Journal of Thermal Engineering, Vol. 5, No. 1, pp. 93-99, January, 2019
Yildiz Technical University Press, Istanbul, Turkey

Some properties of the topological spaces generated from the simple undirected graphs

Hatice Kübra Sarı[1]

ABSTRACT

Graph theory, which is used effectively in many fields from science to liberal arts, has very important place in our lives. As a result of this, the topological structure of the graphs is studied by many researchers. In this paper, we investigate the topological spaces generated by the graphs. The states of being an accumulation point and an interior point of a point in these spaces are examined. It is defined that relative topology on a subgraph of a graph. It is shown that this topology is different from the topology generated by this subgraph. Moreover, using the minimal adjacencies of vertices set of a graph, necessary and sufficient conditions for being T_0-space, T_1-space and Hausdorff space of the topological space generated from this graph are presented. This enables to examine whether the topological space is T_0, T_1 and Hausdorff without obtaining the topology generated from the graph.

Keywords: *Simple Undirected Graphs, Topological Spaces, Graph Theory, Relative Topology.*

1. INTRODUCTION

Graph theory is a branch of mathematics that deals with graphs and all quantitative and qualitative objects, concepts and phenomena associated with graphs. Graphs are mathematical structures consist of vertices and edges. They are used to model binary relationships between objects in a particular collection. A graph consists of vertices representing objects and edges linking these vertices. Graph theory was firstly proposed by Euler in 1736 [4]. Recently, the theory has been successfully applied to areas in different disciplines [9,10,11]. Since the theory is based on relational combinations, it has essential role in representing combinatorial objects and mathematical combinations. Rough set theory is also based on relational combinations. Thus, it has been studied that the relationship between rough sets and graphs by some researchers [6]. In [8], authors have defined vertice-centered metric topology on vertices set of a connected undirected graph. They have studied some properties of this metric topologies.

Applications of graph theory are used effectively to solutions of many problems in many fields. As a result of the widespread use of the theory, its topological structure has been a matter of curiosity. Some researchers have generated the topologies from graphs using various methods. In 2013, M. Amiri et. al. have created a topology using vertices of an undirected graph [2]. K. A. Abdu and A. Kılıçman have investigated the topologies generated from directed graphs in 2018 [1]. In 2020, H. K. Sarı and A. Kopuzlu have generated from simple undirected graphs without isolated vertices [7].

In this paper, it is investigated fistly some topological notions such as accumulation point and interior point in the topological spaces generated from graphs by H. K. Sarı and A. Kopuzlu. Then relative topology on a subgraph of a graph is defined. It is shown that this relative topology is not same the topology generated from this subgraph. Finally, conditions of being T_0, T_1 and Hausdorff space of the topological space generated from a graph is presented.

This paper is organized as follows. Section 2 reviews some fundamental concepts related to topological spaces and graphs. In Section 3, we investigate some properties of the topological space generated from the simple undirected graphs without isolated vertices. The results obtained from the work are presented in Section 4.

2. PRELIMINARIES

In this section, it is presented that some fundamental notions used in work.

2.1. Topological Concepts

[1] *Department of Mathematics, Faculty of Science, University of Ataturk, Erzurum, 25240, Turkey*
**E-mail address: haticekubra4659@gmail.com*
Orcid id: https://orcid.org/0000-0001-5110-1256
Manuscript Received 14 April 2021, Accepted 5 July 2021

Definiton 2.1.1.[5] Let X be a topological space. A point $x \in X$ is called an accumulation point of a subset A of X iff every open set U containing x contains a point of A different from x, That is,

$$U \text{ open}, x \in U \text{ implies } (U \backslash \{x\}) \cap A \neq \emptyset.$$

The set of accumulation points of the subset A is called derived set of A and it is denoted by A'.

Definition 2.1.2. [5] Let A be a subset of a topological space X. A point $x \in A$ is called an interior point of A if x belong to an open set U contained in A:

$$x \in U \subseteq A, \text{ where } U \text{ is open.}$$

Definition 2.1.3. [5] Let (X, τ) be a topological space and A be a non-empty subset of X. The class τ_A of all intersections of A with τ-open subsets of X is a topology on A. It is called the relative topology on A and (A, τ_A) is called a subspace of (X, τ). The class τ_A is defined as follow:

$$\tau_A = \{A \cap U : U \in \tau_A\}.$$

Definition 2.1.4. [5] Given a topological space X.

i) A topological space X is called a T_0-space iff it satisfies the following axiom:

[T_0] For any pair of distinct points in X, there exists an open set containing one of the points but not the other.

ii) A topological space X is called a T_1-space iff it satisfies the following axiom:

[T_1] For any pair of distinct points $x, y \in X$, there exists open sets U and V such that $x \in U, y \notin U$ and $y \in V, x \notin V$.

iii) A topological space X is called a Hausdorff space or T_2-space iff it satisfies the following axiom:

[T_2] Each pair of distinct points $x, y \in X$ belong respectively to disjoint open sets. The other words, there exists open sets U and V such that $x \in U, y \in V$ and $U \cap V = \emptyset$.

Theorem 2.1.1. [5] Let $\mathcal{B}$ be a class of subsets of a non-empty set X. Then $\mathcal{B}$ is a base for some topology on X iff it holds the following properties:

i) $X = \bigcup\{B : B \in \mathcal{B}\}$.

ii) For any $B_1, B_2 \in \mathcal{B}$, $B_1 \cap B_2$ is the union of elements of $\mathcal{B}$.

2.2. Graph Theory

Definition 2.2.1. [3] A graph G is an ordered pair $(V(G), E(G))$ consisting of a set $V(G)$ of vertices and a set $E(G)$, disjoint from $V(G)$, of edges, together with an incidence function ψ_G that associates with each edge of G an unordered pair of (not necessarily distinct) vertices of G. If V (and so E) is finite, G is finite graph.

A graph whose edges set is directed is called a directed graph. Otherwise, the graph is an undirected graph. The graphs we will used in our study are undirected graphs.

Definition 2.2.2. [3] A graph G' is called a subgraph of a graph G if $V(G') \subseteq V(G), E(G') \subseteq E(G)$ and ψ_G is the restriction of $\psi_{G'}$ to $E(G')$.

Definition 2.2.3. [3] A loop is an edge with identical ends. If there exist two or more edge linking same pair of vertices, then these edges are called parallel edges. A simple graph is a graph that has no loops or parallel edges.

Definition 2.2.4. [7] Given a graph $G = (V, E)$. The set of vertices becoming adjacent to a vertice x of G is called adjacency of x and it is denoted $A_G(x)$. The minimal adjacency of x is defined as follow:

$$[x]_G = \bigcap_{x \in A_G(v)} A_G(v).$$

Theorem 2.2.1. [7] Let $G = (V, E)$ be a simple undirected graph without isolated vertices. Then the class $\beta_G = \{[x]_G : x \in V\}$ is a base for a topology on V.

Definition 2. 2.5. [7] Given a simple undirected graph $G = (V, E)$ without isolated vertices. Then the topology generated by the class $\beta_G = \{[u]_G : u \in V\}$ is called the topology generated from G.

3. THE SOME TOPOLOGICAL NOTION IN TOPOLOGICAL SPACES GENERATED BY SIMPLE UNDIRECTED GRAPHS

Theorem 3.1. Let $G = (V, E)$ be a simple undirected graph without isolated vertices, τ_G be the topology generated from the graph G and $A \subset V$. Then $x \in V$ is an accumulation point of the subset A if and only if $([x]_G \backslash \{x\}) \cap A \neq \emptyset$.

Proof: The base of the topology generated from G and the topology generated from G are as follows, respectively:

$$\beta_G = \{[x]_G : x \in V\}$$

$$\tau_G = \{U \subseteq V : U = \cup_{x \in V' \subseteq V} [x]_G, V' \subseteq V\}.$$

Let $x \in V$ be an accumulation point of the subset A. Then, for every open set U containing x, it is obtained that

$$(U \backslash \{x\}) \cap A \neq \emptyset.$$

Hence, it is seen that

$$\big((\cup_{x \in V' \subseteq V} [x]_G) \backslash \{x\}\big) \cap A \neq \emptyset.$$

Thus, it is obtained that

$$([x]_G \backslash \{x\}) \cap A \neq \emptyset.$$

Conversely, we assume that $([x]_G \backslash \{x\}) \cap A \neq \emptyset$, for $x \in V$. Then, it is obtained that

$$\big((\cup_{x \in V' \subseteq V} [x]_G) \backslash \{x\}\big) \cap A \neq \emptyset.$$

Thus, for every $x \in U \in \tau_G$, we have

$$(U \backslash \{x\}) \cap A \neq \emptyset.$$

Consequently, x is an accumulation point of A.

Theorem 3.2. Let $G = (V, E)$ be a simple undirected graph without isolated vertices, τ_G be topology generated from the graph G and $A \subset V$. $x \in A$ is an interior point if and only if $[x]_G \subseteq A$.

Proof: The base of τ_G is in the form of:

$$\beta_G = \{[x]_G : x \in V\}.$$

Let $x \in A$ be an interior point of A. Then, there exists an element U of τ_G such that $x \in U \subseteq A$. Since $U = \cup_{x \in V' \subseteq V} [x]_G$, it is seen that $x \in \cup_{x \in V' \subseteq V} [x]_G \subseteq A$. Thus, it is obtained that

$$[x]_G \subseteq A.$$

Conversely, let $[x]_G \subseteq A$. Since $[x]_G \in \beta_G$, it is seen that$[x]_G \in \tau_G$. Thus, it is obtained that $x \in [x]_G \subseteq A$. Hence, it is said to $x \in A$ is an interior point of A.

We shall consider the following example using above theorems which enables us to find the interior and derivative set of a subset of the topological space generated from a graph without founding the topology.

Example 3.1. Given graph $G = (V, E)$ in the Figure 2. Let us consider the topological space (V, τ_G).

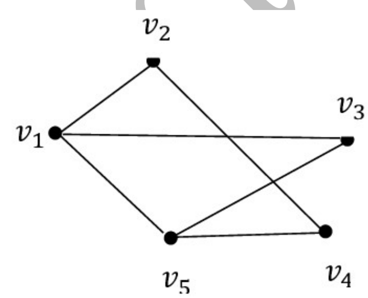

Figure 2. The Graph G

The minimal adjacencies of vertices of G are as follows:

$$[v_1]_G = \{v_1\}, [v_2]_G = \{v_2, v_5\}, [v_3]_G = \{v_3\}, [v_4]_G = \{v_1, v_4\}, [v_5]_G = \{v_5\}.$$

Given the subset $U = \{v_2, v_3, v_4\} \subseteq V$. Let us investigate accumulation points of U. Since $([v_1]_G \backslash \{v_1\}) \cap U = \emptyset$, v_1 is not an accumulation point of U. Similarly, when the points of v_2, v_3, v_4, v_5 is considered, it is seen that they are not accumulation points of U. Consequently, derivative set of U is found as follow:

$$U' = \emptyset.$$

Let us now investigate interior points of U. Since $[v_2]_G \nsubseteq U$, v_2 is not an interior point of U. Since $[v_3]_G \subseteq U$, v_3 is an interior point of U. Since $[v_4]_G \nsubseteq U$, v_4 is not an interior point of U. Consequently, the set of interior points of U is found as follow:

$$U^o = \{v_3\}.$$

Theorem 3.3. Given a simple undirected graph $G = (V, E)$ without isolated vertices. Let $G' = (V', E')$ be a subgraph of G, where $V' \subseteq V$ and $E' \subseteq E$. Then the class $(\beta_G)_{G'} = \{V' \cap [x]_G : [x]_G \in \beta_G\}$ is a base for a topology on V'.

Proof: Firstly, we shall show that $\cup_{x \in V} (V' \cap [x]_G) = V'$. It is known that

$$\bigcup_{x\in V}(V'\cap[x]_G)=V'\cap(\bigcup_{x\in V}[x]_G).$$

Since β_G is a base for the topology τ_G, it is obtained that $\bigcup_{x\in V}[x]_G=V$, for every $x\in V$. Thus, we have that

$$\bigcup_{x\in V}(V'\cap[x]_G)=V'\cap(\bigcup_{x\in V}[x]_G)=V'\cap V=V'.$$

Secondly, we shall show that $(V'\cap[x]_G)\cap(V'\cap[y]_G)$ is the union of elements of $(\beta_G)_{G'}$, for any $V'\cap[x]_G, V'\cap[y]_G\in(\beta_G)_{G'}$. It is known that

$$(V'\cap[x]_G)\cap(V'\cap[y]_G)=V'\cap([x]_G\cap[y]_G).$$

Since β_G is a base for the topology τ_G, there exist some $z\in V$ such that $[x]_G\cap[y]_G=\bigcup_{z\in V}[z]_G$. Thus, it is obtained that

$$\begin{aligned}(V'\cap[x]_G)\cap(V'\cap[y]_G)&=V'\cap([x]_G\cap[y]_G)\\&=V'\cap(\bigcup_{z\in V}[z]_G)\\&=\bigcup_{z\in V}(V'\cap[z]_G).\end{aligned}$$

Consequently, $(\beta_G)_{G'}$ is a base for a topology on V'.

Definition 3.1. Let $G=(V,E)$ be a simple undirected graph without isolated vertices and $G'=(V',E')$ be a subgraph of G. Then the topology generated by $(\beta_G)_{G'}$ is called relative topology on G'. It is denoted $(\tau_G)_{G'}$.

Example 3.2. Given the graph $G=(V,E)$ and the subgraph $G'=(V',E')$ of G in Figure 1. We shall find the relative topology $(\tau_G)_{G'}$on V'.

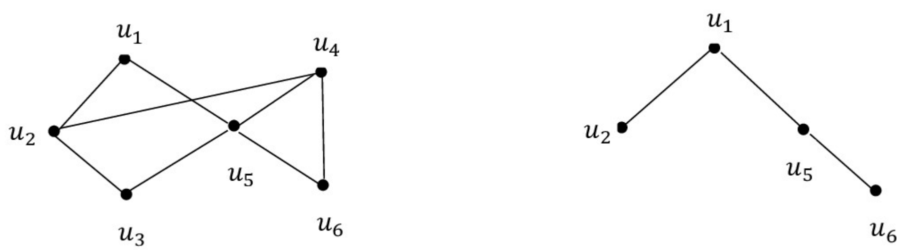

Figure 1. The Graph G and The Subgraph G'

The base of the topology τ_G generated from the graph G in the form of:

$$\beta_G=\{\{u_1,u_3,u_4\},\{u_2,u_5\},\{u_4\},\{u_5\},\{u_6\}\}.$$

Thus, the base of the relative topology on G' is found as follow:

$$(\beta_G)_{G'}=\{\{u_1\},\{u_2,u_5\},\{u_5\},\{u_6\}\}.$$

Hence, the relative topology $(\tau_G)_{G'}$ is in the form of:

$$(\tau_G)_{G'}=\{V',\emptyset,\{u_1\},\{u_5\},\{u_6\},\{u_1,u_5\},\{u_1,u_6\},\{u_2,u_5\},\{u_5,u_6\},\{u_1,u_2,u_5\},\{u_2,u_5,u_6\},\{u_1,u_5,u_6\}\}.$$

Now, we investigate topology generated from the subgraph G' of G. The base of $\tau_{G'}$ is as follow:

$$\beta_{G'}=\{\{u_1\},\{u_2,u_5\},\{u_5\},\{u_1,u_6\}\}.$$

Then $\tau_{G'}$ is in the form of:

$$\tau_{G'}=\{V',\emptyset,\{u_1\},\{u_5\},\{u_1,u_6\},\{u_2,u_5\},\{u_1,u_5\},\{u_1,u_2,u_5\},\{u_1,u_5,u_6\}\}.$$

It is clearly seen that $(\tau_G)_{G'}$ is different from $\tau_{G'}$.

As seen above example, the relative topology on the subgraph G' of G is not same the topology generated from the subgraph G' of G.

Theorem 3.4. Let $G=(V,E)$ be a simple undirected graph without isolated vertices and τ_G be the topology generated from G. The topological space (V,τ_G) is a T_0-space if and only if for each pair x,y of distinct points of V, it is satisfied that $y\notin[x]_G$ or $x\notin[y]_G$.

Proof: Suppose (V,τ_G) is a T_0-space and $x,y\in V$. Then there exists an open set U_1such that $x\in U_1, y\notin U_1$ or an open set U_2 such that $y\in U_2, x\notin U_2$. Since $[x]_G\in\beta_G$, it is seen that $x\in[x]_G\subseteq U_1$. Similarly, it is seen that $y\in[y]_G\subseteq U_2$. Accordingly, we obtain that

$$y\notin[x]_G \text{ or } x\notin[y]_G.$$

Conversely, suppose $y\notin[x]_G$ or $x\notin[y]_G$, for each pair x,y of distinct points of V. Then there exists the open set $[x]_G$ such that $x\in[x]_G, y\notin[x]_G$ or the open set $[y]_G$ such that $y\in[y]_G, x\notin[y]_G$. Thus, (V,τ_G) is a T_0-space.

Theorem 3.5. Let $G = (V, E)$ be a simple undirected graph without isolated vertices and τ_G be the topology generated from G. The topological space (V, τ_G) is a T_1-space if and only if for each pair x, y of distinct points of V, it is satisfied that $y \notin [x]_G$ and $x \notin [y]_G$.

Proof: Suppose that (V, τ_G) is a T_1-space. Let $x, y \in V$. Then there exist open sets U_1 and U_2 such that $x \in U_1, y \notin U_1$ and $y \in U_2, x \notin U_2$. Since $[x]_G \in \beta_G$, it is seen that $x \in [x]_G \subseteq U_1$. Similarly, it is seen that $y \in [y]_G \subseteq U_2$. Accordingly, we obtain that

$$y \notin [x]_G \text{ and } x \notin [y]_G.$$

Conversely, suppose $y \notin [x]_G$ and $x \notin [y]_G$, for each pair x, y of distinct points of V. Then there exist open sets $[x]_G$ and $[y]_G$ such that $x \in [x]_G, y \notin [x]_G$ and $y \in [y]_G, x \notin [y]_G$. Thus, (V, τ_G) is a T_1-space.

Theorem 3.6. Let $G = (V, E)$ be a simple undirected graph without isolated vertices and τ_G be the topology generated from G. The topological space (V, τ_G) is a Hausdorff space if and only if $[x]_G \cap [y]_G = \emptyset$, for each pair x, y of distinct points of V.

Proof: Let (V, τ_G) be a Hausdorff space and $x, y \in V$. Then there exist open sets U_1 and U_2 such that $x \in U_1, y \in U_2$ and $U_1 \cap U_2 = \emptyset$, for each pair of x, y of distinct points of V. Since $[x]_G \in \beta_G$, it is seen that $x \in [x]_G \subseteq U_1$. Similarly, it is seen that $y \in [y]_G \subseteq U_2$. Accordingly, we obtain that

$$[x]_G \cap [y]_G = \emptyset.$$

Conversely, suppose $[x]_G \cap [y]_G = \emptyset$, for each pair x, y of distinct points of V. Then, since there exist open sets $[x]_G$ and $[y]_G$ such that $x \in [x]_G$, $y \in [y]_G$ and $[x]_G \cap [y]_G = \emptyset$, (V, τ_G) is a Hausdorff space.

In following example, we shall examine whether the topology generated from a simple undirected graph is a T_0, T_1 and Hausdorff space using above theorems.

Example 3.3. Let us examine whether the topological space (V, τ_G) generated from the graph G given in Figure 2 is T_0, T_1 and Hausdorff space.

Since it is obtained that $v_i \notin [v_j]_G$ or $v_j \notin [v_i]_G$, for each pair of v_i, v_j of distinct points of V, V is a T_0-space. But V is not a T_1-space, since $v_1 \in [v_4]_G$, for $v_1, v_4 \in V$. Moreover, $[v_i]_G \cap [v_j]_G \neq \emptyset$, some $v_i, v_j \in V$. Hence, V is not a Hausdorff space.

4. CONCLUSION

In this paper, the topological space generated from simple undirected graphs without isolated vertices is studied. It is presented necessary and sufficient condition for a point in this graph to be an accumulation point and an interior point. Thus, when a simple undirected graph $G = (V, E)$ and a subset of V are given, interior of this subset and its derivative set can be easily found without needing to obtain the topology generated from this graph. Relative topology on a subgraph of a graph is defined. It is shown that this relative topology is not the topology generated from this subgraph. Moreover, conditions to be T_0, T_1 and Hausdorff space for the topological space generated from a graph is given by using the minimal adjacencies of vertices set of this graph.

REFERENCES

[1] K. A. Abdu, A. Kılıçman, Topologies on the Edges Set of Directed Graphs, J. Math. Computer Sci., 18 (2018), 232-241.

[2] S. M. Amiri, A. Jafarzadeh, H. Khatibzadeh, An Alexandroff Topology on Graphs, Bulletin of the Iranian Mathematical Society. 39 (4) (2013), 647-662.

[3] J.A. Bondy, U.S.R. Murty, Graph Theory, Springer, Berlin, 2008.

[4] L. Euler, Solutio problematis ad geometriam situs pertinentis. Commentarii Academiae Scientiarum Imperialis Petropolitanae, 8 (1736), 128-140.

[5] S. Lipschutz, Schaum's Outline of Theory and Problems of General Topology, Mcgraw-Hill Book Company, New York, St. Louis, San Francisco, Toronto,Sydney, 1965.

[6] H. K. Sarı, A. Kopuzlu, A Note on a Binary Relation corresponding to a Bipartite Graph, ITM Web of Conferences 22 (2018) 01039.

[7] H. K. Sarı, A. Kopuzlu, On Topological Spaces Generated by Simple Undirected Graphs, Aims Matehamtics, 5 (6) (2020), 5541-5550.

[8] H. K. Sarı, A Kopuzlu, The vertice-centered metric topologies generated from the connected undirected graphs,

Malaysian Journal of Mathematical Sciences 15(2): 243–252 (2021).
[9] A. Şahin, Dichromatic polynomial for graph of a (2, n)-torus knot, Applied Mathematics and Nonlinear Sciences, https://doi.org/10.2478/amns.2020.2.00068.
[10] B. Şahin, A. Şahin, On total vertex-edge domination, TWMS J. App. Eng. Math. V.9, No.1, Special Issue, 2019, pp. 128-133.
[11] A. Şahin, Coloring in graphs of twist knots,Numerical Methods for Partial Differential Equations, (2020), 1-8. https://doi.org/10.1002/num.22714.