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Research Article

Target attractor formed via fractional feedback control

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ABSTRACT

We discuss here the stabilization problem for an ordinary differential equation (ODE) dynamical model. To make such a control, one can form a Kolesnikov's subset attracting the phase trajectories to its neighborhood in the phase space via defining the appropriate feedback signal. Kolesnikov's target attractor algorithm provides the exponential convergence, but at the same time it demands the permanent power supply pumping the energy to the system even if the control goal is achieved.

To decrease the power cost of Kolesnikov's control, we re-formulate the feedback in the form of Caputo's fractional derivative. In this case the solution to the ODE together with the feedback control signal could be found with the Rida-Arafa method based on the generalized Mittag-Leffler function.

We prove that for the certain constraints over the initial condition and the target stabilization level, the integer-dimensional Kolesnikov algorithm can be replaced with the fractional target attractor feedback to provide the minimal power cost.

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INTRODUCTION

The stabilization problem for an ordinary differential equation (ODE) dynamical model is one of the key applications of control theory. It involves a variety of different feedback algorithms, including optimal and sub-optimal gradient approaches [1, 2]. From the point of physics the basic idea of a gradient method is developing in the dynamical system a sort of friction force at the target level of the controlled parameter. This friction-type force provides a necessary decay of the system dynamics to achieve the control goal at the constant level (stabilization) or at the certain time-dependent function level (tracking). The gradient methods are relatively universal and very flexible for the adaptation to a wide spectrum of applied problems [3]. The typical handicap of gradient algorithms is a certain error in the achievement of the control goal.

An alternative way is Kolesnikov's 'synergetic' feedback which forms a subset attracting the phase trajectories to the

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target neighborhood in the phase space via the appropriate control signals [4]. Kolesnikov's target attractor algorithm provides an exponentially fast convergence of the controlled parameters, but at the same time it demands the permanent power supply pumping the energy to the system, even if the control goal is achieved.

For many practical applications the way to minimize the control power supply may lead us to the fractal space [5, 6]. To decrease the power cost of Kolesnikov's control, we re-formulate here the feedback in the form of Caputo's fractional derivative [7]. In this case the solution to the ODE together with the feedback control signal could be found with the Rida-Arafa method [8] based on the generalized Mittag-Leffler function [9].

We demonstrate our approach with a simplified toy model. We prove that for the certain constraints over the initial condition and the target stabilization level, the integer-dimensional Kolesnikov algorithm can be replaced successfully with the fractional target attractor feedback to provide the minimal power cost.

TOY MODEL FOR CONTROL

First, let's define a simplified 'toy' model, in which we will be able to investigate our approach analytically. Let's consider a car traveling along the straight line with the velocity v. We compose the second Newtonian law for the car acceleration, via the time-dependent engine acceleration $u(t)$ and the viscous friction term bv in RHS, where b is a positive coefficient:

$$\frac{dv}{dt} = u(t) - bv(t)\,;\; b = \text{const} > 0. \tag{1}$$

For simplicity we normalized the mass: $m = 1$. Here the engine acceleration u plays a role of control parameter. The control goal is to stabilize the velocity v at the certain constant level v_*:

$$v \to v_* \text{ as } t \to \infty. \tag{2}$$

We will apply Kolesnikov's method for the stabilization problem (2).

KOLESNIKOV'S TARGET ATTRACTOR FEEDBACK AND ITS POWER COST

According to Kolesnikov's approach, we need to form in the phase space of our dynamical system (1) the attracting manifolds ψ_s as functions of control object state variables [4]. These manifolds serve as a subset referring the goal function:

$$\psi_s(x_1, \ldots, x_n) = 0, \tag{3}$$

where $\{x_1,\ldots x_n\}$ are the set of the n-dimensional phase space parameters. Particularly, for our model (1) we consider the goal function as:

$$\frac{d\psi(t)}{dt} = -\frac{1}{T}\psi(t)\,,\; T = \text{const} > 0. \tag{4}$$

The constant T stands here for the typical scale of Kolesnikov's control. Eq.(4) has the solution:

$$\psi(t) = \psi(0)e^{-t/T}. \tag{5}$$

Now for the purpose of stabilization (2) let's define the goal function ψ as:

$$\psi(t) = v(t) - v_*. \tag{6}$$

Eq.(6) corresponds to the Kolesnikov's target attractor locking the phase trajectories in the neighborhood of the stabilization level v_*. The substitution of (6) into (1) provides the explicit form of the control parameter:

$$u(t) = bv(t) - \frac{1}{T}[v(t) - v_*]\,. \tag{7}$$

Thus, the control signal (7) drives the system (1) towards the goal (2) exponentially fast. The control problem is solved.

Let's now evaluate the power cost of Kolesnikov's control. First of all, we can easily see that the control signal is not off even if we are closed to our control goal:

$$u(t) \to \mathrm{b}v_* \text{ as } t \to \infty. \tag{8}$$

The power cost for the control (7) can be evaluated as:

$$P(t) = u(t)v(t). \tag{9}$$

By (7) we obtain:

$$P(t) = bv^2(t) - \frac{1}{T}v(t)[v(t) - v_*]\,, \tag{10}$$

with the asymptotic value: $P_* = bv_*^2$. Let's try to minimize the control power cost $P(t)$ via re-formulating the feedback (4) in the fractal space.

KOLESNIKOV'S FRACTIONAL FEEDBACK

We make the definition of Kolesnikov's fractional feedback:

$$D_t^{\alpha}\psi_{\alpha}(t) = -\frac{1}{T^{\alpha}}\psi_{\alpha}(t)\,;\; T = \text{const} > 0, \tag{11}$$

using here Caputo's fractional derivative [7]:

$$D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{df(\tau)/d\tau}{(t-\tau)^{\alpha}} d\tau\ ;\ 0 < \alpha < 1, \quad (12)$$

with the Γ-function:

$$\Gamma(\alpha) = \int_0^{+\infty} e^{-x} x^{\alpha-1} dx. \quad (13)$$

The positive constant T in (11) again has the dimension of time, as in (4). The goal function $\psi_{\alpha}(t)$ corresponds to (6).

To solve (11), we apply here the Rida-Arafa method [8] using the generalized Mittag-Leffler functions in the form of series [9]:

$$E_{\alpha}(ax^n) = \sum_{n=0}^{\infty} a^n \frac{x^{n\alpha}}{\Gamma(n\alpha+1)}. \quad (14)$$

By substitution the Mittag-Leffler series (14) into the fractional DE (11) we obtain:

$$\psi_{\alpha}(t) = \nu_* \sum_{n=0}^{\infty} (-1)^n \frac{(t/T)^{n\alpha}}{\Gamma(n\alpha+1)} = \nu_* E_{\alpha}\left(-\left(\frac{t}{T}\right)^{\alpha}\right). \quad (15)$$

Here the factor ν_* provides the correct velocity dimension of the goal function (15).

Now we are ready to fix the free fractional dimension parameter α by minimizing the power cost (9) for Kolesnikov's fractional feedback. Substituting the (15) into (9) we get:

$$\begin{aligned} P_{\alpha}(t) &= [b\nu_* + b\psi_{\alpha}(t) + \dot{\psi}_{\alpha}(t)] \cdot [\nu_* + \psi_{\alpha}(t)] = \\ &= b\nu_*^2 + 2by_{\alpha}(t) + \dot{y}_{\alpha}(t)\,. \end{aligned} \quad (16)$$

In (16) we used the notation:

$$y_{\alpha}(t) = \nu_* \psi_{\alpha}(t) + \frac{\psi_{\alpha}^2(t)}{2}. \quad (17)$$

The first term $b\nu_*^2$ in RHS(16) is a constant. To minimize the rest of P_{α} as a function of time t, let's get rid of the other two terms. We chose:

$$2by_{\alpha}(t) + \dot{y}_{\alpha}(t) \cong 0. \quad (18)$$

that could be evaluated asymptotically as:

$$y_{\alpha}(t) \cong y_{\alpha}(0) e^{-2bt}. \quad (19)$$

From another hand, to evaluate the control power cost, we use here the asymptotic properties of the generalized Mittag-Leffler functions [10], then we obtain the leading term as:

$$\psi_{\alpha}(t) \approx \frac{\nu_*}{\alpha} e^{-t/T}. \quad (20)$$

Making the asymptotic comparison of (19) and (20):

$$\frac{\nu_*}{\alpha} e^{-t/T} \propto \left[\nu_* \psi_{\alpha}(0) + \frac{\psi_{\alpha}^2(0)}{2}\right] e^{-2bt}, \quad (21)$$

from the exponents we conclude that:

$$T = 2b. \quad 22)$$

The factors multiplied by the exponents in (21) also must correspond each to another, such that we should chose α as:

$$\alpha = \frac{2\nu_*^2}{\nu^2(0) - \nu_*^2}. \quad (23)$$

Thus, we found that the fractional dimension parameter minimizing the Kolesnikov's control power cost (9) for the model (1) depends on the target level of stabilization ν_* (2). From another hand, our approach has a fractional dimension constraint:

$$\alpha < 1, \text{ then } \nu_* < \frac{\nu(0)}{\sqrt{3}}. \quad (24)$$

Only for such a domain of the target stabilization levels our approach can be successful.

CONCLUSIONS

For the certain constraints over the initial conditions and the target stabilization level, the integer-dimensional Kolesnikov algorithm can be replaced with the fractional target attractor feedback to provide the minimal power cost.

Our fractional re-formulation of Kolesnikov's control can be applied to a variety of physics and engineering models.

The approach proposed in the paper is robust. It is stable under the relatively small external perturbations. The choice of the fractional dimension parameter α depends on the stabilization level and on the initial conditions for the dynamical ODE. The asymptotic behavior of the solution defines also the typical control time scale T.

AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

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