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ON THE REGULARIZED TRACE OF THE DIFFERENTIAL OPERATOR EQUATION GIVEN IN A FINITE INTERVAL

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SONLU BİR ARALIKTA VERİLMİŞ DİFERANSİYEL OPERATÖR DENKLEMİNİN DÜZENLİ İZİ ÜZERİNE

ÖZET

Bu çalışmada, sonlu bir aralıkta verilmiş sınırsız operatör katsayılı ikinci mertebeden diferansiyel operatörün düzenli izi için bir formül bulunmuştur.
Anahtar Sözcükler: Hilbert uzayı, Kendine eş operator, çekirdek operatör

ABSTRACT

In this work; a formula for the regularized trace of second order differential operator, which is given in a finite interval and with unbounded operator coefficient, is found.
Keywords: Hilbert space, Self-adjoints operator, Kernel operator

1. INTRODUCTION

Let H be a separable Hilbert space. We denote the inner product in H by $(.,.)_H$ and the norm in H by $\|.\|_H$. The function f is strongly measurable belonging to H defined on $[0,\pi]$ and satisfies the condition $\int_0^\pi \|f(x)\|_H^2 \, dx < \infty$ The set of all funtions f is denoted by $H_1 = L_2(H;[0,\pi])$. If the inner product of arbitrary two elements f and g of the space H_1 is defined as

$$(f,g) = \int_0^\pi (f(x), g(x))_H \, dx \tag{1.1}$$

then H_1 becomes a separable Hilbert space, [1].The norm in the space H_1 is denoted by $\|.\|$. $s_\infty(H)$ denotes the set of compact operators from H to H. If $B \in s_\infty(H)$ then B^*B is a non-negative self-adjoint operator and $(B^*B)^{1/2} \in s_\infty(H)$, [2]. Let the non-zero eigenvalues of the

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operator $(B^*B)^{1/2}$ be $s_1 \ge s_2 \ge ... \ge s_k$ $(0 \le k \le \infty)$. Here, each eigenvalue is repeated according to multiplicity. The numbers of $s_1, s_2, ..., s_k$ are called s-numbers of the operator B. If $k < \infty$ then $s_j = 0$ $(j = k+1, k+2, ...)$ will be accepted. s-numbers of the operator B is also denoted by $s_k(B)$ (k=1,2,...). If B is a normal operator, that is B*B=BB* then $s_k(B) = |l_k(B)|$ (k=1,2,...), [2].
Here, $l_1(B), l_2(B), ..., l_k(B)$ are the non-zero eigenvalues of the operator B. s_1 or $s_1(H)$ is the set of all the operators $B \in s_\infty(H)$ the s-numbers of which satisfy the condition

$\sum_{k=1}^{\infty} s_k(B) < \infty$ .The set s_1 is a separable Banach space [2] with respect to the norm

$$\|B\|_{s_1(H)} = \sum_{k=1}^{\infty} s_k(B) \qquad B \in s_1 \tag{1.2}$$

An operator is called a kernel operator if it belongs to $s_1(H)$. If the operator $A \in s_1(H)$ and the operator $B: H \to H$ is linear and bounded then $AB, BA \in s_1(H)$ and

$$\|BA\|_{s_1(H)} \le \|B\|\|A\|_{s_1(H)} , \quad \|AB\|_{s_1(H)} \le \|B\|\|A\|_{s_1(H)} \tag{1.3}$$

[2]. If $B \in s_1(H)$ and $\{e_k\}_1^\infty \subset H$ is any orthonormal basis, the series $\sum_{k=1}^{\infty} (Be_k, e_k)_H$ is convergent and the sum of the series $\sum_{k=1}^{\infty} (Be_k, e_k)_H$ does not depend on the choice of the basis $\{e_k\}_1^\infty$. The sum of the series $\sum_{k=1}^{\infty} (Be_k, e_k)_H$ is said to be matrix trace and is denoted by trB. If $A, B \in s_1(H)$ and a, b is any scalar then

$$tr(aA + bB) = a trA + b trB \ , \ trA^* = \overline{trA} \text{ , and } trB = \sum_{k=1}^{u(A)} l_k(B) \tag{1.4}$$

In the last equality, each eigenvalue is added according to its own algebraic multiplicity number. $u(A)$ denotes the sum of algebraic multiplicity of non-zero eigenvalues of the operator B, [2]. The sum of the series $\sum_{k=1}^{u(A)} l_k(B)$ is said to be the spectral trace of the operator B.Recall that a self-adjoint operator is said to have purely-discrete spectrum if its spectrum consist of eigenvalues $\{l_i\}_1^\infty$ of finite multiplicity and $\lim_{n\to\infty} |l_n| = \infty$.

Let us consider the differential expression in the space $H_1 = L_2(H, [0, p])$

$$l_0(y) = -y''(x) + Ay(x) \tag{1.5}$$

Here, a densely defined operator $A: D(A) \to H$ in the space H satisfies the conditions

$A = A^* \ge I, \qquad A^{-1} \in s_\infty(H)$, (I is identity operator in H)

Let $g_1 \le g_2 \le ... \le g_j \le ...$ be the eigenvalues of the operator A and $j_1, j_2, ..., j_j, ...$ be the orthonormal eigenvectors corresponding to these eigenvalues.
Moreover, D_0 denotes the set of the functions y(x) satisfying the conditions:

(1) y(x) has continuous derivative of the second order with respect to the norm in the space H in the interval $[0, p]$

(2) Ay(x) is continuous with respect to the norm in the space H.

(3) $y'(0) = y'(p) = 0$

Here $\overline{D_0} = H_1$ ($\overline{D_0}$ denoted by closure of D_0) and the operator $L_0' = D_0 \to H_1$, $L_0' y = l_0(y)$ is symmetric. The eigenvalues of L_0' are $k^2 + g_j$ (k=0,1,2,...; j=1,2,...) and the orthonormal eigenvectors corresponding to these eigenvalues are $M_k \cos kx.j_j$ (k=0,1,2,...; j=1,2,...) .Here,

$$M_k = \begin{cases} \sqrt{p^{-1}} & ;if \quad k = 0 \\ \sqrt{2p^{-1}} & ;if \quad k = 1,2,... \end{cases} \tag{1.6}$$

As seen, the orthonormal eigenvectors system of the symmetric operator L_0' is an orthonormal basis in the space H_1.

Let Q(x) be an operator function satisfying the following conditions:

(1) Q(x) has weak derivative of the second order in the interval $[0, p]$. The operator function $Q''(x)$ is weakly measurable,and for every $x \in [0, p]$, $Q^{(i)}(x) : H \to H \quad (i = 0,1,2)$ are self-adjoint compact operators.

(2) The functions $\left\| Q^{(i)}(x) \right\|_{s_1(H)} \; (i = 0,1,2)$ are bounded and measurable in the interval $[0, p]$.

(3) For every $f \in H \; \int_0^p (Q(x) f, f)_H \, dx = 0$.

In this work,we find a formula for the eigenvalues of the operators $L_0 = \overline{L_0'}$ and $L = L_0 + Q$ and this formula is said to be regularized trace formula.

The regularized trace formulas for scalar differential operators are studied in [3],[4],[5] and in many other works. The list of the works on the subjects is given in [6] and [7], but a small number of these works are on the regularized trace of differential operators with operator coefficient.

In [8], the regularized trace of the Sturm-Lioville operator with bounded operator coefficient is calculated. In [9], a formula for the regularized trace of the difference of two Sturm-Lioville operators which is given in half-axis with the bounded operator coefficient is found. In [10], a formula for the regularized trace of the Sturm-Liouville operator under Dirichlet boundary conditions with unbounded operator coefficient, is found. In [11], the regularized trace of a singular differential operator of second order with bounded operator coefficient is investigated. In [12] and [13], the formulas for the regularized traces of differential operators with bounded operator coefficient are found.

2. SOME RELATIONS ABOUT THE EIGENVALUES AND RESOLVENTS

In this section, we will prove that the operators L_0 and L are self-adjoint and we will find some relations about the eigenvalues and resolvents of the operators L_0 and L .

Theorem 2.1. Every symmetric closed operator , the eigenvectors system of which is closed is self-adjoint.

Proof . Let H be a separable Hilbert space. Let $B : D(B) \to H$ be a symmetric operator with $D(B) \subset H$, $\{e_i\}_1^\infty$ be an orthonormal system consisting of the eigenvectors of the operator B and l also be an nonreal number. Since $(B-lI)^{-1}$ is a bounded closed operator , the linear manifold $D((B-lI)^{-1})=R(B-lI)$ is closed. That is, the linear manifold $R(B-lI)$ is a subspace of H. On the other hand , since the subspace $R(B-lI)$ contains the closed system $\{e_i\}_1^\infty$, then $R(B-lI)=H$ similarly , $R(B-\bar{l}I)=H$

In this case, as well known, the operator B is self-adjoint. The Theorem 2.1 is proved. •

Since the eigenvectors system of the symmetric operator L_0' is closed, according to the Theorem 2.1. the operator $L_0=\overline{L_0'}$ is self-adjoint and since the bounded operator $Q:H_1\to H_1$ is self-adjoint , the operator $L=L_0+Q$ is also self-adjoint.

The operators L_0 and L have purely-discrete spectrum. Let the eigenvalues of the operators L_0 and L be $m_1\le m_2\le\ldots\le m_n\le\ldots$ and $l_1\le l_2\le\ldots\le l_n\le\ldots$ respectively.

By using [14] , we can prove the following theorem:

Theorem 2.2. If $g_j\sim aj^a$ as $j\to\infty$ that is

$$\lim_{j\to\infty}\frac{g_j}{aj^a}=1 \text{ , then as } n\to\infty \quad l_n, m_n\sim dn^{\frac{2a}{2+a}} \qquad (d>0)$$

By using Theorem 2.2. , it is easily seen that the sequence $\{m_n\}$ has a subsequence $m_{n_1}<m_{n_2}<\ldots<m_{n_m}<\ldots$ such that

$$m_k-m_{n_m}\ge d_0\left(k^{\frac{2a}{2+a}}-n_m^{\frac{2a}{2+a}}\right) \qquad (k=n_m,n_m+1,n_m+2,\ldots)$$

Let $R_l^0=(L_0-lI)^{-1}$ and $R_l=(L-lI)^{-1}$ be the resolvents of the operators L_0 and L respectively. If $a>2$ by Theorem 2.2 , R_l^0 and R_l are compact operators for $l\ne m_n, l_n$ (n=1,2,3...) . In this case

$$tr(R_l-R_l^0)=trR_l-trR_l^0=\sum_{k=1}^{\infty}\left(\frac{1}{l_k-l}-\frac{1}{m_k-l}\right) \tag{2.1}$$

[2]. Let $b_m=2^{-1}(m_{n_m}+m_{n_m+1})$. It easy to see that for the large value of m the inequalities

$m_{n_m}<b_m<m_{n_m+1}$, $l_{n_m}<b_m<l_{n_m+1}$ are satisfied and the series

$$\sum_{k=1}^{\infty}\left(\frac{1}{l_k-l}\right) , \quad \sum_{k=1}^{\infty}\left(\frac{1}{m_k-l}\right)$$

are uniform convergent on the circle $|l|=b_m$. Therefore by (2.1)

$$\sum_{k=1}^{n_m}(l_k-m_k)=-\frac{1}{2pi}\int_{|l|=b_m} l\,tr(R_l-R_l^0)dl \tag{2.2}$$

Lemma 2.1. If $g_j\sim a\cdot j^a$ $(0<a<\infty,\ 2<a<\infty)$ as $j\to\infty$ then

$$\left\|R_l^0\right\|_{s_1(H_1)}\le const.n_m^{1-d} \qquad \left(d=\frac{a-2}{a+2}\right) \text{ on the circle } |l|=b_m .$$

Proof . For $l \notin \{m_k\}_{k=1}^{\infty}$ since R_l^0 is a normal operator then

$$\left\|R_l^0\right\|_{s_1(H_1)} = \sum_{k=1}^{\infty} \frac{1}{|m_k - l|}$$

[2]. Since $|l| = b_m = 2^{-1}(m_{n_m} + m_{n_m+1})$ then

$$\left\|R_l^0\right\|_{s_1(H_1)} \le \sum_{k=1}^{\infty} \frac{1}{\left||l| - m_k\right|} \le \sum_{k=1}^{n_m} \frac{2}{m_{n_m} + m_{n_m+1} - 2m_k} + \sum_{k=n_m+1}^{\infty} \frac{2}{2m_k - m_{n_m} - m_{n_m+1}}$$

$$\le \sum_{k=1}^{n_m} \frac{2}{m_{n_m+1} - m_k} + \sum_{k=n_m+1}^{\infty} \frac{2}{m_k - m_{n_m}} \tag{2.3}$$

is obtained. By using the Theorem 2.2, we limit the sums on the right hand side of the inequality above:

$$\sum_{k=1}^{n_m} \frac{1}{m_{n_m+1} - m_k} < \frac{n_m}{m_{n_m+1} - m_{n_m}} \le \frac{n_m}{d_0[(n_m+1)^{1+d} - n_m^{\ 1+d}]} < \frac{n_m}{d_0 n_m^d} = d_0^{-1} n_m^{1-d} \tag{2.4}$$

$$\sum_{k=n_m+1}^{\infty} \frac{1}{m_k - m_{n_m}} \le d_0^{-1} \sum_{k=n_m+1}^{\infty} \frac{1}{k^{1+d} - n_m^{1+d}}$$

$$= \frac{1}{d_0[(n_m+1)^{1+d} - n_m^{\ 1+d}]} + d_0^{-1} \sum_{k=n_m+2}^{\infty} \frac{1}{k^{1+d} - n_m^{1+d}} \tag{2.5}$$

Moreover

$$\sum_{k=n_m+2}^{\infty} \frac{1}{k^{1+d} - n_m^{1+d}} \le \int_{n_m+1}^{\infty} \frac{dx}{x^{1+d} - n_m^{1+d}}$$

and it is easily shown that $\int_{n_m+1}^{\infty} \frac{dx}{x^{1+d} - n_m^{1+d}} \le d^{-1} n_m^{-\frac{d^2}{1+d}}$.

Considering the last two inequalities in (2.5)

$$\sum_{k=n_m+1}^{\infty} \frac{1}{m_k - m_{n_m}} \le \frac{1}{d_0[(n_m+1)^{1+d} - n_m^{\ 1+d}]} + \frac{n_m^{-\frac{d^2}{1+d}}}{d_0 d} \le \frac{2}{d_0 d} \tag{2.6}$$

By (2.3) , (2.4) and (2.6)

$\left\|R_l^0\right\|_{s_1(H_1)} \le \frac{6}{d_0 d} \cdot n_m^{1-d}$ is found. Lemma 2.1 is proved • .

Lemma 2.2. If $g_j \sim a \cdot j^a$ $(0 < a < \infty,\ 2 < a < \infty)$ as $j \to \infty$ and Q is a bounded self adjoint operator from H_1 to H_1 then, $|l| = b_m$ and for the large values of m $\|R_l\| \le const \cdot n_m^{-d}$

Proof . Since the eigenvalues of the kernel operator R_l are $\left\{(l_k - l)^{-1}\right\}_{k=1}^{\infty}$ then

$$\|R_l\| = \max_k \left\{|l_k - l|^{-1}\right\} \tag{2.7}$$

For $|l| = b_m$

$$\left\| l_k \right| - \left| l \right\| = \left| \left| l_k \right| - \frac{1}{2} \cdot \left(m_{n_m} + m_{n_m+1} \right) \right| = \frac{1}{2} \left| m_{n_m} + m_{n_m+1} - 2 \left| l_k \right| \right| \tag{2.8}$$

$$m_{n_m} + m_{n_m+1} - 2\left| l_k \right| \geq m_{n_m} + m_{n_m+1} - 2\, l_{n_m} = m_{n_m+1} - m_{n_m} + 2\left(m_{n_m} - l_{n_m} \right)$$

$k \leq n_m$ and for the large values of m, since $\left| l_k \right| \langle\, l_{n_m}$ then

$$m_{n_m} + m_{n_m+1} - 2\left| l_k \right| \geq m_{n_m+1} - m_{n_m} - 2\left| m_{n_m} - l_{n_m} \right| > m_{n_m+1} - m_{n_m} - c \tag{2.9}$$

Here c is a constant. $k \geq n_m + 1$ and for the large values of m, since $\left| l_k \right| = l_k \geq l_{n_m+1}$ then

$$\begin{aligned} 2\left| l_k \right| - m_{n_m} - m_{n_m+1} &\geq 2\, l_{n_m+1} - m_{n_m} - m_{n_m+1} \\ &= 2\left(l_{n_m+1} - m_{n_m+1} \right) + m_{n_m+1} - m_{n_m} \\ &\geq m_{n_m+1} - m_{n_m} - 2\left| l_{n_m+1} - m_{n_m+1} \right| \qquad \geq m_{n_m+1} - m_{n_m} - c \end{aligned} \tag{2.10}$$

On the other hand, since $\lim\limits_{m \to \infty} \left(m_{n_m+1} - m_{n_m} \right) = \infty$ by (2.8) , (2.9) and (2.10)

$\left\| l_k \right| - \left| l \right\| \geq \frac{1}{4} \left(m_{n_m+1} - m_{n_m} \right)$ is found. By using the Theorem 2.2

$$\left\| l_k \right| - \left| l \right\| \geq \frac{d_0}{4} \left[(n_m + 1)^{1+d} - n_m^{1+d} \right] > \frac{d_0}{4} n_m^d \quad \text{or} \qquad \left\| l_k \right| - l \right|^{-1} < \frac{4}{d_0} n_m^{-d}$$

is obtained. By (2.7) and the last inequality $\left\| R_l \right\| \leq \frac{4}{d_0} \cdot n_m^{-d}$ is found. Lemma is proved.•

We will use the last two lemmas to prove the following theorem.

3. THE FORMULA FOR THE REGULARIZED TRACE OF THE OPERATOR L

This is a well known formula for the resolvents of the operators L_0 and L:

$$R_l = R_l^0 - R_l Q R_l^0 \quad (l \in r(L) \cap r(L_0)) .$$

By using the last formula and (2.2) , it can be shown that

$$\sum_{k=1}^{n_m} (l_k - m_k) = \sum_{j=1}^{p} D_{mj} + D_m^{(p)} \tag{3.1}$$

Here

$$D_{mj} = \frac{(-1)^j}{2pij} \int_{|l| = b_m} tr\left[(Q R_l^0)^j \right] dl \tag{3.2}$$

$$D_m^{(p)} = \frac{(-1)^p}{2pi} \int_{|l| = b_m} l \cdot tr\left[R_l (Q R_l^0)^{p+1} \right] dl \tag{3.3}$$

Theorem 3.1. If $g_j \sim a \cdot j^a \quad (0 < a < \infty, \ \ 2 < a < \infty)$ as $j \to \infty$, and the operator function Q(x) satisfies the conditions (1) and (2) then

$$\lim_{m \to \infty} D_{mj} = 0 \qquad (j \geq 2)$$

Proof. According to the formula (3.2)

$$\left| D_{mj} \right| \leq \frac{1}{2pj} \int_{|l| = b_m} \left| tr (Q R_l^0)^j \right| \left| dl \right| \leq \frac{1}{2pj} \int_{|l| = b_m} \left\| Q R_l^0 \right\|_{s_1(H_1)} \left\| Q R_l^0 \right\|^{j-1} \left| dl \right|$$

$$\left|D_{mj}\right| \le \frac{1}{2pj} \int_{|l|=b_m} \|Q\| \cdot \left\|R_l^0\right\|_{s_1(H_1)} \left\|QR_l^0\right\|^{j-1} |dl|$$

$$\le \frac{1}{2pj} \int_{|l|=b_m} \|Q\|^j \cdot \left\|R_l^0\right\|_{s_1(H_1)} \left\|R_l^0\right\|^{j-1} |dl| \le const \int_{|l|=b_m} \left\|R_l^0\right\|_{s_1(H_1)} \left\|R_l^0\right\|^{j-1} |dl| \qquad (3.4)$$

Since $R_l = R_l^0$ for $Q \equiv 0$ according to Lemma 2.2

$$\left\|R_l^0\right\| \le const \cdot n_m^{-d} \qquad (3.5)$$

By using Lemma 2.1 and the inequalities (3.4) , (3.5)

$$\left|D_{mj}\right| \le const \int_{|l|=b_m} n_m^{1-d} \cdot n_m^{-d(j-1)} \cdot |dl| \le const \cdot m_{n_m} \cdot n_m^{1-dj}$$

is obtained. Since $m_{n_m} \le const \cdot n_m^{1+d}$, then $\left|D_{mj}\right| \le const \cdot n_m^{2-d(j-1)}$ is found. As seen; if $j \ge \left[\left|2d^{-1}\right|\right] + 2$ then

$$\lim_{m\to\infty} D_{mj} = 0 \qquad (3.6)$$

It is necessary to show that the equality above is satisfied for j=2,3,..., $\left[\left|2d^{-1}\right|\right]+1$ to complete the proof . Let us show for j=2 . D_{m2} satisfies the following inequality

$$D_{m2} = \sum_{k=1}^{n_m} \sum_{j=n_m+1}^{\infty} \left(m_k - m_j\right)^{-1} \left(j_k, Qj_j\right)_{H_1} \left(Qj_j, j_k\right)_{H_1}$$. Therefore

$$\left|D_{m2}\right| \le \sum_{j=n_m+1}^{\infty} \left[\left(m_j - m_{n_m}\right)^{-1} \sum_{k=1}^{\infty} \left|\left(Qj_j, j_k\right)_{H_1}\right|^2\right] \qquad (3.7)$$

Since $\{j_k\}_{k=1}^{\infty}$ is an orthonormal basis in the space H_1 and the equality

$$\lim_{n\to\infty} \sum_{j=n_m+1}^{\infty} \left(m_j - m_{n_m}\right)^{-1} = 0 \qquad (3.8)$$

is satisfied, by (3.7) and (3.8) $\lim_{n\to\infty} D_{m2} = 0$ is obtained. •

Theorem 3.2. If $g_j \sim a \cdot j^a$ $(0 < a < \infty,\ 2 < a < \infty)$ as $j \to \infty$ and Q(x) satisfies the conditions (1), (2), (3) then the formula in the form

$$\lim_{m\to\infty} \sum_{k=1}^{n_m} \left(l_k - m_k\right) = \frac{1}{4}\left[trQ(0) + trQ(p)\right]$$

is satisfied for the regularized trace of the operator L.

Proof. According to the formula (3.2)

$$D_{m1} = -\frac{1}{2pi} \int_{|l|=b_m} tr(QR_l^0)\,dl \qquad (3.9)$$

Since the orthonormal eigenvectors system $\{\Psi_p(x)\}_1^{\infty}$ according to the eigenvalues $\{m_p\}_1^{\infty}$ of the operator L_0 is an orthonormal basis of the space H_1 then $tr(QR_l^0) = \sum_{p=1}^{\infty} (QR_l^0 \Psi_p, \Psi_p)_{H_1}$. If the equality above is written into the equality (3.9),

$$R_l^0 \Psi_p = (L_0 - lI)^{-1} \cdot \Psi_p = (m_p - l)^{-1} \Psi_p \text{ and } \quad m_{n_m} < b_m < m_{n_m+1}$$

are considered, then $D_{m1} = -\frac{1}{2pi} \int_{|l|=b_m} \left[\sum_{p=1}^{\infty} (m_p - l)^{-1} (Q\Psi_p, \Psi_p)_{H_1} \right] dl$

$$= \sum_{p=1}^{\infty} (Q\Psi_p, \Psi_p)_{H_1} \cdot \frac{1}{2pi} \int_{|l|=b_m} \frac{dl}{l - m_p} = \sum_{p=1}^{n_m} (Q\Psi_p, \Psi_p)_{H_1} \qquad (3.10)$$

is obtained. Since the orthonormal eigenvectors according to the eigenvalues $k^2 + g_j$ (k=0,1,2,...;j=1,2,...) of the operator L_0 are $M_k \cos kx.j_j$ (k=0,1,2,...;j=1,2,...) respectively then $\Psi_p(x) = M_{k_p} \cos k_p x.j_{j_p}$ (p=1,2,...) Therefore by (3.10)

$$D_{m1} = \sum_{p=1}^{n_m} \int_0^p \left(Q(x) M_{k_p} \cos k_p x j_{j_p}, M_{k_p} \cos k_p x j_{j_p} \right)_H dx$$

$$= \frac{1}{2} \sum_{p=1}^{n_m} M_{k_p}^2 \int_0^p (1 + \cos 2k_p x) \left(Q(x) j_{j_p}, j_{j_p} \right)_H dx$$

is found.Q(x) satisfies the condition (3) and $M_k = \sqrt{2p^{-1}}$ (k=1,2,...) is considered,by the last relation

$$D_{m1} = \frac{1}{p} \sum_{p=1}^{n_m} \int_0^p \cos 2k_p x (Q(x) j_{j_p}, j_{j_p})_H dx \qquad (3.11)$$

is found. If the operator function Q(x) satisfies the conditions (1) and (2), the multiple series $\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \int_0^p \left(Q(x) j_j, j_j \right)_H \cos 2kx dx$ is absolute convergent. In this case as known

$$\lim_{m \to \infty} \sum_{p=1}^{n_m} \int_0^p \cos 2k_p x (Q(x) j_{j_p}, j_{j_p})_H dx = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \int_0^p \left(Q(x) j_j, j_j \right)_H \cos 2kx dx$$

By (3.11) and the last equality

$$\lim_{m \to \infty} D_{m1} = \frac{1}{p} \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \int_0^p \left(Q(x) j_j, j_j \right)_H \cos 2kx dx$$

$$= \frac{1}{4} \sum_{j=1}^{\infty} \left\{ \sum_{k=1}^{\infty} \left[\frac{2}{p} \int_0^p \left(Q(x) j_j, j_j \right)_H \cos kx dx \right] \cos k.0 \right.$$

$$\left. + \sum_{k=1}^{\infty} \left[\frac{2}{p} \int_0^p \left(Q(x) j_j, j_j \right)_H \cos kx dx \right] \cos kp \right\}$$

is found. If we consider that Q(x) satisfies the condition (3) , the sums according to k on the right hand side of the last relation are the values at the point 0 and p of the Fourier series of the function $\left(Q(x) j_j, j_j \right)_H$ having second order derivative according to the functions $\{\cos kx\}_{k=0}^{\infty}$ in the interval $[0, p]$ respectively. Therefore

$$\lim_{m \to \infty} D_{m1} = \frac{1}{4} \sum_{j=1}^{\infty} \left[\left(Q(0) j_j, j_j \right)_H + \left(Q(p) j_j, j_j \right)_H \right]$$

or

$$\lim_{m\to\infty} D_{m1} = \frac{1}{4}\left[trQ(0) + trQ(p)\right] \tag{3.12}$$

similar to the proof of the equality (3.6) the formula

$$\lim_{m\to\infty} D_m^{(p)} = 0 \qquad \left(p \rangle \frac{3(a+2)}{a-2}\right) \tag{3.13}$$

can be proved. By the formulas (3.1) , (3.12) , (3.13) and Theorem 3.1

$$\lim_{m\to\infty} \sum_{k=1}^{n_m} \left(l_k - m_k\right) = \frac{1}{4}\left[trQ(0) + trQ(p)\right]$$

is obtained. The Theorem is proved.

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