz}^k \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \widehat{C_{13}^k} & \widehat{C_{23}^k} & \widehat{C_{36}^k} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx}^k \\ \varepsilon_{yy}^k \\ \gamma_{xy}^k \end{bmatrix} + \begin{bmatrix} \widehat{C_{55}^k} & \widehat{C_{45}^k} & 0 \\ \widehat{C_{45}^k} & \widehat{C_{44}^k} & 0 \\ 0 & 0 & \widehat{C_{33}^k} \end{bmatrix} \begin{bmatrix} \tau_{xz}^k \\ \tau_{yz}^k \\ \sigma_{zz}^k \end{bmatrix} \tag{13}$$

where

$$\widehat{C_{ii}^k} = \frac{\overline{C_{jj}^k}}{\Delta} \quad ; \quad \widehat{C_{ij}^k} = \frac{-\overline{C_{ij}^k}}{\Delta} \quad ; \quad \Delta = \overline{C_{55}^k}\,\overline{C_{44}^k} - \overline{C_{45}^k}^2 \quad ; \quad i, j(i \neq j) = 4,5 \tag{14}$$

$$\widehat{C_{33}^k} = \frac{1}{\overline{C_{33}^k}} \tag{15}$$

e_n^u and e_p^u are respectively out of plane and in plane strains in Eq.(6), which are derived from displacement field (u) and obtained by;

$$e_n^u = \begin{bmatrix} \gamma_{xz}^k \\ \gamma_{yz}^k \\ \epsilon_{zz}^k \end{bmatrix} = \begin{bmatrix} \frac{\partial u^k}{\partial z} + \frac{\partial w^k}{\partial x} \\ \frac{\partial v^k}{\partial z} + \frac{\partial w^k}{\partial y} \\ \frac{\partial w^k}{\partial z} \end{bmatrix} \quad ; \quad e_p^u = \begin{bmatrix} \varepsilon_{xx}^k \\ \varepsilon_{yy}^k \\ \gamma_{xy}^k \end{bmatrix} = \begin{bmatrix} \frac{\partial u^k}{\partial x} \\ \frac{\partial v^k}{\partial y} \\ \frac{\partial u^k}{\partial y} + \frac{\partial v^k}{\partial x} \end{bmatrix} \tag{16}$$

4. COMPARISONS OF MULTILAYERED THEORIES IN THE LITERATURE

Carrera's [34] research could be seen as the most detailed one which compares Ren [30], Di Sciuva [31], Idlbi, Karama and Touratier [32] and Pagano's [12] 3-D exact solution with his own studies obtained by LWM-m (Layerwise Model Mixed Analysis), ESLM-m (Equivalent Single Layer Model Mixed Analysis), LWM-d (Layerwise Model Displacement Analysis) and ESLM-d (Equivalent Single Layer Model Displacement Analysis).

Ren [30] extended Lekhnitskii [29]'s approach that was originally for beams, and which describes interlaminar continuous transverse shear stress as well as zigzag effects, to plates. Ren [30] expressed continuous transverse shear stresses between layers ($\tau_{xz}^{k-1} = \tau_{xz}^{k}$) and ($\tau_{yz}^{k-1} = \tau_{yz}^{k}$) as;

$$\tau_{xz}^k = \xi_x(x,y)a^k(z) + \eta_x(x,y)c^k(z) \tag{17.a}$$

$$\tau_{yz}^k = \xi_y(x,y)b^k(z) + \eta_y(x,y)g^k(z) \tag{17.b}$$

where detailed derivation of coeefficients $a^k(z)$, $b^k(z)$, $c^k(z)$ and $g^k(z)$ in Eqs. (17.a-17.b) could be obtained in Ren [30].

Di Sciuva [31] proposed a discrete-layer plate model based on the piecewise cubic approximation of the in-plane displacement across the plate thickness;

$$\widetilde{u_\alpha} = u_\alpha - z\,\partial_\alpha w + f_{\alpha\tau}(z)g_\tau \quad ; \ \alpha \text{ and } \tau = 1,2 \tag{18.a}$$

$$\widetilde{u_3} = w \tag{18.b}$$

where u_α and w are the displacements in the x_α and z directions respectively; g_τ are the transverse shear rotations for z=0. ∂_α (.) represents the partial derivative $\partial(.)/\partial\alpha$. The functions $f_{\alpha\tau}(z)$ specify the distribution of the transverse shear strains along the thickness as;

$$f_{\alpha\tau}(z) = \delta_\alpha^\tau f(z) + \textstyle\sum_{k=1}^{N-1} a_{\alpha\tau k}(z - z_k)Y_k \tag{19.a}$$

with a cubic function f(z);

$$f(z) = z\left(\delta_F - \delta_T \frac{4}{3h^2} z^2\right) \tag{19.b}$$

where δ_α^τ is the Kronecker delta ($\delta_\alpha^\tau = 1$ if $\alpha = \tau$; $\delta_\alpha^\tau = 0$ if $\alpha \neq \tau$); $a_{\alpha\tau k}$ are parameters to be determined; z_k are the coordinates of the k th layer; Y_k is the heaviside unit function (it has the value 0 for $z < z_k$ and 1 for $z \geq z_k$). The tracers δ_F and δ_T identify the contributions through classical plate theory – CLT ($\delta_F = \delta_T = 0$) ; FSDT ($\delta_F = 1, \delta_T = 0$); HSDT ($\delta_F = \delta_T = 1$); RHSDT ($\delta_F = 1, \delta_T = 0$) plate models.

Di Sciuvia [31] determined $a_{\alpha\tau k}$ parameters in Eq. (19.a) by satisfying the interlaminar continous transverse shear stresses ($\tau_{xz}^{k-1} = \tau_{xz}^{k}$) and $\left(\tau_{yz}^{k-1} = \tau_{yz}^{k}\right)$.

Another study, Carrera[34] compared the results with, belongs to Idlbi, Karama and Touratier [32] that compared the accuracy of three shear deformation laminated plate theories proposed by Reissner [33], Reddy [34] and Touratier's [35] sinus approach with the elasticity solution given by Pagano[12].

Just similar as Eq.(18.a); the kinematics suitable for shear bending is written in Idlbi, Karama and Touratier [32] as;

$$U_\alpha = -zw_{,\alpha} + f(z)\gamma_\alpha^0 \quad ; \quad \alpha = 1\,,2 \tag{20.a}$$

$U_3 = w$ (20.b)

where γ_α^0 are the transverse shear strains at the mid plane of the plate.

Idlbi, Karama and Touratier [32] compared the three shear deformation laminated plate theories by choosing f(z)=z for Reissner [33]; f(z)=z($1-4z^2/3h^2$) for Reddy [34] and f(z)=$\frac{h}{\pi}$ Sin($\frac{\pi z}{h}$) for Touratier's [35] sinus approach.

The interlayer continuity conditions, given in Beakou and Touratier [36] were then introduced to Touratier's [35] sinus approach that was the best model among those compared, in [32].

The three dimensional elasticity solutions of Pagano [12] for a simply supported, symmetric three ply (0/90/0) rectangular plate (b=3a) under sinusoidal loading are used to assess the theories of Ren [30], Di Sciuva [31], Idlbi, Karama and Touratier [32]. The layer material coefficients used in Pagano [12] are; $E_L = 25x10^6$psi (172 GPa), $E_T = 10^6$psi (7GPa), $G_{LT} = 0.5x10^6$psi (3.4 GPa), $\mu_{LT} = \mu_{LT} = 0.25$, $G_{TT} = 0.2x10^6$psi (1.4 GPa) where L signifies the direction parallel to the fibres, T the transverse direction and μ is the Poisson's ratio.

Carrera[37] made a full Equivalent Single Layer Mixed (ESLM) description in which Murakami's [23] zig-zag functions is used for displacement field through total thickness as;

$$u = u_0 + (-1)^k \zeta_k u_z + z^r u_r \quad , r = 1,2,\ldots,N \tag{21}$$

where subscript Z refers to the introduced zig-zag term in z power expansion whrere linear and higher order distributions in the z-direction are introduced by the r-polynomials and $\zeta_k = 2\, z_k/\, h_k$ is a non dimensioned layer coordinate in which z_k is the physical coordinate of the k-layer whose thickness is h_k. The exponent k in $(-1)^k$ changes the sign of the zig-zag term in each layer which permits the discontinuity of the first derivative of the displacement variables in the z-direction. As can be seen from Eq.(21) that the unknown variables u_0, u_z and u_r are layer (k) independent.

Carrera[37] expressed the stress variables in terms of the displacement variables in ESLM description by using a weak form of Hooke's law which can also be seen as the first integrand in Eq.(6) as;

$$\int (e_n^u - e_n^\sigma)\delta\sigma_n dv = 0 \tag{22}$$

By substituting Eq.(16) for e_n^u and Eq.(13) for e_n^σ in Eq.(22); it leads to a relation between transverse stress and displacement variables and the stress variables could be easily expressed in terms of displacement unknowns.

Carrera [24-25] has also used Layerwise Mixed (LM) description by RMVT which does not require any zig-zag function for the simulation of the zig-zag effects.

Carrera [34] compared his studies with exact solution of Pagano [12] and the other available analyses of Ren [30], Di Sciuva [31], Idlbi, Karama and Touratier [32], which is given in Table 1. The problem considered in Table 1. is a three layered (0/90/0) rectangular (b=3a) simply supported plate bent by a transverse bisinusoidal distribution of normal pressure (p) just as the same case in Pagano[12] mentioned above.

Acronyms have been introduced to denote the different analysis of Carrera[34] in Table 1. Three characters have been used to build up these acronyms. The first character can be L or E which states Layerwise or Equivalent Single Layer Analysis and the second one can be M or D which states Mixed or classical analysis on the basis of Displacement formulation. The third character can assume the numbers 1,2,3 or 4 which state the order N of the stress and displacement fields. For instance, LM3 means Layer-wise Mixed analysis with cubic stress and displacement field in each layer. A suffix has been added to the acronyms for the transverse stress results as –A, -H and –I which denote stresses obtained by the assumed model, by Hooke's law and by integration of the 3-D indefinite equilibrium equations, respectively.

Calculations are performed for normalized central deflection $\bar{w}=w \times 100 \frac{E_T h^3}{pa^4}$ and normalized shear stresses $\overline{\tau_{xz}} = \tau_{xz}/(\frac{pa}{h})$ where E_T is transverse elasticity modulus; h is the total thickness of the plate; a is the width of the plate and p is the normal pressure applied.

Table 1. Comparisons of multilayered plate analyses with exact solution [12]

†	$\bar{w}=w \times 100 \frac{E_T h^3}{pa^4}$				$\overline{\tau_{xz}} = \tau_{xz}/(\frac{pa}{h})$		
a/h	**4**	**10**	**20**		**4**	**10**	**20**
Exact [12]	2.820	0.919	0.610		0.387	0.420	0.434
[32]	2.729	0.918	0.609		0.378	0.441	0.451
[31]-linear	2.717	0.881	0.599		0.366	0.419	-----
[31]-cubic	2.757	0.919	0.610		0.329	0.420	-----
[30]	2.800	0.920	-----		0.317	0.415	-----
Layerwise Model Mixed Analysis (LM)							
LM4	2.821	0.919	0.610	A	0.386	0.420	0.434
				H	0.386	0.420	0.434
LM3	2.822	0.919	0.610	A	0.387	0.420	0.434
				H	0.473	0.426	0.439
				I	0.387	0.420	0.434
LM2	2.825	0.919	0.610	A	0.394	0.422	0.435
				H	0.396	0.421	0.434
				I	0.391	0.420	0.434
LM1	2.730	0.910	0.608	A	0.347	0.417	0.433
				H	0.365	0.458	0.479
				I	0.396	0.418	0.434
Layerwise Model Displacement Analysis (LD)							
LD4	2.821	0.919	0.610	H	0.386	0.420	0.434
				I	0.387	0.421	0.434
LD3	2.821	0.919	0.610	H	0.390	0.420	0.434
				I	0.386	0.420	0.434
LD2	2.798	0.918	0.610	H	0.459	0.419	0.434
				I	0.389	0.420	0.434
LD1	2.721	0.899	0.604	H	0.356	0.420	0.434
				I	0.395	0.421	0.435
Equivalent Single Layer Model Mixed Analysis (EM)							
EM3	2.815	0.919	0.610	A	0.422	0.427	0.441
				H	0.442	0.426	0.440
EM2	2.767	0.906	0.606	A	0.365	0.429	0.444
				H	0.355	0.424	0.438
EM1	2.839	0.915	0.606	A	0.399	0.459	0.476
				H	0.368	0.435	0.450
Equivalent Single Layer Model Displacement Analysis (ED)-Not included Zig-zag Effect							
ED4	2.625	0.867	0.595	H	0.612	0.307	0.312
				I	0.378	0.427	0.596
ED3	2.627	0.867	0.595	H	0.613	0.307	0.312
				I	0.378	0.427	0.436
ED2	2.035	0.752	0.566	H	0.399	0.158	0.158
				I	0.437	0.439	0.439
ED1	2.051	0.750	0.563	H	0.414	0.158	0.158
				I	0.437	0.439	0.439

† $E_L/E_T = 25,\ G_{LT}/E_T = G_{Lz}/E_T = 0.50,\ G_{TT}/E_T = 0.20, \upsilon_{LT} = \upsilon_{Lz} = \upsilon_{TT} = 0.25$

5. SUMMARY AND CONCLUDING REMARKS

The following points could be obtained as a result of the review of multilayered structure literature as;

- Although layerwise theories (LD and LM) suffer from major computational drawback due to the heavy algebra; they lead to excellent aggreement with respect to three dimensional exact solutions.
- Mixed models either ESLM-mixed (Equivalent Single Layer Model – mixed) or LWM-mixed (Layerwise Model – mixed) show better performance than classical displacement models. (LWM-displacement and ESLM-displacement)
- Since ESLM – displacement type analysis does not describe zig-zag effects; are not in sufficient aggrement to the exact solution and needs a posteriori procedure for computing transverse shear stresses. The accuracy of ESLM analysis is also very much subordinate to laminate lay-outs and to the mechanical properties of the lamina.

In conclusion; Layerwise Mixed Model by Reissner's Mixed Variational Theorem (RMVT) that is, post processing procedures are not required for interlaminar transverse shear stresses, leads to better three dimensional description of stress and strain fields of multilayered structures than classical analysis formulated with only displacement variables.

LIST OF SYMBOLS AND ACRONYMS

Symbols

a, b, h	: Plate / shell geometrical parameters (length, width and thickness)
N_L	: Number of constituent layers of multilayered structures
k	: Sub/super-script used to denote parameters related to the k-layer
σ_p	: In-plane stresses
σ_n	: Normal (transverse) stresses
e^u	: Strain matrix derived from displacements (u)
e^σ	: Strain matrix derived from stresses (σ)
E_L	: Longitudinal elasticity modulus
E_T	: Transverse elasticity modulus
G_{LT}	: Shear modulus in the longitudinal-transverse plane

Acronyms

RMVT	: Reissner's Mixed Variation Theorem
LWT	: Layerwise Theory
ESLT	: Equivalent Single Layer Theory
FSDT	: First Order Shear Deformation Theory
HSDT	: Higher Order Shear Deformation Theory
CLT	: Classical Lamination Theory
IC	: Interlaminar Continuity
ZZ	: Zig-Zag Effect
KE	: Kinematic Equations
PBC	: Primary Boundary Conditions
CE	: Constitutive Equations
FBC	: Flux Boundary Conditions

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