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FACTORIZATION PROPERTIES IN POLYNOMIAL EXTENSION OF UFR'S

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ABSTRACT

We investigate the factorization properties on the polynomial extension $A[X]$ of A where A is a UFR and show that $A[X]$ is a U-BFR for any UFR A. We also consider the ring structure $A+XI[X]$ where A is a UFR.
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TÇA HALKALARIN POLİNOM GENİŞLEMELERİNDE ÇARPANLARA AYIRMA ÖZELLİKLERİ

ÖZET

A bir TÇA halka (tektürlü çarpanlara ayrılabilen halka) olmak üzere A 'nın polinom genişlemesi $A[X]$ üzerindeki çarpanlara ayırma özelliklerini araştırıyoruz ve herhangi bir TÇA halka A için $A[X]$ 'in U-KÇA halka (U-Kısıtlı Çarpanlarına Ayrılabilen Halka) olduğunu gösteriyoruz. A bir TÇA olmak üzere $A+XI[X]$ yapısındaki halkaları da göz önüne alıyoruz.
Anahtar Sözcükler : Çarpanlara ayırma, Polinom halkaları.

1. INTRODUCTION

One of the main problem in ring theory is to determine whether the polynomial extension of any commutative ring R possesses or not the properties belonging to R. The purpose of this paper is to investigate which factorization properties is exactly satisfied in the polynomial extension $R[X]$ of R where R is a UFR.

Let R be a commutative ring with identity. Any elements $a,b \in R$ are *associate*, denoted by $a \sim b$, if $a|b$ and $b|a$, that is, $(a)=(b)$. A nonunit $a \in R$ is an *irreducible* (or an *atom*) if $a = bc \;\Rightarrow\; a \sim b$ or $a \sim c$. Hence 0 is irreducible if and only if R is an integral domain. R is *atomic* if each nonzero nonunit element of R is a finite product of irreducible elements. A principal ideal ring (PIR) is called a *special principal ideal ring* (SPIR) if it has only one prime ideal $P \neq R$ and P is nilpotent, that is, $P^n = (0)$ for some integer $n > 0$. R is said to

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be *presimplifiable* if $x = xy \Rightarrow x = 0$ or $y \in U(R)$. SPIR's and integral domains are examples of presimplifiable rings.

Classical irreducible decomposition may cause the bad behavior of factorization for some elements in a commutative ring because of nontrivial idempotents. For example $5 = 5^n$ for every positive integer n in $\mathbb{Z}_{20}$. A U-decomposition which is introduced by Fletcher [9] eliminates this bad behavior of a factorization. A *U-decomposition* of $r \in R$ is a factorization

$$r = (p'_1 \ldots p'_k)(p_1 \ldots p_n)$$

such that

(i) the p'_i's and p_j's are irreducible

(ii) $p'_i(p_1 \ldots p_n) = (p_1 \ldots p_n)$ for $i = 1, \ldots, k$ and

(iii) $p_j(p_1 \ldots p_{j-1}p_{j+1} \ldots p_n) \neq (p_1 \ldots p_n)$ for $j = 1, \ldots, n$.

Let $r = (p'_1 \ldots p'_k)(p_1 \ldots p_n)$ be a U-decomposition of $r \in R$. Then the product $p_1 \ldots p_n$ is called the *relevant part* and the other is the *irrelevant part*. Any irreducible decomposition can be rearranged to a U-decomposition. Two U-decompositions

$$r = (p'_1 \ldots p'_k)(p_1 \ldots p_n) = (q'_1 \ldots q'_l)(q_1 \ldots q_m)$$

are *associates* if (i) $n = m$ and (ii) p_i and q_i are associates for $i = 1, \ldots, n$, after a suitable change in the order of the factors in the relevant parts. R is called *unique factorization ring* (UFR) if each nonunit has a U-decomposition, that is, R is atomic and any two U-decompositions of a nonunit element of R are associates. Fletcher shows that R is a UFR if and only if R is a finite direct product of UFD's and SPIR's [10].

There are a family of factorization properties which are weaker than unique factorization. For a detailed study of these factorization properties in integral domain see Refs. [4,5,6]. In this paper we consider these factorization properties for a commutative ring with zero divisors. Any commutative ring R is called a *bounded factorization ring* (BFR) if for each nonzero nonunit $a \in R$, there exist a natural number $N(a)$ so that for any factorization $a = a_1 \ldots a_n$ of a where each a_i is a nonunit we have $n < N(a)$. If we replace the condition that p'_i's and p_j's are irreducible by p'_i's and p_j's are nonunits in the definition of U-decomposition, we have the definition of *U-factorization*. A commutative ring R is a U-BFR if for each nonzero nonunit $a \in R$, there exist a natural number $N(a)$ so that for each U-factorization of a, $a = (a_1, \ldots, a_n)(b_1, \ldots, b_m)$, $m < N(a)$ [1]. R is a UFR $\Rightarrow$ R is a U-BFR. R is called a *U-half-factorial ring* (U-HFR) if R is U-atomic (that is, every nonzero nonunit has a U-factorization in which all the relevant divisors are irreducible) and if $a = (a_1, \ldots, a_n)(b_1, \ldots, b_m) = (c_1, \ldots, c_l)(d_1, \ldots, d_s)$ are two U-factorizations with b_i, d_j irreducible, then $m = s$. It is clear that UFR $\Rightarrow$ U-HFR. For these factorization properties see [1,7,8].

For an integral domain D it is well known that D is a UFD $\Leftrightarrow$ $D[X]$ is a UFD. But if zero divisors are present the situation is not so clear. In [2] Anderson and Markanda show that $R[X]$ is a UFR if and only if R is a finite direct product of UFD's. In other words $R[X]$ is not a UFR if R is a SPIR. In fact R is not even a U-HFR if R is a SPIR. For example the element $2X^2$ has two distinct irreducible decomposition (and U-decomposition) in $\mathbb{Z}_4$ which is a UFR, namely

$$2X^2 = 2(2 + X^2) = 2.X.X \ .$$

In here we continue to investigate these factorization properties in $R[X]$ where R is a UFR and give a positive result for the concepts a U-BFR and a BFR. More precisely we show that if R is a SPIR then $R[X]$ is a BFR and if R is a UFR then $R[X]$ is a U-BFR. We also consider these factorization properties in the rings $R+XI[X]$ for any ideal I of R where R is a UFR.

In [12] Gonzalez, Pelerin and Robert show that $A+XI[X]$ is a HFD (half-factorial domain) if and only if I is a prime ideal of A for the domain case where A is a UFD. In here we investigate which factorization properties are satisfied in the rings $A+XI[X]$ for any ideal I of A where A is a UFR and show that $A+XI[X]$ is always a U-BFR for any ideal I of A.
For any undefined terminology or notations, see [11].

2. BOUNDED FACTORIZATION PROPERTIES ON $A[X]$

Suppose A is a SPIR and $P=(p)$ is the unique prime ideal of A, where $P^n=(0)$ and n is the smallest integer which satisfies $P^n=(0)$. If $n=1$ then A is a field. So from now on, we assume that $n>1$ and A,P and n will be as above unless otherwise stated.

Proposition 2.1: Let A be a SPIR. If $f(X)=a_0+a_1X+...+a_mX^m\in A[X]$ is any irreducible element in $A[X]$ then $f(X)$ is one of the following forms up to associate:

(i) $f(X)=p$

(ii) $f(X)=X$

(iii) $f(X)=a_0+a_1X+...+a_{k-1}X^{k-1}+X^k+a_{k+1}X^{k+1}+...+a_mX^m$ for some $1\le k\le m$ where $a_0,a_{k+1},...,a_m\in P$

(iv) $f(X)=1+a_1X+...+a_{k-1}X^{k-1}+X^k+a_{k+1}X^{k+1}+...+a_mX^m$ for some $1\le k\le m$ where $a_{k+1},...,a_m\in P$.

Proof : Suppose $a_i\in P$ for all $0\le i\le m$. We can write $a_i=pa_i'$ $\exists a_i'\in A$. Then $f(X)=p(a_0'+a_1'X+...+a_m'X^m)$. Since $f(X)$ is irreducible in $A[X]$, $f(X)\sim p$ or $f(X)\sim f'(X)$ where $f'(X)=a_0'+a_1'X+...+a_m'X^m$.

$$f(X)\sim p \;\Rightarrow\; p=f(X).c(X)\;\; \exists c(X)\in A[X]$$
$$\Rightarrow\; f(X)=f(X).c(X)f'(X).$$

Since $A[X]$ is presimplifiable and $0\ne f(X)$, $c(X)f'(X)$ is a unit in $A[X]$, and hence $f'(X)$ is a unit in $A[X]$. So we may take $f(X)=p$ up to associate. If $f(X)\sim f'(X)$ then

$$f'(X)=f(X).d(X)\;\;\exists d(X)\in A[X]\;\Rightarrow\; f(X)=p.f(X).d(X)$$
$$\Rightarrow\; p.d(X)\in U(A).$$

But this is a contradiction since p is not a unit. Therefore if all $a_i\in P$ then $f(X)=p$ up to unit.
Now suppose $f(X)\notin P[X]$. Consider the constant term of $f(X)$:

Case I: Suppose $a_0=0$. Then $f(X)=X.f'(X)$ $\exists f'(X)\in A[X]$. So either $f(X)\sim X$ or $f(X)\sim f'(X)$. $f(X)\sim f'(X)$ gives X is a unit as above which is a contradiction. So $f(X)\sim X$ and $f'(X)$ is a unit in $A[X]$. Hence we take $f(X)=X$ up to associate.
Now $a_0\ne 0$. Then either $a_0\in P$ or $a_0\in U(A)$.

Case II: Let $a_0 \in P$. Then for some $1 \le k \le m$ $a_k \notin P$ since $f(X) \notin P[X]$. So $a_k \notin P \Rightarrow a_k \in U(A)$. So we may take $a_k = 1$.

Case III: Let $a_0 \in U(A)$. We know that any element of $A[X]$ is a unit in $A[X]$ if and only if the constant term is a unit in A and all other coefficients are nilpotents in A. Any nonunit element of A is a nilpotent. So a_k must be a unit in A for some $1 \le k \le m$ since $f(X)$ is not a unit in $A[X]$. So again we may take $a_k = 1$.

Proposition 2.2: If A is a SPIR then $A[X]$ is a BFR.

Proof: Let $f(X) = a_0 + a_1 X + ... + a_s X^s \in A[X]$ be a nonzero nonunit element of $A[X]$. First note that $A[X]$ is presimplifiable since 0 is primary in A [3]. Collecting all the factor p in each coefficient a_i, we can write uniquely each a_i as $a_i = p^{\alpha_i} u_i$ where $0 \le \alpha_i \le n$ and $u_i \in U(A)$. Then $f(X) = p^{\alpha_0} u_0 + p^{\alpha_1} u_1 X + ... + p^{\alpha_s} u_s X^s \in A[X]$. Since $f(X) \ne 0$ the number of irreducible factor p in any irreducible decomposition of $f(X)$ is at most $n-1$. Now we show that in any factorization of $f(X)$ into a product of irreducible elements the number of irreducible factors as in Proposition 2.1 is at most $\deg f(X) = s$. It is obvious that the number of irreducible factor $X's$ are at most s. Let $f(X) = p \, ... \, pX \, ... \, Xf_1 \, ... \, f_t$ be an irreducible decomposition of $f(X)$ where f_i is irreducible as in Proposition 2.1 other than p and X. Let

$$f_1 = a_0 + ... + a_{k_1 - 1} X^{k_1 - 1} + X^{k_1} + a_{k_1 + 1} X^{k_1 + 1} + ... + a_l X^l ,$$
$$f_2 = b_0 + ... + b_{k_2 - 1} X^{k_2 - 1} + X^{k_2} + a_{k_2 + 1} X^{k_2 + 1} + ... + a_m X^m$$

where a_{k_1} and b_{k_2} are the last coefficients of f_1 and f_2 which is equal to 1, respectively. Consider the coefficient of $X^{k_1 + k_2}$ of the product $f_1 f_2$:

$$\sum_{j=0}^{k_1+k_2} a_j b_{k_1 + k_2 - j} .$$

$$\text{If } j > k_1 \Rightarrow a_j \in P \Rightarrow a_j b_{k_1+k_2-j} \in P ,$$
$$\text{if } j < k_1 \Rightarrow b_{k_1+k_2-j} \in P \Rightarrow a_j b_{k_1+k_2-j} \in P ,$$
$$\text{if } j = k_1 \Rightarrow a_j = b_{k_1+k_2-j} = 1 \Rightarrow a_j b_{k_1+k_2-j} = 1 .$$

So

$$\sum_{j=0}^{k_1+k_2} a_j k_{k_1+k_2-j} = 1 + pr , \ \exists r \in A .$$

Since $1 + pr \notin P$, $1 + pr$ is a unit in A, and hence not a zero element. Hence by induction on t we can see that the coefficient of $X^{k_1 + \dots + k_t}$ in the product $f_1 \, ... \, f_t$ is not zero where $k_i \ge 1$. So if $t > s$ then $\deg(f(X)) < \deg(p \, ... \, pX \, ... \, Xf_1 \, ... \, f_t)$ which is a contradiction. Thus $t \le s$ and hence $A[X]$ is a BFR.

Following theorem is in [1]. For a commutative ring R, $a \in R$ is called U-*bounded* if $\sup\{m \mid a = (a_1,...,a_n)(b_1,...,b_m) \text{ is a U-factorization of } a\} < \infty$.

Theorem 2.3: Let $R_1, R_2, ..., R_n$ be commutative rings, $n > 1$, and let $R = R_1 \times ... \times R_n$. Then R is a U-BFR $\Leftrightarrow$ each R_i is a U-BFR and 0_{R_i} is U-bounded. Hence 0_{R_i} is U-bounded.

Now we can state the main result of this paper as a corollary.

Corollary 2.4: If A is a UFR then $A[X]$ is a U-BFR.

Proof : Since A is a UFR, A is a finite direct product of UFD's and SPIR's, say $A = A_1 \times ... \times A_n$. Then $A[X] = A_1[X] \times ... \times A_n[X]$. If A_i is a UFD then clearly $A_i[X]$ is UFD and hence a U-BFR. If A_i is a SPIR then $A_i[X]$ is a U-BFR by Proposition 2.2. Hence by Theorem 2.3, $A[X]$ is U-BFR.

Lemma 2.5: Let $A \subset B$ be an extension of commutative rings. If B is a BFR and $U(B) \cap A = U(A)$ then A is also a BFR.

Proof: Let $a \in A$ be a nonzero nonunit and let $a = a_1 \ ... \ a_n$ be any factorization of a into nonunits. Then $a_i \notin U(B)$ for $i = 1, ... , n$. Hence $a = a_1 \ ... \ a_n$ is a factorization of a in B. Since B is a BFR, $n \le N(a)$ for some positive integer $N(a)$. So A is a BFR.

Proposition 2.6: If A is a UFR then $A + XI[X]$ is a U-BFR for any ideal I of A.

Proof : Let $A = A_1 \times ... \times A_n$, a finite direct product of UFD's and SPIR's. Then I is of the form $I = I_1 \times ... \times I_n$ where each I_i is an ideal of A_i. So

$$\begin{aligned} A + XI[X] &= (A_1 \times ... \times A_n) + X(I_1 \times ... \times I_n)[X] \\ &= (A_1 + XI_1[X]) \times ... \times (A_n + XI_n[X]) \end{aligned}$$

If A_i is a UFD then by Lemma 2.5 $A_i + XI_i[X]$ is a BFR since $A_i + XI_i[X] \subseteq A_i[X]$ and $U(A_i + XI_i[X]) = U(A_i[X]) = U(A_i)$. If A_i is a SPIR then $A_i + XI_i[X]$ is again a BFR since $A_i[X]$ is a BFR by Proposition 2.2 and $U(A_i[X]) \cap (A_i + XI_i[X]) = U(A_i + XI_i[X])$. Clearly 0_{A_i} is always U-bounded in each case. Hence $A + XI[X]$ is a U-BFR for every ideal *I* of A by Theorem 2.4.

3. RESULTS AND DISCUSSION

In this paper we concentrate on polynomial extension of a UFR with zero divisors to investigate factorization properties related to it. We show that if A is a UFR then $A[X]$ is always a U-BFR and $A + XI[X]$ is always a U-BFR for any ideal *I* of A. We do not know these results are remain valid if the term "U-BFR" is substituted for "UFR".

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