σ Journal of Engineering and Natural Sciences
Mühendislik ve Fen Bilimleri Dergisi
Sigma
2005/1

FREE VIBRATION ANALYSIS OF TIMOSHENKO BEAMS UNDER VARIOUS BOUNDARY CONDITIONS

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Geliş/Received: 14.09.2004 Kabul/Accepted: 02.02.2005

ABSTRACT

Free vibration of Timoshenko beams having different boundary conditions is analyzed. The Lagrange equations are used to examine the free vibration characteristics of Timoshenko beams. The constraint conditions of supports are taken into account by using Lagrange multipliers. In the study, for applying the Lagrange equations, trial functions denoting the deflection and the rotation of the cross-section of the beam are expressed in the polynomial form. By using the Lagrange equations, the problem is reduced to the solution of a system of algebraic equations. The first eight eigenvalues of Timoshenko beam are calculated and tabulated for different thickness-to-length ratios. It is believed that the tabulated results will prove useful to designers and provide a reference against which other researchers can compare their results.
Keywords: Free Vibrations of Timoshenko Beams, Lagrange Equations, Lagrange Multipliers.

TİMOSHENKO KİRİŞLERİNİN SERBEST TİTREŞİMLERİNİN FARKLI MESNET ŞARTLARI ALTINDA İNCELENMESİ

ÖZET

Bu çalışmada farklı mesnet şartlarına sahip Timoshenko kirişlerinin serbest titreşimleri incelenmiştir. Problemin çözümü için Lagrange denklemleri kullanılmıştır. Problemde mesnet şartları Lagrange çarpanları kullanılarak sağlanmıştır. Lagrange denklemlerinin uygulanması için kirişin düşey yerdeğiştirmelerini ve kiriş kesitlerinin dönmelerini ifade eden çözüm fonksiyonlarının oluşturulmasında polinomlar kullanılmıştır. Lagrange denklemleri kullanılarak problem cebrik denklem sisteminin çözümüne indirgenmiştir. Timoshenko kirişinin ilk sekiz moduna ait özdeğerleri farklı narinlik oranları (kiriş yüksekliği/kiriş açıklığı) için tablolar halinde verilmiştir. Tablolaştırılan sonuçların tasarımcılar için faydalı olacağı ve diğer araştırmacıların sonuçlarını karşılaştırmada referans olacağı düşünülmektedir.
Anahtar Sözcükler: Timoshenko kirişlerinin serbest titreşimleri, Lagrange denklemleri, Lagrange çarpanları.

1. INTRODUCTION

Vibrations of beams are of considerable interest to the engineers designing mechanical and structural systems. Many researchers have investigated the free vibration analysis of beams having various boundary conditions and based on the Bernoulli-Euler beam theory (for example [1-4]). The well-known Bernoulli-Euler beam theory states that plane sections remain plane after deformation, regarding transverse shear strain to be neglected. Although this theory is very useful for slender beams and columns, it does not give accurate solutions for thick beams. In the

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Timoshenko beam theory, the normality assumption of the Bernoulli-Euler theory is relaxed and a constant state of transverse shear strain with respect to the thickness coordinate is included. The Timoshenko beam theory requires shear correction factors to compensate for the error due to this constant shear stress assumption.

Lee and Schultz [5] applied the pseudospectral method to the eigenvalue analysis of Timoshenko beams and axisymmetric circular Mindlin plates. In [5], clamped, simply, free and sliding boundary conditions of Timoshenko beams are treated and numerical results are presented for different thickness-to-length ratios. Zhou [6] used the Rayleigh-Ritz method for the free vibration of multi-span beams. In [6], the static Timoshenko beam functions which are composed of a set of transverse deflection functions and a set of rotational angle functions are developed as the trial functions. Rossi et al. [7] have solved analytically the problem of free vibrations of beams carrying elastically mounted concentrated masses. Farghaly [8] has investigated the natural frequencies and the critical buckling load coefficients for multi-span Timoshenko beam.

In the present study, the free vibration of Timoshenko beams is analyzed by using the Lagrange equations with the trial functions in the polynomial form denoting the deflection and the rotation of the cross-section of the beam. The constraint conditions of the supports are taken into account by using Lagrange multipliers. The convergence study is based on the numerical values obtained for various numbers of polynomial terms. In the numerical examples, the first eight eigenvalues of the Timoshenko beams are determined for the different thickness-to-length ratios. The accuracy of the results is established by comparison with previously published accurate results for the free vibration analysis of the Timoshenko beams.

2. THEORY AND FORMULATIONS

Consider a straight uniform single-span Timoshenko beam of length L, depth h and width b, having rectangular cross-section depicted in Fig. 1. A Cartesian coordinate system (x, y, z) is defined on the central axis of the beam, where the x axis is taken along the central axis, with the y axis in the width direction and the z axis in the depth direction. Also, the origin of the coordinate system is chosen at the mid-point of the total length of the beam.

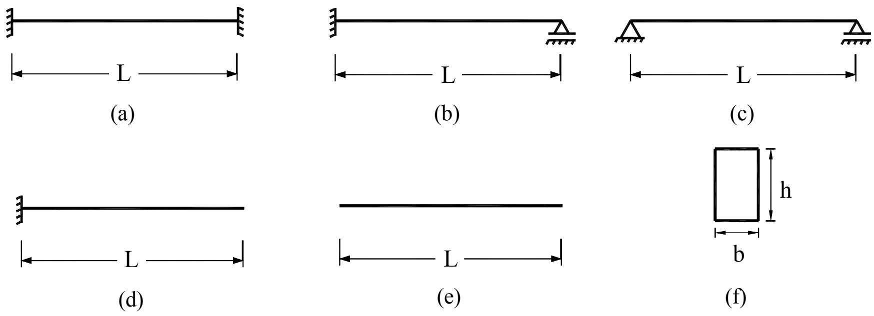

Figure 1. (a) Clamped-clamped, (b) clamped-pinned, (c) pinned-pinned, (d) clamped-free, (e) free-free Timoshenko beams, (f) cross-section of the beams

The Timoshenko beam theory is based on the following displacement fields

$$u_x(x,z,t) = -z\psi(x,t)$$
$$u_z(x,z,t) = u_z(x,t)\,, \tag{1}$$

where $u_z(x,t)$ is the transverse displacement of a point on the beam reference plane and $\psi(x,t)$ is the rotation of a normal to the reference plane about *y*-axis.
The strains and stresses in the Timoshenko beam theory are

$$e_x = -z\frac{d\psi}{dx}, \quad \gamma_{xz} = \frac{du_z}{dx} - \psi \tag{2a}$$

$$\sigma_{xx} = E\,e_x, \quad \sigma_{xz} = k_s\,G\,\gamma_{xz} \tag{2b}$$

where *E* is the Young's modulus, *G* is the transverse shear modulus and k_s is a constant that accounts for non-uniform shear stress distribution through the thickness.
The strain energy of the beam in Cartesian coordinates is

$$U = \frac{1}{2}\int_{-L/2}^{L/2}\int_A (\sigma_{xx}\,e_x + \sigma_{xz}\,\gamma_{xz})\;dA\;dx\;. \tag{3}$$

Substituting Eq. (2b) into Eq. (3) leads to

$$U = \frac{b}{2}\int_{-L/2}^{L/2}\int_{-h/2}^{h/2} (e_x\,E\,e_x + k_s\,\gamma_{xz}G\,\gamma_{xz})\;dz\;dx\;. \tag{4}$$

With the help of Eqs. (2a) and (4), the strain energy of the beam at any time can be expressed as

$$U = \frac{1}{2}\int_{-L/2}^{L/2} D_{xx}\left(\frac{d\psi(x,t)}{dx}\right)^2 dx + \frac{1}{2}\int_{-L/2}^{L/2} k_s A_{xz}\left(\frac{du_z(x,t)}{dx} - \psi(x,t)\right)^2 dx \tag{5}$$

where

$$D_{xx} = b\int_{-h/2}^{h/2} E\,z^2\,dz, \quad A_{xz} = b\int_{-h/2}^{h/2} G\;dz\;. \tag{6}$$

D_{xx} and A_{xz} in Eq. (6) can be expressed as follows;

$$D_{xx} = EI(x), \quad A_{xz} = GA(x) \tag{7}$$

where $I(x)$ and $A(x)$ are the moment of inertia and the area of the cross-section. Rewriting Eq. (5) at any time in terms of the above expression gives

$$U = \frac{1}{2}\int_{-L/2}^{L/2}\left[EI(x)\left(\frac{d\psi(x,t)}{dx}\right)^2 + k_s GA(x)\left(\frac{du_z(x,t)}{dx} - \psi(x,t)\right)^2\right] dx\;. \tag{8}$$

It follows from Eq. (1) that the velocities take the form

$$v_x = \frac{du_x(x,z,t)}{dt} = -z\frac{d\,\psi(x,t)}{dt}, \quad v_z = \frac{du_z(x,t)}{dt}\;. \tag{9}$$

The kinetic energy of the beam at any time is

$$T = \frac{1}{2}\int_{-L/2}^{L/2}\int_A \left[\rho(z)(v_x^2 + v_z^2)\right] dA\;dx\;, \tag{10}$$

where $\rho(z)$ is the mass of the beam per unit volume. Substituting Eq. (9) into Eq. (10) leads to

$$T = \frac{b}{2}\int_{-L/2}^{L/2}\int_{-h/2}^{h/2}\left[z^2\left(\frac{d\,\psi(x,t)}{dt}\right)^2 + \left(\frac{du_z(x,t)}{dt}\right)^2\right]\rho(z)\,dz\,dx\;. \tag{11}$$

By defining the following cross-sectional inertial coefficients

$$\left(J_A,\ J_D\right) = b\int_{-h/2}^{h/2}(1,\ z^2)\,\rho(z)\,dz \tag{12}$$

the kinetic energy of the beam at any time is

$$T = \frac{1}{2}\int_{-L/2}^{L/2} J_A\left(\frac{du_z(x,t)}{dx}\right)^2 dx + \frac{1}{2}\int_{-L/2}^{L/2} J_D\left(\frac{d\psi(x,t)}{dx}\right)^2 dx\ . \tag{13}$$

The J_A and J_D expressions are given as follows by using the moment of inertia *I* and the area *A* of the cross-section;

$$J_A = \rho A(x),\quad J_D = \rho I(x)\ . \tag{14}$$

Rewriting Eq. (13) at any time by using Eq. (14) gives

$$T = \frac{1}{2}\int_{-L/2}^{L/2}\left[\rho A(x)\left(\frac{du_z(x,t)}{dt}\right)^2 + \rho I(x)\left(\frac{d\psi(x,t)}{dt}\right)^2\right] dx\ . \tag{15}$$

The functional of the problem is

$$I = T - U\ . \tag{16}$$

Introducing the following non-dimensional parameters

$$x_1 = \frac{x}{L},\quad \overline{w} = \frac{u_z}{L},\quad \overline{\psi} = \psi\ , \tag{17}$$

the potential and kinetic energy of the beam can be written at any time as

$$U = \frac{1}{2}\int_{-1/2}^{1/2}\left[\frac{EI(x_1)}{L}\left(\frac{d\overline{\psi}(x_1,t)}{dx_1}\right)^2 + k_s GA(x_1)L\left(\frac{d\,\overline{w}(x_1,t)}{dx_1} - \overline{\psi}(x_1,t)\right)^2\right] dx_1\ , \tag{18}$$

$$T = \frac{1}{2}\int_{-1/2}^{1/2}\left[\rho A(x_1)L^3\left(\frac{d\overline{w}(x_1,t)}{dt}\right)^2 + \rho I(x_1)L\left(\frac{d\overline{\psi}(x_1,t)}{dt}\right)^2\right] dx_1\ . \tag{19}$$

It is known that some expressions satisfying geometrical boundary conditions are chosen for $\overline{w}(x_1,t)$ and $\overline{\psi}(x_1,t)$ and by using the Lagrange equations, the natural boundary conditions are also satisfied. Therefore, by using the Lagrange equations and by assuming the transverse displacement $\overline{w}(x_1,t)$ and the rotation of cross-sections $\overline{\psi}(x_1,t)$ to be representable by a linear series of admissible functions and adjusting the coefficients in the series to satisfy the Lagrange equations, approximate solutions are found for the displacement and the rotation functions. For applying the Lagrange equations, the trial functions $\overline{w}(x_1,t)$ and $\overline{\psi}(x_1,t)$ are approximated by space-dependent polynomial terms $x_1^0,\ x_1^1,\ x_1^2,\dots,x_1^M$ and time-dependent generalized displacement coordinates $\overline{A}_m(t)$ and $\overline{B}_m(t)$. Thus

$$\overline{w}(x_1,t) = \sum_{m=0}^{M}\overline{A}_m(t)\,x_1{}^m \tag{20a}$$

$$\overline{\psi}(x_1,t) = \sum_{m=0}^{M}\overline{B}_m(t)\,x_1{}^m\ . \tag{20b}$$

The constraint conditions of the supports are satisfied by using the Lagrange multipliers. Therefore, it is not necessary at first for these functions to satisfy the geometrical boundary conditions. As it is known, there is no need for these functions to satisfy the natural boundary conditions. However, if the functions are chosen to satisfy the natural boundary conditions, rate of convergence also increase. The constraint conditions of the beams are given as follows:
For the clamped-clamped beam (Fig. 1a)

$$\bar{w}\left(x_{1S_1}, t\right)=0,\ \bar{w}\left(x_{1S_2}, t\right)=0,\ \bar{\psi}\left(x_{1S_1}, t\right)=0,\ \bar{\psi}\left(x_{1S_2}, t\right)=0\ , \tag{21a}$$

for the clamped-pinned beam (Fig. 1b)

$$\bar{w}\left(x_{1S_1}, t\right)=0,\ \bar{w}\left(x_{1S_2}, t\right)=0,\ \bar{\psi}\left(x_{1S_1}, t\right)=0\ , \tag{21b}$$

for the pinned-pinned beam (Fig. 1c)

$$\bar{w}\left(x_{1S_1}, t\right)=0,\ \bar{w}\left(x_{1S_2}, t\right)=0\ , \tag{21c}$$

for the clamped-free beam (Fig. 1d)

$$\bar{w}\left(x_{1S_1}, t\right)=0,\ \bar{\psi}\left(x_{1S_1}, t\right)=0\ , \tag{21d}$$

and there is no constraint conditions for the free-free beam (Fig. 1e). In Eqs. (21a-d), x_{1S_i} denotes the location of the i th support.

The Lagrange multipliers formulation of the considered problem necessities the construction of the Lagrangian functional. The Lagrangian functional of the problem is obtained as follows:

$$L = I + L_m \tag{22}$$

where,
for the clamped-clamped beam (Fig. 1a)

$$L_m = \alpha_1\, \bar{w}\left(x_{1S_1}, t\right) + \alpha_2\, \bar{w}\left(x_{1S_2}, t\right) + \beta_1\, \bar{\psi}\left(x_{1S_1}, t\right) + \beta_2\, \bar{\psi}\left(x_{1S_2}, t\right), \tag{23a}$$

for the clamped-pinned beam (Fig. 1b)

$$L_m = \alpha_1\, \bar{w}\left(x_{1S_1}, t\right) + \alpha_2\, \bar{w}\left(x_{1S_2}, t\right) + \beta_1\, \bar{\psi}\left(x_{1S_1}, t\right), \tag{23b}$$

for the pinned-pinned beam (Fig. 1c)

$$L_m = \alpha_1\, \bar{w}\left(x_{1S_1}, t\right) + \alpha_2\, \bar{w}\left(x_{1S_2}, t\right), \tag{23c}$$

for the clamped-free beam (Fig. 1d)

$$L_m = \alpha_1\, \bar{w}\left(x_{1S_1}, t\right) + \beta_1\, \bar{\psi}\left(x_{1S_1}, t\right), \tag{23d}$$

In Eqs. (23a-d), α_i, β_i quantities are the Lagrange multipliers which are the support force reactions and support moment reactions in the considered problem.

The Lagrange equations are given as follows;

$$\frac{\partial L}{\partial \Omega_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\Omega}_k} = 0\ , \qquad k = 1, 2, ..., 2M + N \tag{24}$$

where the overdot stands for the partial derivative with respect to time, *N* is the number of the Lagrange multipliers and

$$\Omega_k = \bar{A}_k \qquad k = 1, 2, ..., M$$
$$\Omega_k = \bar{B}_k \qquad k = M+1, ..., 2M \tag{25a}$$

and, for the clamped-clamped beam (Fig. 1a)

$$\Omega_{2M+1} = \alpha_1 \,,\; \Omega_{2M+2} = \alpha_2 \,,\; \Omega_{2M+3} = \beta_1 \,,\; \Omega_{2M+4} = \beta_2 \tag{25b}$$

for the clamped-pinned beam (Fig. 1b)

$$\Omega_{2M+1} = \alpha_1 \,,\; \Omega_{2M+2} = \alpha_2 \,,\; \Omega_{2M+3} = \beta_1 \,,\; \Omega_{2M+4} = 0 \tag{25c}$$

for the pinned-pinned beam (Fig. 1c)

$$\Omega_{2M+1} = \alpha_1 \,,\; \Omega_{2M+2} = \alpha_2 \,,\; \Omega_{2M+3} = 0 \,,\; \Omega_{2M+4} = 0 \tag{25d}$$

for the clamped-free beam (Fig. 1d)

$$\Omega_{2M+1} = \alpha_1 \,,\; \Omega_{2M+2} = \beta_1 \,,\; \Omega_{2M+3} = 0 \,,\; \Omega_{2M+4} = 0 \tag{25e}$$

for the free-free beam (Fig. 1e)

$$\Omega_{2M+1} = 0 \,,\; \Omega_{2M+2} = 0 \,,\; \Omega_{2M+3} = 0 \,,\; \Omega_{2M+4} = 0 \,. \tag{25f}$$

The time-dependent generalized displacement coordinates for the free vibration of the beam can be expressed as follows:

$$\bar{A}_m(t) = A_m \, e^{i\,\omega\,t} \,, \tag{26a}$$

$$\bar{B}_m(t) = B_m \, e^{i\,\omega\,t} \,. \tag{26b}$$

In Eqs. (26a-b), $\bar{A}_m$ and $\bar{B}_m$ are complex variables containing a phase angle. Dimensionless amplitudes of the displacement and normal rotation of a cross-section of the beam can be expressed as follows;

$$w(x_1) = \sum_{m=0}^{M} A_m \, x_1^{\;m} \,, \tag{27a}$$

$$\psi(x_1) = \sum_{m=0}^{M} B_m \, x_1^{\;m} \,. \tag{27b}$$

Introducing the following non-dimensional parameters

$$\lambda^2 = \frac{\rho A \omega^2 L^4}{EI}, \;\; \kappa = \frac{k_s G A L^2}{EI}, \;\; \mu = \frac{h^2}{12 L^2} \tag{28}$$

and by using Eq. (24), the following simultaneous sets of linear algebraic equations are obtained which can be expressed in the following matrix forms

$$[A]\{D\} - \lambda^2 [B]\{D\} = \{0\} \,, \tag{29}$$

where $[A]$ and $[B]$ are the coefficient matrices obtained by using Eq. (24) and

In Eq. (30), $(x^{k-1})'$ is the first derivative of the x^{k-1} and the vector $\{D\}$ in Eq. (29) is defined by

$$A_{km} = \int_{-L/2}^{L/2} \kappa\,(x^{k-1})'(x^{m-1})'\mathrm{d}\,\mathrm{x} \qquad k = 1, 2, ..., M\ , \quad m = 1, 2, ..., M$$

$$A_{km} = \int_{-L/2}^{L/2} \kappa\,(x^{k-1})'(x^{m-1})\mathrm{d}\,\mathrm{x} \qquad k = 1, 2, ..., M \quad m = M+1, ...,2M$$

$$A_{km} = \int_{-L/2}^{L/2} \kappa\,(x^{k-1})(x^{m-1})'\mathrm{d}\,\mathrm{x} \qquad k = M+1, ...,2M\ , m = 1, 2, ..., M$$

$$A_{km} = \int_{-L/2}^{L/2} \left[\kappa\, x^{k-1}x^{m-1} + (x^{k-1})'(x^{m-1})'\right]\mathrm{d}\,\mathrm{x} \qquad k = M+1, ...,2M\ , m = M+1, ...,2M$$

$$B_{km} = \int_{-L/2}^{L/2} x^{k-1}x^{m-1}\mathrm{d}\,\mathrm{x} \qquad k = 1, 2, ..., M\ , \quad m = 1, 2, ..., M$$

$$B_{km} = 0 \qquad k = 1, 2, ..., M\ , \quad m = M+1, ...,2M$$

$$B_{km} = 0 \qquad k = M+1, ...,2M\ , m = 1, 2, ..., M$$

$$B_{km} = \int_{-L/2}^{L/2} \mu\, x^{k-1}x^{m-1}\mathrm{d}\,\mathrm{x} \qquad k = M+1, ...,2M\ , m = M+1, ...,2M \qquad (30)$$

$$D_k = A_k \qquad k = 1, 2, 3...., M$$
$$D_k = B_k \qquad k = M+1, ..., 2M \qquad (31)$$

The size of matrices $[A]$ and $[B]$ is $(2M+N)\times(2M+N)$ and the size of vector $\{D\}$ is $(2M+N)$**.** The total number of unknown coefficients is $(2M+N)$. Again, the number of equations which can be written by using Eq. (24) is $(2M+N)$, which is given in matrix form by Eq. (29). Therefore, the total number of these equations is equivalent to the total number of unknown coefficients and these unknowns can be determined by using above-mentioned equations. The eigenvalues (characteristic values) λ are found from the condition that the determinant of the system of equations given by Eq. (29) must vanish. Moreover, the other components of matrices $[A]$ and $[B]$ are obtained from the boundary conditions and the other components of the vector $\{D\}$ are given in appendix at the end of the paper.

3. NUMERICAL RESULTS

The first eight eigenvalues of the Timoshenko beam with clamped-clamped, clamped-pinned, pinned-pinned, clamped-free, free-free boundary conditions are given in Tables 2-6 for the different thickness-to-length ratios. In order to compare the obtained results with the existing results, the classical solutions based on the Bernoulli-Euler beam theory and the results of the Pseudospectral method given in the Ref. [5] are added to the tables. Convergence study of the Timoshenko beam with pinned-pinned boundary conditions is carried out for $h/L = 0.05$ and the results are given in Table 1.

It is not necessary to give the E, G and A values of the beam in the calculations. Referring to the relationship between the E and G as

$$\kappa = \frac{6k_s}{(1+\nu)}\frac{L^2}{h^2} \tag{32}$$

where ν is the Poisson's ratio. In all of the following calculations, the rectangular cross-sectional beams with shear correction factor $k_s = 5/6$, the Poisson's ratio $\nu = 0.3$ and different thickness-to-length ratios ranging from $h/L = 0.002$ to 0.2 are considered.

Table 1. The convergence study of the first eight dimensionless frequency parameters λ_i of the pinned-pinned Timoshenko beams for h/L=0.05.

M	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8
6	3.13536	6.24422	10.8167	15.6169	-	-	-	-
8	3.13498	6.23142	9.31090	12.4148	19.1151	-	-	-
10	3.13498	6.23136	9.25602	12.1887	15.3761	18.6607	-	-
12	3.13498	6.23136	9.25537	12.1814	15.0103	17.7638	21.3842	24.9885
14	3.13498	6.23136	9.25536	12.1813	14.9929	17.6837	20.3467	22.9914
16	3.13498	6.23136	9.25536	12.1813	14.9926	17.6810	20.2463	22.7017
18	3.13498	6.23136	9.25536	12.1812	14.9926	17.6802	20.2445	22.6809

It is observed from the Table 1 that, the natural frequencies decrease as the number of the polynomial terms increases: It means that the convergence to the exact value is from above. Namely, by increasing the number of the polynomial terms, the exact value can be approached from above. It should be remembered that energy methods always overestimate the fundamental frequency, so with more refined analyses, the exact value can be approached from above.

From here on, the number of the polynomial terms M is taken as 18 in all of the numerical investigations.

It can be deduced that the results obtained from the present study are in good aggrement with those of Lee and Schultz [5] as given in the Tables 2-4.

It is known that, the eigenvalues obtained by using first order or higher order beam theories are lower than the corresponding eigenvalues obtained by the classical beam theory. As seen from the Tables 2-6, the eigenvalues of the beams decrease with the increase of thickness-to-length ratio.

For example, the fifth eigenvalue of the pinned-pinned beam is 15.7066 for $h/L = 0.002$, it is 15.6996 for $h/L = 0.005$, 15.6749 for $h/L = 0.01$, 15.5784 for $h/L = 0.02$, 14.9926 for $h/L = 0.05$, 13.6131 for $h/L = 0.1$ and 11.2219 for $h/L = 0.2$.

However, the two solutions are very close to each other for small values of h/L. For instance, the differences between the results of the two theories are very small when h/L is less than 0.02. Moreover, the difference of the value of the eigenvalue of the classical beam theory and the Timoshenko beam theory increases for increasing mode numbers. For example, while the value of the eigenvalue of the clamped-clamped Timoshenko beam for $h/L = 0.2$ is 89 percent of the classical theory for the first mode, it is 70 for the fourth mode, and 51 for the eighth mode.

Table 2. The comparison study of the first eight dimensionless frequency parameters λ_i of the pinned-pinned Timoshenko beams for different thickness-to-length ratios.

Methods	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8
Classical Solution	3.14159	6.28319	9.42478	12.5664	15.7080	18.8496	21.9911	25.1327
h/L=0.002								
Present	3.14158	6.28311	9.42449	12.5656	15.7066	18.8471	21.9878	25.0625
PS*	3.14158	6.28310	9.42449	12.5657	15.7066	18.8473	21.9875	25.1273
h/L=0.005								
Present	3.14152	6.28265	9.42298	12.5621	15.6996	18.8351	21.9687	25.1025
PS	3.14153	6.28265	9.42298	12.5621	15.6997	18.8352	21.9684	25.0988
h/L=0.01								
Present	3.14132	6.28105	9.41760	12.5494	15.6749	18.7925	21.9013	25.0022
PS	3.14133	6.28106	9.41761	12.5494	15.6749	18.7926	21.9011	24.9988
h/L=0.02								
Present	3.14052	6.27470	9.39629	12.4993	15.5784	18.6280	21.6444	24.6249
PS	3.14053	6.27471	9.39632	12.4994	15.5784	18.6282	21.6443	24.6227
h/L=0.05								
Present	3.13498	6.23136	9.25536	12.1812	14.9926	17.6802	20.2445	22.6809
PS	3.13498	6.23136	9.25537	12.1813	14.9926	17.6810	20.2447	22.6862
h/L=0.1								
Present	3.11567	6.09066	8.84048	11.3430	13.6131	15.6769	17.5700	19.1928
PS	3.11568	6.09066	8.84052	11.3431	13.6132	15.6790	17.5705	19.3142
h/L=0.2								
Present	3.04533	5.67155	7.83949	9.65693	11.2219	12.5971	13.0323	13.4442
PS	3.04533	5.67155	7.83952	9.65709	11.2220	12.6022	13.0323	13.4443

*Pseudospectral Method in Ref. [5].

Table 3. The comparison study of the first eight dimensionless frequency parameters λ_i of the clamped-clamped Timoshenko beams for different thickness-to-length ratios.

Methods	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8
Classical Solution	4.73004	7.85320	10.9956	14.1372	17.2788	20.4204	23.5619	26.7035
h/L=0.002								
Present	4.72997	7.85294	10.9949	14.1358	17.2765	20.4166	23.5574	26.7011
PS*	4.72998	7.85295	10.9950	14.1359	17.2766	20.4168	23.5567	26.6960
h/L=0.005								
Present	4.72962	7.85161	10.9916	14.1293	17.2650	20.3983	23.5298	26.6618
PS	4.72963	7.85163	10.9917	14.1294	17.2651	20.3985	23.5292	26.6567
h/L=0.01								
Present	4.72839	7.84689	10.9799	14.1061	17.2244	20.3336	23.4328	26.5242
PS	4.72840	7.84690	10.9800	14.1062	17.2246	20.3338	23.4325	26.5192
h/L=0.02								
Present	4.72347	7.82816	10.9339	14.0154	17.0675	20.0866	23.0678	26.0130
PS	4.72350	7.82817	10.9341	14.0154	17.0679	20.0868	23.0682	26.0086
h/L=0.05								
Present	4.68987	7.70351	10.6399	13.4611	16.1586	18.7316	21.1825	23.5193
PS	4.68991	7.70352	10.6401	13.4611	16.1590	18.7318	21.1825	23.5168
h/L=0.1								
Present	4.57951	7.33121	9.85595	12.1453	14.2323	16.1478	17.9214	19.3788
PS	4.57955	7.33122	9.85611	12.1454	14.2324	16.1487	17.9215	19.5723
h/L=0.2								
Present	4.24198	6.41793	8.28526	9.90363	11.3486	12.6357	13.4567	13.8115
PS	4.24201	6.41794	8.28532	9.90372	11.3487	12.6402	13.4567	13.8101

*Pseudospectral Method in Ref. [5].

Table 4. The comparison study of the first eight dimensionless frequency parameters λ_i of the free-free Timoshenko beams for different thickness-to-length ratios.

Methods	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8
Classical Solution	4.73004	7.85320	10.9956	14.1372	17.2788	20.4204	23.5619	26.7035
h/L=0.002								
Present	4.7300	7.85303	10.9951	14.1360	17.2768	20.4165	23.5529	26.6751
PS*	4.7300	7.85304	10.9952	14.1362	17.2770	20.4174	23.5575	26.6970
h/L=0.005								
Present	4.72982	7.85216	10.9927	14.1309	17.2672	20.4012	23.5278	26.6411
PS	4.72982	7.85217	10.9928	14.1311	17.2678	20.4022	23.5341	26.6630
h/L=0.01								
Present	4.72916	7.84906	10.9841	14.1129	17.2334	20.3472	23.4402	26.5220
PS	4.72918	7.84908	10.9843	14.1131	17.2350	20.3483	23.4516	26.5436
h/L=0.02								
Present	4.72658	7.83677	10.9505	14.0424	17.1077	20.1409	23.1358	26.0772
PS	4.72659	7.83679	10.9508	14.0426	17.1078	20.1415	23.1394	26.0979
h/L=0.05								
Present	4.70873	7.75402	10.7332	13.6036	16.3500	18.9781	21.4813	23.8446
PS	4.70873	7.75404	10.7332	13.6040	16.3550	18.9813	21.4834	23.8654
h/L=0.1								
Present	4.64849	7.49717	10.1254	12.5074	14.6680	16.6352	18.4371	20.0782
PS	4.64849	7.49719	10.1255	12.5076	14.6682	16.6358	18.4375	20.0959
h/L=0.2								
Present	4.44958	6.80256	8.77284	10.4093	11.7940	12.8162	13.5583	13.6517
PS	4.44958	6.80257	8.77287	10.4094	11.7942	12.8163	13.5584	13.6520

*Pseudospectral Method in Ref. [5].

Table 5. The first eight dimensionless frequency parameters λ_i of the clamped-free Timoshenko beams for different thickness-to-length ratios.

Methods	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8
Classical Solution	1.8751	4.6941	7.8548	10.9960	14.1371	17.2787	20.4203	23.5619
h/L=0.002								
Present	1.8751	4.6940	7.8545	10.9949	14.1360	17.2765	20.4168	23.5547
h/L=0.005								
Present	1.8751	4.6937	7.8534	10.9921	14.1301	17.2662	20.3996	23.5290
h/L=0.01								
Present	1.8750	4.6927	7.8495	10.9820	14.1093	17.2294	20.3393	23.4387
h/L=0.02								
Present	1.8748	4.6888	7.8340	10.9423	14.0283	17.0871	20.1102	23.0985
h/L=0.05								
Present	1.8732	4.6620	7.7303	10.6861	13.5309	14.9025	18.9780	21.3312
h/L=0.1								
Present	1.8677	4.5724	7.4153	10.5733	12.6524	14.4452	16.1224	16.5083
h/L=0.2								
Present	1.8465	4.2852	6.6112	10.1580	12.4559	12.7887	13.3540	14.3551

Table 6. The first eight dimensionless frequency parameters λ_i of the clamped-pinned Timoshenko beams for different thickness-to-length ratios.

Methods	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8
Classical Solution	3.927	7.069	10.2101	13.352	16.4933	19.6349	22.7765	25.9181
h/L=0.002								
Present	3.9265	7.0684	10.2097	13.3508	16.4916	19.6319	22.7726	25.9165
h/L=0.005								
Present	3.9264	7.0676	10.2074	13.3458	16.4825	19.6169	22.7495	25.8825
h/L=0.01								
Present	3.9258	7.0646	10.1992	13.3283	16.4504	19.5638	22.6681	25.7638
h/L=0.02								
Present	3.9234	7.0530	10.1668	13.2595	16.3256	19.3601	22.3731	25.3580
h/L=0.05								
Present	3.9071	6.9747	9.9562	12.8306	15.5852	18.2150	20.7217	23.1063
h/L=0.1								
Present	3.8517	6.7305	9.3658	11.7583	13.9329	15.9194	17.7500	19.2987
h/L=0.2								
Present	3.6656	6.0726	8.0743	9.7860	11.2866	12.6191	13.1417	13.9660

4. CONCLUSIONS

The free vibration of the Timoshenko beams have been investigated for different thickness-to-length ratios. The obtained eigenvalues for the Timoshenko beams having various boundary conditions are compared with the previously published results. Using the Lagrange equations with the trial functions in the polynomial form and satisfying the constraint conditions by the use of Lagrange multipliers is a very good way for studying the free vibration characteristics of the beams. Numerical calculations have been carried out to clarify the effects of the thickness-to-length ratio on the eigenvalues of the beams. It is observed from the investigations that the results of the classical and the Timoshenko beam theory are very close to each other for small values of h/L. However, as the thickness-to-length ratio becomes larger, the results of the classical theory and the Timoshenko beam theory differ from each other significiantly.

All of the obtained results are very accurate and may be useful to other researchers so as to compare their results.

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APPENDIX

A1. For clamped-clamped boundary condition.

$$A_{km} = x_{S\,m}^{k-1} \qquad k = 1, 2, ..., M \;; \qquad m = 2M+1,\ 2M+2 \tag{A1}$$

$$A_{km} = x_{S\,m}^{k-1} \qquad k = M+1, ..., 2M \;; \qquad m = 2M+3,\ 2M+4 \tag{A2}$$

$$A_{km} = x_{S\,k}^{m-1} \qquad k = 2M+1,\ 2M+2 \;; \qquad m = 1, 2, ..., M \tag{A3}$$

$$A_{km} = x_{S\,k}^{m-1} \qquad k = 2M+3,\ 2M+4 \;; \qquad m = M+1, ..., 2M \tag{A4}$$

$$A_{km} = 0 \qquad k = 1, 2, ..., M \;; \qquad m = 2M+3,\ 2M+4 \tag{A5}$$

$$A_{km} = 0 \qquad k = M+1, ..., 2M \;; \qquad m = 2M+1,\ 2M+2 \tag{A6}$$

$$A_{km} = 0 \qquad k = 2M+1,\ 2M+2 \;; \qquad m = M+1, ..., 2M \tag{A7}$$

$$A_{km} = 0 \qquad k = 2M+3,\ 2M+4 \;; \qquad m = 1, 2, ..., M \tag{A8}$$

$A_{km}=0$	$k, m = 2M+1,.....\ 2M+4$		(A9)
$B_{km}=0$	$k = 1, 2, ..., 2M$;	$m = 2M+1,.....\ 2M+4$	(A10)
$B_{km}=0$	$k = 2M+1,.....\ 2M+4$;	$m = 1, 2, ..., 2M$	(A11)
$B_{km}=0$	$k, m = 2M+1,.....\ 2M+4$		(A12)
$D_k=\alpha_1$	$k = 2M+1$,		(A13)
$D_k=\alpha_2$	$k = 2M+2$		(A14)
$D_k=\beta_1$	$k = 2M+3$,		(A15)
$D_k=\beta_2$	$k = 2M+4$		(A16)

A2. For clamped-pinned boundary condition.

$A_{km}=x_{S\,m}^{k-1}$	$k = 1, 2, ..., M$;	$m = 2M+1,\ 2M+2$	(A17)
$A_{km}=x_{S\,m}^{k-1}$	$k = M+1, ..., 2M$;	$m = 2M+3$	(A18)
$A_{km}=x_{S\,k}^{m-1}$	$k = 2M+1,\ 2M+2$;	$m = 1, 2, ..., M$	(A19)
$A_{km}=x_{S\,k}^{m-1}$	$k = 2M+3$;	$m = M+1, ...,2M$	(A20)
$A_{km}=0$	$k = 1, 2, ..., M$;	$m = 2M+3$	(A21)
$A_{km}=0$	$k = M+1, ..., 2M$;	$m = 2M+1,\ 2M+2$	(A22)
$A_{km}=0$	$k = 2M+1,\ 2M+2$;	$m = M+1, ...,2M$	(A23)
$A_{km}=0$	$k = 2M+3$;	$m = 1, 2, ..., M$	(A24)
$A_{km}=0$	$k, m = 2M+1,.....\ 2M+3$		(A25)
$B_{km}=0$	$k = 1, 2, ..., 2M$;	$m = 2M+1,.....\ 2M+3$	(A26)
$B_{km}=0$	$k = 2M+1,.....\ 2M+3$;	$m = 1, 2, ..., 2M$	(A27)
$B_{km}=0$	$k, m = 2M+1,.....\ 2M+3$		(A28)
$D_k=\alpha_1$	$k = 2M+1$,		(A29)
$D_k=\alpha_2$	$k = 2M+2$		(A30)
$D_k=\beta_1$	$k = 2M+3$,		(A31)

A3. For pinned-pinned boundary condition.

$A_{km}=x_{S\,m}^{k-1}$	$k = 1, 2, ..., M$;	$m = 2M+1,\ 2M+2$	(A32)
$A_{km}=x_{S\,k}^{m-1}$	$k = 2M+1,\ 2M+2$;	$m = 1, 2, ..., M$	(A33)
$A_{km}=0$	$k = M+1, ..., 2M$;	$m = 2M+1,\ 2M+2$	(A34)
$A_{km}=0$	$k = 2M+1,\ 2M+2$;	$m = M+1, ...,2M$	(A35)
$A_{km}=0$	$k, m = 2M+1,\ 2M+2$		(A36)
$B_{km}=0$	$k = 1, 2, ..., 2M$;	$m = 2M+1,\ 2M+2$	(A37)
$B_{km}=0$	$k = 2M+1,\ 2M+2$;	$m = 1, 2, ..., 2M$	(A38)
$B_{km}=0$	$k, m = 2M+1,\ 2M+2$		(A39)
$D_k=\alpha_1$	$k = 2M+1$,		(A40)
$D_k=\alpha_2$	$k = 2M+2$		(A41)

A3. For clamped-free boundary condition.

$A_{km}=x_{S\,m}^{k-1}$	$k=1,2,....,M$;	$m=2M+1$	(A42)
$A_{km}=x_{S\,m}^{k-1}$	$k=M+1,....,2M$;	$m=2M+2$	(A43)
$A_{km}=x_{S\,k}^{m-1}$	$k=2M+1$;	$m=1,2,....,M$	(A44)
$A_{km}=x_{S\,k}^{m-1}$	$k=2M+2$;	$m=M+1,....,2M$	(A45)
$A_{km}=0$	$k=1,2,....,M$;	$m=2M+2$	(A46)
$A_{km}=0$	$k=M+1,....,2M$;	$m=2M+1$	(A47)
$A_{km}=0$	$k=2M+1$;	$m=M+1,....,2M$	(A48)
$A_{km}=0$	$k=2M+2$;	$m=1,2,....,M$	(A49)
$A_{km}=0$	$k,m=2M+1,\ 2M+2$		(A50)
$B_{km}=0$	$k=1,2,....,2M$;	$m=2M+1,\ 2M+2$	(A51)
$B_{km}=0$	$k=2M+1,\ 2M+2$;	$m=1,2,....,2M$	(A52)
$B_{km}=0$	$k,m=2M+1,\ 2M+2$		(A53)
$D_k=\alpha_1$	$k=2M+1$,		(A54)
$D_k=\beta_1$	$k=2M+2$		(A55)