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Research Article

Application of müntz-legendre polynomials for solving complex differential equations

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ABSTRACT

This paper has obtained the numerical solutions of complex differential equations using the Müntz-Legendre Polynomials. The technique was performed on test problem. Then, different technical error analyses were applied to the test problem. Finally, when exact solutions and numerical solutions were compared with tables and graphs, it was realized that our method is practical and reliable.

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INTRODUCTION

A complex differential equation started to attract more attention after the second half of the 20th century by many scientists. These equations are vital due to their application for modeling some phenomena in real life and arise in different scientific fields such as applied mathematics, physics, and engineering. In most cases, the exact analytical solution of these equations does not exist or may not be solved directly. Therefore, it is crucial to provide the correct numerical solution. Some studies were performed about the solutions for these equations. Sezer and Yalçınbaş [1] used Taylor method, Yüzbaşı et al.[2] applied Bessel polynomial method, Dusunceli and Celik obtained different solutions by Legendre[3,4], Hermite[5], and Fibonacci[6] polynomial methods. Düz solved these equations using two methods: the Fourier transform method[7] and the reduced differential transform method[8]. Also, Lü et al. [9]. examined growth and uniqueness related to complex differential equations.

Solving different types of differential equations using orthogonal function families has an increasing place in science and engineering. Laguerre, Legendre, Hermite, and Chebyshev functions are frequently used among these functions. The applications of Müntz-Legendre polynomials from the family of orthogonal polynomials have also started to be used in recent years. Maccarthy et al.[10] and Borwein et al.[11] introduced and investigated these polynomials. Yüzbaşı et al.[12] obtained the solutions of singular perturbed problems and Rahimkhani et al.[13] obtained the solution of the Bagley-Torvik equation by using the Müntz-Legendre polynomials. For further reading[14–20].

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Our aim in this study is to find new approximate solutions for linear complex differential equations defined by equation (1-2) below using müntz-legendre polynomials.

$$\sum_{n=0}^{m} P_n(z)u^{(n)}(z) = g(z) \tag{1}$$

with the initial conditions

$$u^{(t)}(\alpha) = \vartheta_t \qquad t = 0,1,\ldots,m-1 \tag{2}$$

where $u(z)$ is unknown function and $u^{(0)}(z) = u(z)$, $P_n(z)$, $g(z)$ are analytical functions in the circular domain which $D = \{z = x + iy,\ z \in C, |z - z_0| \leq r,\ r \in R^+\}$, $\alpha, z_0 \in D$, $MS - V_t$ is appropriate complex or real constant.

The article is designed as follows; some preliminaries and notations are introduced in section 2. In section 3, Müntz-Legendre polynomials are defined. In section 4, the proposed numerical technique is introduced. In section 5, the method is applied to the test problem. Finally, the conclusions have been given.

PRELIMINARIES AND NOTATIONS

In this section, some important definitions and theories will be given.

Definition 1: The first and second order derivatives of $u(z)$ where $z = x + iy$ are as follows.

$$\frac{du}{dz} = \frac{1}{2}\left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right)$$

$$\frac{d^2u}{dz^2} = \frac{1}{4}\left(\frac{\partial^2 u}{\partial x^2} - 2i\frac{\partial u}{\partial x \partial y} - \frac{\partial^2 u}{\partial y^2}\right)$$

Definition 2(Conway): A function $u: A \to C$ (C, complex plane) is analytic if u is continuously differentiable on A.

Theorem 1: a and b real-valued functions defined on a region A and suppose that a and b have continuous partial derivatives. Then $u: A \to C$ defined by $u(x + iy) = a(x,y) + ib(x,y)$ is analytic if a and b satisfy the Cauchy-Riemann equations. the Cauchy-Riemann equations are defined as follows.

$$\frac{\partial a}{\partial x} = \frac{\partial b}{\partial y} \text{and} \frac{\partial a}{\partial y} = -\frac{\partial b}{\partial x}$$

Theorem 2: If the series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ has a radius of convergence $R > 0$ then $u(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ is analytic in $B(z_0,R)$.

MÜNTZ-LEGENDRE POLYNOMIALS

The n-th Müntz-Legendre polynomial of $M(\Lambda)$ is defined as follows[10-11]

$$M_n(t) = \frac{1}{2\pi i}\int_D \prod_{k=0}^{n-1} \frac{x + \lambda_k + 1}{x - \lambda_k} \frac{t^x}{x - \lambda_n} dx,$$

where the simple contour D surrounds all the zeros of the denominator in the integral.

When the Müntz sequence provides the following conditions,

$$\lambda_k > -\frac{1}{2}, (n \in N_0), \lambda_k \neq \lambda_j, k \neq j.$$

The Müntz-Legendre polynomials on the interval [0,1] are as follows,

$$M_n(t;\Lambda) = \sum_{k=0}^{n} c_{k,n} t^{\lambda_k}, c_{k,n} = \frac{\prod_{j=0}^{n-1}(\lambda_k + \lambda_j + 1)}{\prod_{j=0, j\neq k}^{n}(\lambda_k - \lambda_j)}, \tag{3}$$
$$c_{0,0} = 1, (n \in N).$$

The Müntz-Legendre polynomials satisfy the orthogonality condition,

$$\int_0^1 M_n(t)M_m(t)dt = \frac{\delta_{n,m}}{(2\lambda_n + 1)},\ \ n \geq m,$$

where $\delta(n,m)$ is the Kronecker symbol.

The Müntz-Legendre polynomials satisfy the following recursive formula

$$M_n(t) = M_{n-1}(t) - (\lambda_n + \lambda_{n-1} + 1)t^{\lambda_n}\int_t^1 x^{-\lambda_n - 1} M_{n-1}(x)dx,$$
$$(t \in (0,1]).$$

In this article, we'll consider $\lambda_k = k$ and on the interval [0,1]. In this case, the first five polynomials can be written as follows.

$$M_0(t) = 1$$
$$M_1(t) = -1 + 2t$$
$$M_2(t) = 1 - 6t + 6t^2$$
$$M_3(t) = -1 + 12t - 30t^2 + 20t^3$$
$$M_4(t) = 1 - 20t + 90t^2 - 140t^3 + 70t^4$$

Numerical Technique

The approximate solution of equation (1) under the conditions (2) is considered in the form below,

$$u(z)=\sum_{n=0}^{N}a_nM_n(z),\ \ z\in D \tag{4}$$

which is the Müntz- Legendre series of the unknown function $u(z)$, where all of a_n are the Müntz-Legendre coefficients to be determined. The collocation points are used as below,

$$z_{pp}=z_0+\frac{r}{N}pe^{\frac{i\theta}{N}p},\ \ 0<\theta\le 2\pi, r\in R^+,\ p\in 0,1,\ldots,N \tag{5}$$

The Müntz-Legendre polynomials are written in matrix form as follows.

$$M(z) = [M_0(z-z_0)\quad M_1(z-z_0)\quad \ldots\quad M_N(z-z_0)] \tag{6}$$

The desired solution $u(z)$ of equation (4) is considered as below,

$$u(z) = M(z)A;\quad A = [a_0\quad a_1\quad \ldots\quad a_N]^T \tag{7}$$

On the other hand, $M(z)$ can be rewritten as below,

$$M(z) = Z(z)B^T, \tag{8}$$

where $Z(z) = [1\quad z\quad z^2\quad \ldots\quad z^N]$ and $B = \begin{bmatrix} c_{0,0} & 0 & 0 & \cdots & 0 \\ c_{0,1} & c_{1,1} & 0 & \cdots & 0 \\ c_{0,2} & c_{1,2} & c_{2,2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{0,N} & c_{1,N} & c_{2,N} & \cdots & c_{N,N} \end{bmatrix}.$

The relations between the matrix $Z(z)$ and its derivatives $Z'(z), Z^{(2)}(z), \ldots, Z^{(n)}(z)$ can be written as follows,

$$\begin{gathered} Z'(z) = Z(z)K^T \\ Z^{(2)}(z) = Z(z)(K^T)^2, \\ \vdots \\ Z^{(n)}(z) = Z(z)(K^T)^n \end{gathered}$$

where,

$$K=\begin{bmatrix} 0 & 0 & 0 & 0 & \ldots & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 3 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{N+1\times N+1}$$

Using relations (7) and (8) the following equation is obtained.

$$u^{(n)}(z) = M^{(n)}(z)A^T = Z^{(n)})(z)B^TA^T = Z(z)(K^T)^nB^TA^T. \tag{9}$$

The equation is formed as follows, substituting the collocation points (5) with equation (1).

$$\sum_{n=0}^{m}P_n(z_{pp})u^{(n)}(z_{pp})=g(z_{pp}). \tag{10}$$

Finally, using equations (9)–(10), the fundamental matrix equation is obtained.

$$\sum_{n=0}^{m}\sum_{p=0}^{N}P_nZ(z_{pp})(K^T)^nB^TA^T=\sum_{p=0}^{N}G_p, \tag{11}$$

Where

$$P_n=\begin{bmatrix} P_n(z_{00}) & 0 & \cdots & 0 \\ 0 & P_n(z_{11}) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & P_n(z_{NN}) \end{bmatrix} \text{and}\, G_p=\begin{bmatrix} g(z_{00}) \\ g(z_{11}) \\ \vdots \\ g(z_{NN}) \end{bmatrix}$$

In equation (11), A is unkown Müntz-Legendre coefficients. In order to determine the approximate solution, the A coefficient matrix must be determined. Therefore, the matrix equation (11) could be rearranged as below:

$$W.A = G \text{ or } [W;G] = [w_{pq};g_p]\quad p,q = 0,1,\ldots,N \tag{12}$$

where,

$$W=\sum_{n=0}^{m}\sum_{p=0}^{N}P_nZ(z_{pp})(K^T)^nB^T \text{ and } A=\begin{bmatrix}a_0 & a_1 & \ldots & a_N\end{bmatrix}^T$$

Initial conditions can also be formulated as follows by the aid of equation (9)

$$u^{(n)}(\alpha) = Z(\alpha)(K^T)^nB^TA^T = \vartheta_t\qquad t = 0,1,\ldots,m-1.$$

In other words the matrix form of the initial conditions could be rewritten as

$$U_tA = \vartheta_t\qquad t = 0,1,\ldots,m-1$$

where

$$U_t = L(\alpha)(M^T)^t\qquad t = 0,1,\ldots,m-1$$

The augmented form of these equations is as follows,

$$[U_t;\vartheta_t] \qquad t = 0,1,\ldots,m-1 \tag{13}$$

The m rows of (13) by the last m rows of the augmented matrix (12) should be returned to get unknown Müntz-Legendre coefficients. The final version of the augmented matrix is as follows,

$$\left[\tilde{W};\tilde{G}\right]=\begin{bmatrix} w_{00} & w_{01} & \cdots & w_{0N} & ; & g_0 \\ w_{10} & w_{11} & \cdots & w_{1N} & ; & g_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ w_{N-m0} & w_{N-m1} & \cdots & w_{N-mN} & ; & g_N \\ u_{00} & u_{01} & \cdots & u_{0N} & ; & \vartheta_0 \\ u_{10} & u_{11} & \cdots & u_{1N} & ; & \vartheta_1 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ u_{m-10} & u_{m-11} & \cdots & u_{m-1N} & ; & \vartheta_{m-1} \end{bmatrix}$$

ACCURACY OF SOLUTION

In this section, the efficiency of the present method can be controlled easily. The approximate solution of equation(1) and their derivatives are substituted into equation(1). Then approximate results are obtained for

$$E(z_j)=\left|\sum_{n=0}^{m} P_n(z_j)f^{(n)}(z_j)-g(z_j)\right|\cong 0 \tag{14}$$

or

$E(z_j) \le 10^{-k_j}$ (k_j is any positive integer).

If $max10^{-k_j} = 10^{-k}$ (k positive integer) is commanded, then the truncation limit N is extended $E(z_j)$ at each point z_j gets less than the prescribed 10^{-k}.

NUMERICAL EXPERIMENTS

A test problem is given to illustrate the proposed method's accuracy and effectiveness, and all of them are performed on a computer using programs written in Matlab. The absolute error function $e_N(z) = |f(z) - f_N(z)|$ at the selected points of the given domain. Finally, error analysis has been done by N_2 and N_∞ defined below.

$$N_\infty = \|f - f_n\|_\infty = max|f_j - f_{nj}|$$

Table 1. Comparison of the absolute errors $e_N(z_j)$ for various values of Equation (18) for the Re(z) value

Z_i	Exact solution(Reel)	$N = 3$	$N = 5$	$N = 9$	$N = 11$
−1−I	4,19876	4,41243	4,18434	4,19862	4,19876
−0.9(1+i)	4,15272	4,30066	4,14360	4,15264	4,15272
−0.8(1+i)	4,11305	4,21116	4,10757	4,11300	4,11304
−0.7(1+i)	4,07980	4,14146	4,07673	4,07978	4,07980
−0.6(1+i)	4,05295	4,08908	4,05136	4,05294	4,05295
−0.5(1+i)	4,03228	4,05155	4,03154	4,03227	4,03228
−0.4(1+i)	4,01740	4,02639	4,01711	4,01740	4,01740
−0.3(1+i)	4,00773	4,01113	4,00763	4,00773	4,00773
−0.2(1+i)	4,00241	4,00329	4,00239	4,00241	4,00241
−0.1(1+i)	4,00031	4,00041	4,00031	4,00031	4,00031
0(1+i)	4	4	4	4	4,00000
0.1+0.1i	3,99964	3,99958	3,99965	3,99964	3,99964
0.2+0.2i	3,99705	3,99670	3,99705	3,99705	3,99705
0.3+0.3i	3,98956	3,98886	3,98957	3,98956	3,98956
0.4+0.4i	3,97406	3,97360	3,97407	3,97406	3,97406
0.5+0.5i	3,94688	3,94844	3,94695	3,94688	3,94688
0.6+0.6i	3,90385	3,91091	3,90404	3,90385	3,90385
0.7+0.7i	3,84020	3,85853	3,84066	3,84018	3,84020
0.8+0.8i	3,75054	3,78883	3,75150	3,75050	3,75054
0.9+0.9i	3,62891	3,69933	3,63067	3,62881	3,62891
1+1i	3,46869	3,58756	3,47161	3,46851	3,46869

Test problem:

Let us consider the second-order complex differential equation as follows;

$$y''(z) + zy'(z) + z^2y(z) = e^z(2 + 2z + z^2 + z^3)$$
$$y(0) = 1, \qquad y'(0) = 1$$

Exact solution of the equation is $y(z) = 3 - z + 2z^2 + e^z$. Then $z_0 = 0$, $r = 1$ and $\theta = 2\pi$ are taken as zero to determine the collocation points. Using the numerical technique which is introduced in section 3, approximate solutions are obtained for $N = 3,5,10$.

N	3	5	10
L_2	7.81×10^{-1}	1.46×10^{-2}	5.95×10^{-6}
L_∞	5.01×10^{-1}	10.6×10^{-1}	5.75×10^{-6}

CONCLUSIONS

In this study, a numerical technique is applied to obtain approximation solutions for the complex differential equations. By using Müntz-Legendre polynomials, the equation is reduced to a system of an algebraic equation. Based on the result of the test problem, it can be said that, there is a difference in the convergence of the real value between the approximate solutions of the real and imaginary parts of the complex differential equation. As N values are increased, it is seen that the convergence in the real part is faster than the convergence in the imaginer part. However, it has been seen that the method is effective and gives good results.

AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

Table 2. Comparison of the absolute errors $e_N(z_i)$ for various values of Equation (18) for the Im(z) value

Z_i	Exact solution(Im.)	$N = 3$	$N = 5$	$N = 9$	$N = 11$
−1−I	4,69044	4,69498	4,70932	4,72738	4,72738
−0.9(1+i)	3,82152	3,82764	3,83224	3,84115	3,84115
−0.8(1+i)	3,03767	3,04383	3,04338	3,04735	3,04735
−0.7(1+i)	2,34009	2,34538	2,34289	2,34443	2,34443
−0.6(1+i)	1,73011	1,73411	1,73136	1,73184	1,73184
−0.5(1+i)	1,20921	1,21187	1,20969	1,20979	1,20979
−0.4(1+i)	0,77896	0,78047	0,77911	0,77911	0,77911
−0.3(1+i)	0,44107	0,44176	0,44110	0,44110	0,44110
−0.2(1+i)	0,19734	0,19755	0,19734	0,19734	0,19734
−0.1(1+i)	0,04966	0,04969	0,04966	0,04966	0,04966
0(1+i)	9,86076e-32	−1,3422e-16	5,5023e-17	9,083e-15	1,081e-13
0.1+0.1i	0,05033	0,05030	0,05033	0,05033	0,05033
0.2+0.2i	0,20265	0,20244	0,20265	0,20265	0,20265
0.3+0.3i	0,45891	0,45823	0,45890	0,45893	0,45893
0.4+0.4i	0,82094	0,81952	0,82093	0,82109	0,82109
0.5+0.5i	1,29043	1,28812	1,29043	1,29101	1,29101
0.6+0.6i	1,86884	1,86588	1,86889	1,87056	1,87056
0.7+0.7i	2,55729	2,55461	2,55757	2,56164	2,56164
0.8+0.8i	3,35650	3,35616	3,35742	3,36618	3,36619
0.9+0.9i	4,26667	4,27235	4,26905	4,28630	4,28630
1+1i	5,28735	5,30501	5,29265	5,32429	5,32430

Table 3. Comparison of the absolute errors $e_N(z_j)$ for various values of Equation (18) for the Re(z) value

z_j	$N = 3$	$N = 5$	$N = 9$	$N = 11$
−1−I	−2,1367e-01	1,441 e-02	1,448 e-04	3,066e-06
−0.9(1+i)	−1,4793e-01	9,120 e-03	8,125e-05	1,572e-06
−0.8(1+i)	−9,8116e-02	5,471 e-03	4,189e-05	7,492e-07
−0.7(1+i)	−6,1656e-02	3,074 e-03	1,949e-05	3,259e-07
−0.6(1+i)	−3,6132e-02	1,589 e-03	7,960e-06	1,258e-07
−0.5(1+i)	−1,9273e-02	7,355 e-04	2,731e-06	4,129e-08
−0.4(1+i)	−8,9902e-03	2,917 e-04	7,311e-07	1,066e-08
−0.3(1+i)	−3,4051e-03	9,167e-05	1,328e-07	1,882e-09
−0.2(1+i)	−8,8884e-04	1,922e-05	1,204e-08	1,653e-10
−0.1(1+i)	−9,5436e-05	1,590e-06	2,144e-10	2,367e-12
0(1+i)	0	0	0	−4,325e-13
0.1+0.1i	6,2103e-05	−5,2024e-07	1,752e-10	2,537e-12
0.2+0.2i	3,5551e-04	−2,1039e-06	1,141e-08	1,643e-10
0.3+0.3i	7,0515e-04	−4,9639e-06	1,300e-07	1,873e-09
0.4+0.4i	4,5746e-04	−1,7386e-05	7,308e-07	1,052e-08
0.5+0.5i	−1,5564e-03	−6,3734e-05	2,787e-06	4,017e-08
0.6+0.6i	−7,0542e-03	−1,892 e-04	8,329e-06	1,200e-07
0.7+0.7i	−1,8331e-02	−4,599 e-04	2,104e-05	3,030e-07
0.8+0.8i	−3,8283e-02	−9,565 e-04	4,705e-05	6,770e-07
0.9+0.9i	−7,0420e-02	−1,759 e-03	9,593e-05	1,377e-06
1+1i	−1,1886e-01	−2,924 e-03	1,819 e-04	2,601e-06

Table 4. Comparison of the absolute errors $eN\ (z_j)$ for various values of Equation (18) for the Im(z) value

z_j		$N = 5$	$N = 9$	$N = 11$
−1−I	−4,5480e-03	−1,8886e-02	−3,694e-02	−3,694 e-02
−0.9(1+i)	−6,1233e-03	−1,0724 e-02	−1,9636e-02	−1,963 e-02
−0.8(1+i)	−6,1627e-03	−5,7112 e-03	−9,6865e-03	−9,685 e-03
−0.7(1+i)	−5,2899e-03	−2,8065 e-03	−4,3473e-03	−4,3467 e-03
−0.6(1+i)	−3,9997e-03	−1,2427 e-03	−1,7240e-03	−1,7238 e-03
−0.5(1+i)	−2,6598e-03	−4,7809 e-04	−5,7735e-04	−5,7729 e-04
−0.4(1+i)	−1,5141e-03	−1,5042 e-04	−1,5134e-04	−1,5133 e-04
−0.3(1+i)	−6,9143e-04	−3,4564e-05	−2,6936e-05	−2,6934e-05
−0.2(1+i)	−2,1659e-04	−4,4840e-06	−2,3647e-06	−2,3646e-06
−0.1(1+i)	−2,7999e-05	−1,3420e-07	−3,6955e-08	−3,6947e-08
0(1+i)	1,3422e-16	−5,5023e-17	−9,0838e-15	−1,0817e-13
0.1+0.1i	2,7976e-05	−8,6019e-08	−3,6938e-08	−3,6947e-08
0.2+0.2i	2,1517e-04	−1,0618e-07	−2,3641e-06	−2,3646e-06
0.3+0.3i	6,7523e-04	2,32654e-06	−2,6928e-05	−2,6934e-05
0.4+0.4i	1,4231e-03	8,71778e-06	−1,5130 e-04	−1,5133 e-04
0.5+0.5i	2,3125e-03	7,14275e-06	−5,7717e-03	−5,7729 e-03
0.6+0.6i	2,9631e-03	−5,0603e-05	−1,7234e-03	−1,7238 e-03
0.7+0.7i	2,6760 e-03	−2,828 e-04	−4,3459 e-03	−4,3467 e-03
0.8+0.8i	3,3926 e-04	−9,2328 e-04	−9,6835 e-03	−9,6854 e-03
0.9+0.9i	−5,6803e-03	−2,3785e-03	−1,9631 e-02	−1,9635 e-02
1+1i	−1,7656 e-02	−5,2976e-03	−3,6939 e-02	−3,6947 e-02

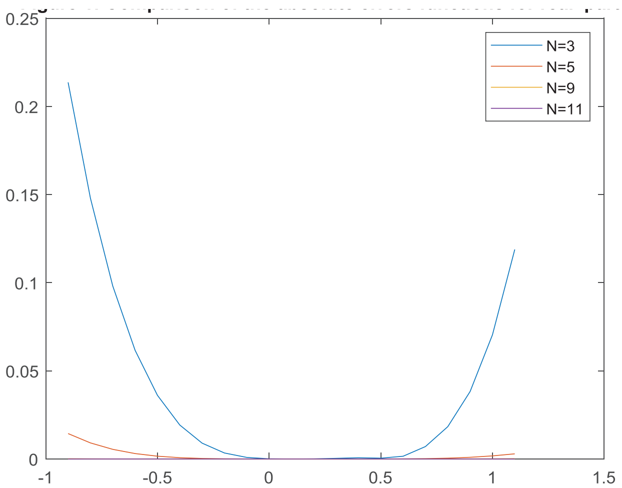

Figure 1. Comparison of the absolute errors functions for real parts.

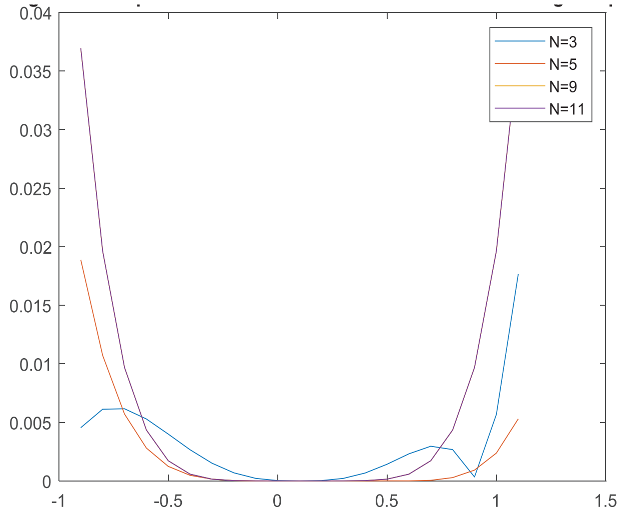

Figure 2. Comparison of the absolute errors functions for imaginer parts.

ETHICS

There are no ethical issues with the publication of this manuscript.

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