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Research Article

A comprehensive survey of dual-generalized complex Fibonacci and Lucas numbers

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ABSTRACT

This paper aims to develop dual-generalized complex Fibonacci and Lucas numbers and obtain recurrence relations. Fibonacci and Lucas's approach to dual-generalized complex numbers contains dual-complex, hyper-dual and dual-hyperbolic situations as special cases and allows general contributions to the literature for all real number $\mathfrak{p}$. For this purpose, Binet's formulas along with Tagiuri's, Hornsberger's, D'Ocagne's, Cassini's and Catalan's identities, are calculated for dual-generalized complex Fibonacci and Lucas numbers. Finally, the results are given, and the special cases for this unification are classified.

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INTRODUCTION

The generalized complex numbers (for details see [1,2]) are defined in the form $z = a_1 + a_2 J$ where $a_1, a_2 \in \mathbb{R}$. Here, J denotes the generalized complex unit where $J^2 = \mathfrak{p}, J \notin \mathbb{R}, \mathfrak{p} \in \mathbb{R}$. This set is analogue to the complex numbers $\mathbb{C}$ for $\mathfrak{p} = -1$, the hyperbolic numbers $\mathbb{H}$ for $\mathfrak{p} = 1$ and the dual numbers $\mathbb{D}$ for $\mathfrak{p} = 0$ (see details in [3,4]). The idea of investigating the number systems by writing the coefficients as elements of the complex, hyperbolic and dual numbers is a fascinating area for researchers. Hence, over the years the various types of number systems have been constructed employ this idea. Hyperbolic numbers with complex coefficients (complex-hyperbolic numbers or hyperbolic-complex numbers) are examined in [1, 5, 6]. n-dimensional hyperbolic-complex and bicomplex numbers are investigated in [7–10], respectively. Dual-complex numbers are presented in [11–13]. In [14], the notion of dual-complex numbers and their holomorphic functions are investigated. Dual-hyperbolic numbers and their algebraic properties are discussed in [13]. Besides, the functions and various matrix representations of dual-hyperbolic numbers

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and complex-hyperbolic numbers are presented in [15]. Furthermore, as an extension of dual numbers, hyper-dual numbers are studied in [16, 17]. Quite a few studies in the literature are related to the types of numbers which consist of different combination of the coefficients, [18–20]. Motivated by the above mentioned papers and using the Cayley-Dickson doubling procedure for construction, the dual-generalized complex ($\mathcal{DGC}$) numbers are investigated in [21] with the form $w = z_1 + z_2\varepsilon$, where z_1, z_2 are generalized complex numbers. Here ε is pure dual unit with conditions $\varepsilon^2 = 0, \varepsilon \neq 0, \varepsilon \notin \mathbb{R}$. In this study various properties and matrix representations are considered. $\mathcal{DGC}$ numbers correspond to the dual-complex, hyper-dual and dual-hyperbolic numbers for the special real values $\mathfrak{p} = -1$, $\mathfrak{p} = 0$ and $\mathfrak{p} = 1$, respectively.

The Fibonacci and Lucas sequences have impressed researchers in other respects for centuries. The properties of Fibonacci and Lucas numbers and their relations can be found in classical studies [22–27]. Several studies in the literature are conducted considering Fibonacci and Lucas numbers, (see some papers in [28–30]).

The Fibonacci sequence and the golden ratio (the ratio of sequential Fibonacci numbers) have been widely discovered in biology, theoretical physics, chemistry, technology and nature. In 2015, it was found in amino acids and codons, the constituent molecules of genetic codes, [31]. It also appears in a ladder network of equal resistors, [32]. The characteristics of particular electrical network can be written as a function of Fibonacci and Lucas numbers, [32–34]. Furthermore, some properties of resonant electronic systems are related to Fibonacci numbers. In 2020, a proposed optical resonant device, with resonant frequencies spaced according to the Fibonacci sequences, demonstrating the analogy with coupled electrical resonant cells was discussed, [35].

It is natural to study Fibonacci and Lucas's versions of the above mentioned type of numbers. By using dual-complex numbers in [14], dual-complex Fibonacci and Lucas numbers are defined in [36] and Binet's formulas, and D'Ocagne, Catalan's and Cassini's identities are obtained. Likewise, dual-complex numbers with generalized Fibonacci and Lucas numbers coefficients are discussed in [37]. In analogy to dual-complex Fibonacci and Lucas numbers, dual-hyperbolic Fibonacci and Lucas numbers and their identities are introduced in [38]. Besides, dual-hyperbolic numbers with generalized Fibonacci and Lucas numbers coefficients are examined in [39, 40]. Hyper-dual generalized Fibonacci numbers are examined in [41]. Additionally, in [42], the researchers obtained some properties of Fibonacci and Lucas numbers by regarding them as a generalized complex Fibonacci and Lucas numbers.

As summarized above, dual-complex, hyper-dual and dual hyperbolic Fibonacci/Lucas numbers are already known in the literature and obtained by considering the value as −1, 0, and 1 for $\mathfrak{p}$. The generalization according to any $\mathfrak{p} \in \mathbb{R}$ is missing. Under these circumstances, the following open problem needs to be answered:

Problem: Is there a probability of occurrence of an extension of dual-complex, hyper-dual and dual-hyperbolic Fibonacci/Lucas numbers for any value of $\mathfrak{p}$? If the answer is affirmative, what algebraic properties and recurrence relations are satisfied?

Based on this open problem, we introduce the theory of $\mathcal{DGC}$ Fibonacci and Lucas numbers. This paper is organized as follows: Section 2 presents basic notations that are used throughout the paper. Section 3 attempts to develop $\mathcal{DGC}$ Fibonacci and Lucas numbers, and also includes the operations on these numbers and several equalities. Also familiar Tagiuri's, Hornsberger's, Binet's formulas, D'Ocagne's, Cassini's and Catalan's identities for these types of numbers are extended and examples are given. This approach shows us the given problem has an affirmative answer and the theory is improved with respect to any real value of $\mathfrak{p}$ ($J^2 = \mathfrak{p}$), not only for −1, 1 and 0. Finally, in Section 4 the results are concluded, the classifications are given and the concrete contributions of the study are discussed.

GENERAL INFORMATION

In this section, we recall some basic notations and results related to Fibonacci/Lucas numbers (see [22-27]) and $\mathcal{DGC}$ numbers (see [21]).

Fibonacci and Lucas Numbers

Fibonacci sequence is defined by $F_{n+1} = F_n + F_{n-1},\ n \geq 1$ with $F_0 = 0,\ F_1 = 1$. The nth Fibonacci number F_n can be formulated by the Binet's formula:

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \tag{1}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Thus, the golden ratio is obtained by using Binet's formula as $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \alpha$. Moreover, nega-Fibonacci numbers are calculated by

$$F_{-n} = (-1)^{n+1} F_n. \tag{2}$$

Analogically, the Lucas sequence is defined by changing the initial numbers in the Fibonacci sequence and written by $L_{n+1} = L_n + L_{n-1},\ n \geq 1$ with $L_0 = 2,\ L_1 = 1$. The nth Lucas number L_n can be formulated by Binet's formula:

$$L_n = \alpha^n + \beta^n. \tag{3}$$

Similar to the Fibonacci numbers, the golden ratio is obtained by using Binet's formula as $\lim_{n\to\infty} \frac{L_{n+1}}{L_n} = \alpha$.

Table 1. Identities for Fibonacci and Lucas numbers

	Fibonacci case	Lucas case
D'Ocagne's	$F_mF_{n+1}-F_{m+1}F_n=(-1)^nF_{m-n}$	$L_mL_{n+1}-L_{m+1}L_n=5(-1)^{n+1}F_{m-n}$
Catalan's	$F_n^2-F_{n+r}F_{n-r}=(-1)^{n-r}F_r^2$	$L_n^2-L_{n+r}L_{n-r}=5(-1)^{n-r+1}F_r^2$
Cassini's	$F_n^{\ 2}-F_{n+1}F_{n-1}=(-1)^{n-1}$	$L_n^{\ 2}-L_{n+1}L_{n-1}=5(-1)^n$

The nega-Lucas numbers can easily be seen by using equation $L_{-n}=(-1)^nL_n$.

Proposition 1. The linear relationships involve the sums or differences of Fibonacci/Lucas numbers and the products of at most 2 of them are given by (see [22–27]):

$$F_{n+r}+F_{n-r}=\begin{cases}F_nL_r, & r=2k\\ F_rL_n, & r=2k+1,\end{cases} \tag{4}$$

$$F_{n+r}-F_{n-r}=\begin{cases}F_rL_n, & r=2k\\ F_nL_r, & r=2k+1,\end{cases} \tag{5}$$

$$F_mF_n-F_{m+r}F_{n-r}=(-1)^{n-r}F_{m+r-n}F_r, \tag{6}$$

$$F_{n+1}^2-F_n^2=F_{n+2}F_{n-1}, \tag{7}$$

$$F_{n+1}^2+F_n^2=F_{2n+1}, \tag{8}$$

$$F_{n+1}^2-F_{n-1}^2=F_{2n}, \tag{9}$$

$$F_nF_m+F_{m+1}F_{n+1}=F_{m+n+1}, \tag{10}$$

$$L_{n+r}+L_{n-r}=\begin{cases}L_nL_r, & r=2k\\ 5F_nF_r, & r=2k+1,\end{cases}$$

$$L_{n+r}-L_{n-r}=\begin{cases}5F_nF_r, & r=2k\\ L_nL_r, & r=2k+1,\end{cases}$$

$$L_mL_n-L_{m+r}L_{n-r}=5(-1)^{n-r+1}F_{m+r-n}F_r,$$

$$L_{n+1}^2-L_n^2=L_{n+2}L_{n-1},$$

$$L_{n+1}^2+L_n^2=5F_{2n+1},$$

$$L_{n+1}^2-L_{n-1}^2=5F_{2n},$$

$$L_nL_m+L_{m+1}L_{n+1}=5F_{m+n+1}.$$

Table 2. Multiplication scheme of $\mathcal{DGC}$ numbers, [21]

	1	J	ε	$J\varepsilon$
1	1	J	ε	$J\varepsilon$
J	J	$\mathfrak{p}$	$J\varepsilon$	$\mathfrak{p}\varepsilon$
ε	ε	$J\varepsilon$	0	0
$J\varepsilon$	$J\varepsilon$	$\mathfrak{p}\varepsilon$	0	0

Theorem 1 (see [26]). The D'Ocagne's, Catalan's and Cassini's identities for Fibonacci and Lucas numbers are given in as Table 1.

$\mathcal{DGC}$ numbers

The set of generalized complex numbers $\mathbb{C}_\mathfrak{p}$ (for details see [1,2]) is defined by:

$$\mathbb{C}_\mathfrak{p}:=\left\{z=a_1+a_2J \mid a_1,a_2\in\mathbb{R},\ J^2=\mathfrak{p},\ J\notin\mathbb{R},\ \mathfrak{p}\in\mathbb{R}\right\}.$$

Considering this set and inspired by the Cayley-Dickson doubling procedure, the set of $\mathcal{DGC}$ numbers is defined as follows (see [21]):

$$\mathbb{DC}_\mathfrak{p}:=\left\{w=z_1+z_2\varepsilon \mid z_1,z_2\in\mathbb{C}_\mathfrak{p},\ \varepsilon^2=0,\ \varepsilon\neq 0,\ \varepsilon\notin\mathbb{R}\right\},$$

where the base elements $\{1,J,\varepsilon,J\varepsilon\}$ satisfy the properties given in Table 2. Here $J\varepsilon$ is called the generalized complex-dual unit.

The operations for $\mathcal{DGC}$ numbers are given by the following, respectively, [21]:

$$\begin{aligned} w_1=w_2 &\Leftrightarrow z_{11}=z_{21}, z_{12}=z_{22},\\ w_1+w_2 &=(z_{11}+z_{21})+(z_{12}+z_{22})\varepsilon,\\ \lambda w_1 &=\lambda z_{11}+\lambda z_{12}\varepsilon,\\ w_1w_2 &=(z_{11}z_{21})+(z_{11}z_{22}+z_{12}z_{21})\varepsilon,\end{aligned}$$

where $w_1=z_{11}+z_{12}\varepsilon$, $w_2=z_{21}+z_{22}\varepsilon\in\mathbb{DC}_\mathfrak{p}$ and $\lambda\in\mathbb{R}$. $\mathbb{DC}_\mathfrak{p}$ is a commutative ring with unity and a vector space over real numbers.

$\mathcal{DGC}$ FIBONACCI AND LUCAS NUMBERS

In this original section, utilizing Section 2, the mathematical formulation of Fibonacci-Lucas numbers with $\mathcal{DGC}$ components is presented and their algebraic properties, linear sums (or differences) and products of at most 2 are obtained for $\mathfrak{p} \in \mathbb{R}$. Finally, we extend important characteristic identities for Fibonacci and Lucas numbers to $\mathcal{DGC}$ Fibonacci and Lucas numbers.

Definition 1. The $\mathcal{DGC}$ Fibonacci and $\mathcal{DGC}$ Lucas numbers are defined by:

$$\tilde{\mathcal{F}}_n = F_n + F_{n+1}J + F_{n+2}\varepsilon + F_{n+3}J\varepsilon \tag{11}$$

and

$$\tilde{\mathcal{L}}_n = L_n + L_{n+1}J + L_{n+2}\varepsilon + L_{n+3}J\varepsilon, \tag{12}$$

where F_n and L_n are the nth Fibonacci and Lucas numbers, respectively.

The set of $\mathcal{DGC}$ Fibonacci and Lucas numbers are denoted by $\mathbb{DC}_{\mathfrak{p}}\mathbb{F}$ and $\mathbb{DC}_{\mathfrak{p}}\mathbb{L}$, respectively. Furthermore, a $\mathcal{DGC}$ Fibonacci number can also be expressed as:

$$\tilde{\mathcal{F}}_n = \left(F_n + F_{n+1}J\right) + \left(F_{n+2} + F_{n+3}J\right)\varepsilon.$$

Definition 2. Let $\tilde{\mathcal{F}}_n, \tilde{\mathcal{F}}_m \in \mathbb{DC}_{\mathfrak{p}}\mathbb{F}$. Then, the equality, the addition, subtraction, scalar multiplication, and multiplication of these numbers are defined by, respectively:

$$\begin{aligned}&\tilde{\mathcal{F}}_n = \tilde{\mathcal{F}}_m \Leftrightarrow F_n = F_m,\ F_{n+1} = F_{m+1},\ F_{n+2} = F_{m+2},\ F_{n+3} = F_{m+3}.\\ &\tilde{\mathcal{F}}_n \pm \tilde{\mathcal{F}}_m = (F_n \pm F_m) + (F_{n+1} \pm F_{m+1})J + (F_{n+2} \pm F_{m+2})\varepsilon\\ &+(F_{n+3} \pm F_{m+3})J\varepsilon,\\ &\lambda\tilde{\mathcal{F}}_n = (\lambda F_n) + (\lambda F_{n+1})J + (\lambda F_{n+2})\varepsilon + (\lambda F_{n+3})J\varepsilon,\ \lambda \in \mathbb{R},\end{aligned} \tag{13}$$

$$\begin{aligned}\tilde{\mathcal{F}}_n \times \tilde{\mathcal{F}}_m = {} & F_nF_m + \mathfrak{p}F_{n+1}F_{m+1} + \left(F_{n+1}F_m + F_nF_{m+1}\right)J\\ & + \left(F_nF_{m+2} + F_{n+2}F_m + \mathfrak{p}(F_{n+1}F_{m+3} + F_{n+3}F_{m+1})\right)\varepsilon\\ & + \left(F_{n+1}F_{m+2} + F_nF_{m+3} + F_{n+3}F_m + F_{n+2}F_{m+1}\right)J\varepsilon.\end{aligned} \tag{14}$$

Table 3. Conjugations and modules

Definition	Formula
Generalized complex conjugation	$\tilde{\mathcal{F}}_n^{\dagger_1} = \left(F_n - F_{n+1}J\right) + \left(F_{n+2} - F_{n+3}J\right)\varepsilon$
Generalized complex module	$\lvert\tilde{\mathcal{F}}_n\rvert_{\dagger_1}^2 = \tilde{\mathcal{F}}_n \times \tilde{\mathcal{F}}_n^{\dagger_1}$
Dual conjugation	$\tilde{\mathcal{F}}_n^{\dagger_2} = \left(F_n + F_{n+1}J\right) - \left(F_{n+2} + F_{n+3}J\right)\varepsilon$
Dual module	$\lvert\tilde{\mathcal{F}}_n\rvert_{\dagger_2}^2 = \tilde{\mathcal{F}}_n \times \tilde{\mathcal{F}}_n^{\dagger_2}$
Coupled conjugation	$\tilde{\mathcal{F}}_n^{\dagger_3} = \left(F_n - F_{n+1}J\right) - \left(F_{n+2} - F_{n+3}J\right)\varepsilon$
Coupled module	$\lvert\tilde{\mathcal{F}}_n\rvert_{\dagger_3}^2 = \tilde{\mathcal{F}}_n \times \tilde{\mathcal{F}}_n^{\dagger_3}$
$\mathcal{DGC}$ conjugation	$\tilde{\mathcal{F}}_n^{\dagger_4} = \left(F_n - F_{n+1}J\right)\left(1 - \frac{F_{n+2} + F_{n+3}J}{F_n + F_{n+1}J}\varepsilon\right)$
$\mathcal{DGC}$ module	$\lvert\tilde{\mathcal{F}}_n\rvert_{\dagger_4}^2 = \tilde{\mathcal{F}}_n \times \tilde{\mathcal{F}}_n^{\dagger_4}$
Anti-dual conjugation	$\tilde{\mathcal{F}}_n^{\dagger_5} = \left(F_{n+2} + F_{n+3}J\right) - \left(F_n + F_{n+1}J\right)\varepsilon$

Definition 3. Let $\tilde{\mathcal{F}}_n \in \mathbb{DC}_{\mathfrak{p}}\mathbb{F}$. The different conjugations and modules in $\mathbb{DC}_{\mathfrak{p}}\mathbb{F}$ can be defined as in Table 3.

Proposition 2. Let $\tilde{\mathcal{F}}_n \in \mathbb{DC}_{\mathfrak{p}}\mathbb{F}$. Then, the properties below can be given:

1) $\tilde{\mathcal{F}}_n + \tilde{\mathcal{F}}_n^{\dagger_1} = 2\left(F_n + F_{n+2}\varepsilon\right)$,

2) $\tilde{\mathcal{F}}_n \times \tilde{\mathcal{F}}_n^{\dagger_1} = F_n^2 - \mathfrak{p}F_{n+1}^2 + 2\left(F_nF_{n+2} - \mathfrak{p}F_{n+1}F_{n+3}\right)\varepsilon$,

3) $\tilde{\mathcal{F}}_n + \tilde{\mathcal{F}}_n^{\dagger_2} = 2\left(F_n + F_{n+1}J\right)$,

4) $\tilde{\mathcal{F}}_n \times \tilde{\mathcal{F}}_n^{\dagger_2} = F_n^2 + \mathfrak{p}F_{n+1}^2 + 2F_nF_{n+1}J$,

5) $\tilde{\mathcal{F}}_n + \tilde{\mathcal{F}}_n^{\dagger_3} = 2\left(F_n + F_{n+3}J\varepsilon\right)$,

6) $\tilde{\mathcal{F}}_n \times \tilde{\mathcal{F}}_n^{\dagger_3} = F_n^2 - \mathfrak{p}F_{n+1}^2 + 2(-1)^n J\varepsilon$,

7) $\left(F_n + F_{n+1}J\right)\tilde{\mathcal{F}}_n^{\dagger_4} = \left(F_n - F_{n+1}J\right)\tilde{\mathcal{F}}_n^{\dagger_2}$
$\quad = F_n^2 - \mathfrak{p}F_{n+1}^2 - \left(F_nF_{n+2} - \mathfrak{p}F_{n+1}F_{n+3}\right)\varepsilon + (-1)^n J\varepsilon$,

8) $\tilde{\mathcal{F}}_n \times \tilde{\mathcal{F}}_n^{\dagger_4} = F_n^2 - \mathfrak{p}F_{n+1}^2$,

9) $\tilde{\mathcal{F}}_n - \varepsilon\tilde{\mathcal{F}}_n^{\dagger_5} = F_n + F_{n+1}J$,

10) $\varepsilon\tilde{\mathcal{F}}_n + \tilde{\mathcal{F}}_n^{\dagger_5} = F_{n+2} + F_{n+3}J$.

It should be noted that Definition 2, Definition 3 and Proposition 2 can be given for $\mathcal{DGC}$ Lucas numbers, similarly.

Let us extend the familiar relations of Fibonacci/Lucas numbers to $\mathcal{DGC}$ Fibonacci/Lucas numbers.

Theorem 2. Let $\tilde{\mathcal{F}}_n \in \mathbb{DC}_{\mathfrak{p}}\mathbb{F}$ and $\tilde{\mathcal{L}}_n \in \mathbb{DC}_{\mathfrak{p}}\mathbb{L}$. Then, the following relations hold for $n, m, r \geq 0$:

1) $\tilde{\mathcal{F}}_n + \tilde{\mathcal{F}}_{n+1} = \tilde{\mathcal{F}}_{n+2}$,

2) $\tilde{\mathcal{F}}_{n+r} + \tilde{\mathcal{F}}_{n-r} = \begin{cases} L_r\tilde{\mathcal{F}}_n, & r = 2k \\ F_r\tilde{\mathcal{L}}_n, & r = 2k+1, \end{cases}$

3) $\tilde{\mathcal{F}}_{n+r} - \tilde{\mathcal{F}}_{n-r} = \begin{cases} F_r\tilde{\mathcal{L}}_n, & r = 2k \\ L_r\tilde{\mathcal{F}}_n, & r = 2k+1, \end{cases}$

4) $\tilde{\mathcal{F}}_m \times \tilde{\mathcal{F}}_n - \tilde{\mathcal{F}}_{m+r} \times \tilde{\mathcal{F}}_{n-r} = (-1)^{n-r}F_{m-n+r}F_r[(1-\mathfrak{p})$
$\quad + J + 3(1-\mathfrak{p})\varepsilon + 3J\varepsilon]$,

5) $\tilde{\mathcal{F}}_n^2 + \tilde{\mathcal{F}}_{n+1}^2 = \tilde{\mathcal{F}}_{2n+1} + \mathfrak{p}F_{2n+3} + F_{2n+2}J$
$\quad + \left(F_{2n+3} + 2\mathfrak{p}F_{2n+5}\right)\varepsilon + 3F_{2n+4}J\varepsilon$,

6) $\tilde{\mathcal{F}}_{n+1}^2 - \tilde{\mathcal{F}}_{n-1}^2 = \tilde{\mathcal{F}}_{2n} + \mathfrak{p}F_{2n+2} + F_{2n+1}J + (F_{2n+2} + 2\mathfrak{p}F_{2n+4})\varepsilon + 3F_{2n+3}J\varepsilon,$

7) $\tilde{\mathcal{F}}_n \times \tilde{\mathcal{F}}_m + \tilde{\mathcal{F}}_{n+1} \times \tilde{\mathcal{F}}_{m+1} = \tilde{\mathcal{F}}_{n+m+1} + \mathfrak{p}F_{n+m+3} + F_{n+m+2}J + (F_{n+m+3} + 2\mathfrak{p}F_{n+m+5})\varepsilon + 3F_{n+m+4}J\varepsilon.$

Proof: With the aid of equations (11) and (12) and identities given in Proposition 1, the following proofs can be given:

1) Considering the definition of Fibonacci number, we have:

$$\begin{aligned}\tilde{\mathcal{F}}_n + \tilde{\mathcal{F}}_{n+1} &= (F_n + F_{n+1}J + F_{n+2}\varepsilon + F_{n+3}J\varepsilon) \\ &\quad + (F_{n+1} + F_{n+2}J + F_{n+3}\varepsilon + F_{n+4}J\varepsilon) \\ &= F_{n+2} + F_{n+3}J + F_{n+4}\varepsilon + F_{n+5}J\varepsilon \\ &= \tilde{\mathcal{F}}_{n+2}.\end{aligned}$$

2) Using equation (4) for r = 2k, we have:

$$\begin{aligned}\tilde{\mathcal{F}}_{n+r} + \tilde{\mathcal{F}}_{n-r} &= L_rF_n + L_rF_{n+1}J + L_rF_{n+2}\varepsilon + L_rF_{n+3}J\varepsilon \\ &= L_r\tilde{\mathcal{F}}_n,\end{aligned}$$

and for r = 2k + 1, we get:

$$\begin{aligned}\tilde{\mathcal{F}}_{n+r} + \tilde{\mathcal{F}}_{n-r} &= F_rL_n + F_rL_{n+1}J + F_rL_{n+2}\varepsilon + F_rL_{n+3}J\varepsilon \\ &= F_r\tilde{\mathcal{L}}_n.\end{aligned}$$

3) Taking into account equation (5) for $r = 2k$, we have:

$$\begin{aligned}\tilde{\mathcal{F}}_{n+r} - \tilde{\mathcal{F}}_{n-r} &= F_rL_n + F_rL_{n+1}J + F_rL_{n+2}\varepsilon + F_rL_{n+3}J\varepsilon \\ &= F_r\tilde{\mathcal{L}}_n,\end{aligned}$$

and for $r = 2k + 1$, we have:

$$\begin{aligned}\tilde{\mathcal{F}}_{n+r} - \tilde{\mathcal{F}}_{n-r} &= L_rF_n + L_rF_{n+1}J + L_rF_{n+2}\varepsilon + L_rF_{n+3}J\varepsilon \\ &= L_r\tilde{\mathcal{F}}_n.\end{aligned}$$

4) Using equations (14) and (4), (5), (6), we have:

$$\begin{aligned}\tilde{\mathcal{F}}_m \times \tilde{\mathcal{F}}_n - \tilde{\mathcal{F}}_{m+r} \times \tilde{\mathcal{F}}_{n-r} &= (-1)^{n-r}[(1-\mathfrak{p})F_{m-n+r} \\ &\quad + (-F_{m-n+r-1} + F_{m-n+r+1})J \\ &\quad + (F_{m-n+r-2} + F_{m-n+r+2} - \mathfrak{p}(F_{m-n+r-2} + F_{m-n+r+2}))\varepsilon \\ &\quad + (-F_{m-n+r-3} + F_{m-n+r-1} - F_{m-n+r+1} + F_{m-n+r+3})J\varepsilon]F_r \\ &= (-1)^{n-r}F_{m-n+r}F_r[(1-\mathfrak{p}) + J + 3(1-\mathfrak{p})\varepsilon + 3J\varepsilon].\end{aligned}$$

5) Calculate $\tilde{\mathcal{F}}_n^2$ and $\tilde{\mathcal{F}}_{n+1}^2$:

$$\begin{aligned}\tilde{\mathcal{F}}_n^2 &= F_n^2 + \mathfrak{p}F_{n+1}^2 + 2F_nF_{n+1}J + 2(F_nF_{n+2} + \mathfrak{p}F_{n+1}F_{n+3})\varepsilon \\ &\quad + 2(F_nF_{n+3} + F_{n+1}F_{n+2})J\varepsilon,\end{aligned} \tag{15}$$

and

$$\begin{aligned}\tilde{\mathcal{F}}_{n+1}^2 &= F_{n+1}^2 + \mathfrak{p}F_{n+2}^2 + 2F_{n+1}F_{n+2}J + 2(F_{n+1}F_{n+3} + \mathfrak{p}F_{n+2}F_{n+4})\varepsilon \\ &\quad + 2(F_{n+1}F_{n+4} + F_{n+2}F_{n+3})J\varepsilon.\end{aligned} \tag{16}$$

Then, by using (15) and (16) and applying equation (8), we obtain:

$$\begin{aligned}\tilde{\mathcal{F}}_n^2 + \tilde{\mathcal{F}}_{n+1}^2 &= F_{2n+1} + \mathfrak{p}F_{2n+3} + 2F_{2n+2}J + 2(F_{2n+3} + \mathfrak{p}F_{2n+5})\varepsilon \\ &\quad + 4F_{2n+4}J\varepsilon \\ &= \tilde{\mathcal{F}}_{2n+1} + \mathfrak{p}F_{2n+3} + F_{2n+2}J + (F_{2n+3} + 2\mathfrak{p}F_{2n+5})\varepsilon \\ &\quad + 3F_{2n+4}J\varepsilon.\end{aligned}$$

6) Using equation (15) for $n \to n-1$, equation (16) and using identities (9), (5), (8), respectively, we have:

$$\begin{aligned}\tilde{\mathcal{F}}_{n+1}^2 - \tilde{\mathcal{F}}_{n-1}^2 &= F_{2n} + \mathfrak{p}F_{2n+2} + 2F_{2n+1}J + 2(F_{2n+2} + \mathfrak{p}F_{2n+4})\varepsilon \\ &\quad + 4F_{2n+3}J\varepsilon \\ &= \tilde{\mathcal{F}}_{2n} + \mathfrak{p}F_{2n+2} + F_{2n+1}J + (F_{2n+2} + 2\mathfrak{p}F_{2n+4})\varepsilon \\ &\quad + 3F_{2n+3}J\varepsilon.\end{aligned}$$

7) Using equation (14) and identity given in equation (10), we obtain:

$$\begin{aligned}\tilde{\mathcal{F}}_n \times \tilde{\mathcal{F}}_m + \tilde{\mathcal{F}}_{n+1} \times \tilde{\mathcal{F}}_{m+1} &= F_{n+m+1} + \mathfrak{p}F_{n+m+3} + 2F_{n+m+2}J \\ &\quad + 2(F_{n+m+3} + \mathfrak{p}F_{n+m+5})\varepsilon + 4F_{n+m+4}J\varepsilon \\ &= \tilde{\mathcal{F}}_{n+m+1} + \mathfrak{p}F_{n+m+3} + F_{n+m+2}J \\ &\quad + (F_{n+m+3} + 2\mathfrak{p}F_{n+m+5})\varepsilon + 3F_{n+m+4}J\varepsilon.\end{aligned}$$

Corollary 1. Using identities 2), 3) and 4) presented in Theorem 2, the following basic identities can be obtained:

- $\tilde{\mathcal{F}}_{n+1} + \tilde{\mathcal{F}}_{n-1} = \tilde{\mathcal{L}}_n,$
- $\tilde{\mathcal{F}}_{n+2} + \tilde{\mathcal{F}}_{n-2} = 3\tilde{\mathcal{F}}_n,$
- $\tilde{\mathcal{F}}_{n+1} - \tilde{\mathcal{F}}_{n-1} = \tilde{\mathcal{F}}_n,$
- $\tilde{\mathcal{F}}_{n+2} - \tilde{\mathcal{F}}_{n-2} = \tilde{\mathcal{L}}_n,$
- $\tilde{\mathcal{F}}_m \times \tilde{\mathcal{F}}_n - \tilde{\mathcal{F}}_{m+1} \times \tilde{\mathcal{F}}_{n-1} = (-1)^{n-1}F_{m-n+1}[(1-\mathfrak{p}) + J + 3(1-\mathfrak{p})\varepsilon + 3J\varepsilon].$

Theorem 3. Let $\tilde{\mathcal{F}}_n \in \mathbb{DC}_{\mathfrak{p}}\mathbb{F}$ and $\tilde{\mathcal{L}}_n \in \mathbb{DC}_{\mathfrak{p}}\mathbb{L}$. Then, the following relations hold for n, m, $r \geq 0$:

1) $\tilde{\mathcal{L}}_n + \tilde{\mathcal{L}}_{n+1} = \tilde{\mathcal{L}}_{n+2},$

2) $\tilde{\mathcal{L}}_{n+r} + \tilde{\mathcal{L}}_{n-r} = \begin{cases} L_r\tilde{\mathcal{L}}_n, & r = 2k \\ 5F_r\tilde{\mathcal{F}}_n, & r = 2k+1, \end{cases}$

3) $\tilde{\mathcal{L}}_{n+r} - \tilde{\mathcal{L}}_{n-r} = \begin{cases} 5F_r\tilde{\mathcal{F}}_n, & r = 2k \\ L_r\tilde{\mathcal{L}}_n, & r = 2k+1, \end{cases}$

4) $\tilde{\mathcal{L}}_m \times \tilde{\mathcal{L}}_n - \tilde{\mathcal{L}}_{m+r} \times \tilde{\mathcal{L}}_{n-r} = 5(-1)^{n-r+1} F_{m-n+r} F_r$
$[(1-\mathfrak{p}) + J + 3(1-\mathfrak{p})\varepsilon + 3J\varepsilon]$,

5) $\tilde{\mathcal{L}}_n^2 + \tilde{\mathcal{L}}_{n+1}^2 =$
$5\left(\tilde{\mathcal{F}}_{2n+1} + \mathfrak{p}F_{2n+3} + F_{2n+2}J + \left(F_{2n+3} + 2\mathfrak{p}F_{2n+5}\right)\varepsilon + 3F_{2n+4}J\varepsilon\right)$,

6) $\tilde{\mathcal{L}}_{n+1}^2 - \tilde{\mathcal{L}}_{n-1}^2 =$
$5\left(\tilde{\mathcal{F}}_{2n} + \mathfrak{p}F_{2n+2} + F_{2n+1}J + \left(F_{2n+2} + 2\mathfrak{p}F_{2n+4}\right)\varepsilon + 3F_{2n+3}J\varepsilon\right)$,

7) $\tilde{\mathcal{L}}_n \times \tilde{\mathcal{L}}_m + \tilde{\mathcal{L}}_{n+1} \times \tilde{\mathcal{L}}_{m+1} =$
$5\left(\tilde{\mathcal{F}}_{n+m+1} + \mathfrak{p}F_{n+m+3} + F_{n+m+2}J + \left(F_{n+m+3} + 2\mathfrak{p}F_{n+m+5}\right)\varepsilon + 3F_{n+m+4}J\varepsilon\right)$.

Proof: By taking into account equations (11) and (12) and identities given in Proposition 1, the proofs can be given.

It should be noted that parts 4) and 7) in Theorem 2 and Theorem 3 are often referred to as Tagiuri's and Hornsberger's identities, respectively.

Corollary 2. By using Theorem 3, the following basic identities can be obtained easily:

- $\tilde{\mathcal{L}}_{n+1} + \tilde{\mathcal{L}}_{n-1} = 5\tilde{\mathcal{F}}_n$,
- $\tilde{\mathcal{L}}_{n+2} + \tilde{\mathcal{L}}_{n-2} = 3\tilde{\mathcal{L}}_n$,
- $\tilde{\mathcal{L}}_{n+1} - \tilde{\mathcal{L}}_{n-1} = \tilde{\mathcal{L}}_n$,
- $\tilde{\mathcal{L}}_{n+2} - \tilde{\mathcal{L}}_{n-2} = 5\tilde{\mathcal{L}}_n$,
- $\tilde{\mathcal{L}}_m \times \tilde{\mathcal{L}}_n - \tilde{\mathcal{L}}_{m+1} \times \tilde{\mathcal{L}}_{n-1} = 5(-1)^n F_{m-n+1}$
$[(1-\mathfrak{p}) + J + 3(1-\mathfrak{p})\varepsilon + 3J\varepsilon]$.

Theorem 4. Let $\tilde{\mathcal{F}}_{-n}$ and $\tilde{\mathcal{L}}_{-n}$ be nega $\mathcal{DGC}$ Fibonacci and Lucas numbers. Then, the following identities can be given for $n \geq 0$:

- $\tilde{\mathcal{F}}_{-n} = (-1)^{n+1}\tilde{\mathcal{F}}_n + (-1)^n L_n (J + \varepsilon + 2J\varepsilon)$,
- $\tilde{\mathcal{L}}_{-n} = (-1)^n \tilde{\mathcal{L}}_n + 5(-1)^{n-1} F_n (J + \varepsilon + 2J\varepsilon)$.

Proof: By using equation (2), and applying identity (4), we have:

$$\begin{aligned}
\tilde{\mathcal{F}}_{-n} &= F_{-n} + F_{-n+1}J + F_{-n+2}\varepsilon + F_{-n+3}J\varepsilon \\
&= (-1)^{n+1}F_n + (-1)^{n+1}F_{n+1}J + (-1)^{n+1}F_{n+2}\varepsilon + (-1)^{n+1}F_{n+3}J\varepsilon \\
&\quad - (-1)^{n+1}F_{n+1}J - (-1)^{n+1}F_{n+2}\varepsilon - (-1)^{n+1}F_{n+3}J\varepsilon + (-1)^n F_{n-1}J \\
&\quad - (-1)^{n-1}F_{n-2}\varepsilon + (-1)^{n-2}F_{n-3}J\varepsilon \\
&= (-1)^{n+1}\tilde{\mathcal{F}}_n + (-1)^n\left(\left(F_{n-1} + F_{n+1}\right)J + \left(F_{n+2} - F_{n-2}\right)\varepsilon \right.\\
&\qquad \left. + \left(F_{n-3} + F_{n+3}\right)J\varepsilon\right) \\
&= (-1)^{n+1}\tilde{\mathcal{F}}_n + (-1)^n L_n (J + \varepsilon + 2J\varepsilon).
\end{aligned}$$

The other part can be proved similarly.

Theorem 5. Let $\tilde{\mathcal{F}}_n \in \mathbb{DC}_{\mathfrak{p}}\mathbb{F}$ and $\tilde{\mathcal{L}}_n \in \mathbb{DC}_{\mathfrak{p}}\mathbb{L}$. Then, for $n \geq 1$, the **Binet's formulas** can be calculated as follows:

- $\tilde{\mathcal{F}}_n = \dfrac{\alpha^* \alpha^n - \beta^* \beta^n}{\alpha - \beta}$,
- $\tilde{\mathcal{L}}_n = \alpha^* \alpha^n + \beta^* \beta^n$,

where $\alpha^* = 1 + \alpha J + \alpha^2\varepsilon + \alpha^3 J\varepsilon$ and $\beta^* = 1 + \beta J + \beta^2\varepsilon + \beta^3 J\varepsilon$.

Proof: By using Binet's formulas for the Fibonacci and Lucas numbers given in equations (1) and (3), Binet's formulas for $\mathcal{DGC}$ Fibonacci and Lucas numbers can be obtained as below:

$$\begin{aligned}
\tilde{\mathcal{F}}_n &= F_n + F_{n+1}J + F_{n+2}\varepsilon + F_{n+3}J\varepsilon \\
&= \frac{\alpha^n - \beta^n}{\alpha - \beta} + \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}J + \frac{\alpha^{n+2} - \beta^{n+2}}{\alpha - \beta}\varepsilon + \frac{\alpha^{n+3} - \beta^{n+3}}{\alpha - \beta}J\varepsilon \\
&= \frac{\alpha^n\left(1 + \alpha J + \alpha^2\varepsilon + \alpha^3 J\varepsilon\right) - \beta^n\left(1 + \beta J + \beta^2\varepsilon + \beta^3 J\varepsilon\right)}{\alpha - \beta} \\
&= \frac{\alpha^*\alpha^n - \beta^*\beta^n}{\alpha - \beta}
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\mathcal{L}}_n &= L_n + L_{n+1}J + L_{n+2}\varepsilon + L_{n+3}J\varepsilon \\
&= \alpha^n + \beta^n + \left(\alpha^{n+1} + \beta^{n+1}\right)J + \left(\alpha^{n+2} + \beta^{n+2}\right)\varepsilon + \left(\alpha^{n+3} + \beta^{n+3}\right)J\varepsilon \\
&= \alpha^n\left(1 + \alpha J + \alpha^2\varepsilon + \alpha^3 J\varepsilon\right) + \beta^n\left(1 + \beta J + \beta^2\varepsilon + \beta^3 J\varepsilon\right) \\
&= \alpha^*\alpha^n + \beta^*\beta^n,
\end{aligned}$$

where $\alpha^* = 1 + \alpha J + \alpha^2\varepsilon + \alpha^3 J\varepsilon$ and $\beta^* = 1 + \beta J + \beta^2\varepsilon + \beta^3 J\varepsilon$.

Let D'Ocagne's, Catalan's and Cassini's identities for $\mathcal{DGC}$ Fibonacci and Lucas numbers be introduced:

Theorem 6. (D'Ocagne's identity) Let $\tilde{\mathcal{F}}_n, \tilde{\mathcal{F}}_m \in \mathbb{DC}_{\mathfrak{p}}\mathbb{F}$ and $\tilde{\mathcal{L}}_n, \tilde{\mathcal{L}}_m \in \mathbb{DC}_{\mathfrak{p}}\mathbb{L}$. Then, for $n, m \geq 0$, the followings can be given:

- $\tilde{\mathcal{F}}_m \times \tilde{\mathcal{F}}_{n+1} - \tilde{\mathcal{F}}_{m+1} \times \tilde{\mathcal{F}}_n = (-1)^n F_{m-n}$
$[(1-\mathfrak{p}) + J + 3(1-\mathfrak{p})\varepsilon + 3J\varepsilon]$,
- $\tilde{\mathcal{L}}_m \times \tilde{\mathcal{L}}_{n+1} - \tilde{\mathcal{L}}_{m+1} \times \tilde{\mathcal{L}}_n = 5(-1)^{n+1} F_{m-n}$
$[(1-\mathfrak{p}) + J + 3(1-\mathfrak{p})\varepsilon + 3J\varepsilon]$.

Proof: By writing $n \to n+1$ and $r = 1$ in identity 4) in Theorem 2 and 3, respectively, the proof is completed.

Theorem 7. (Catalan's identity) Let $\tilde{\mathcal{F}}_n \in \mathbb{DC}_{\mathfrak{p}}\mathbb{F}$ and $\tilde{\mathcal{L}}_n \in \mathbb{DC}_{\mathfrak{p}}\mathbb{L}$. Then, the followings can be given:

- $\tilde{\mathcal{F}}_n^2 - \tilde{\mathcal{F}}_{n+r} \times \tilde{\mathcal{F}}_{n-r} = (-1)^{n-r} F_r^2 [(1-\mathfrak{p}) + J + 3(1-\mathfrak{p})\varepsilon + 3J\varepsilon]$,
- $\tilde{\mathcal{L}}_n^2 - \tilde{\mathcal{L}}_{n+r} \times \tilde{\mathcal{L}}_{n-r} = 5(-1)^{n-r+1} F_r^2 [(1-\mathfrak{p}) + J + 3(1-\mathfrak{p})\varepsilon$
$+ 3J\varepsilon]$.

Proof: By writing $m \to n$ in identity 4) in Theorem 2 and 3, respectively, the identities are proved.

Theorem 8. (Cassini's identity) Let $\tilde{\mathcal{F}}_n \in \mathbb{DC}_{\mathfrak{p}}\mathbb{F}$ and $\tilde{\mathcal{L}}_n \in \mathbb{DC}_{\mathfrak{p}}\mathbb{L}$. Then, the followings can be given:

Table 4. Special cases for $\mathfrak{p} \in \{-1,0,1\}$

Definition	Number	Condition ($\varepsilon^2 = 0$)	Ref.
Dual-complex Fibonacci	$F_n + F_{n+1}i + F_{n+2}\varepsilon + F_{n+3}i\varepsilon$	$J = i, i^2 = -1$	[36,37]
Dual-complex Lucas	$L_n + L_{n+1}i + L_{n+2}\varepsilon + L_{n+3}i\varepsilon$	$J = i, i^2 = -1$	[36,37]
Hyper-dual Fibonacci	$F_n + F_{n+1}\epsilon + F_{n+2}\varepsilon + F_{n+3}\epsilon\varepsilon$	$J = \epsilon, \epsilon^2 = 0, (\varepsilon, \epsilon \neq 0), (\varepsilon\epsilon \neq 0)$	[41]
Hyper-dual Lucas	$L_n + L_{n+1}\epsilon + L_{n+2}\varepsilon + L_{n+3}\epsilon\varepsilon$	$J = \epsilon, \epsilon^2 = 0, (\varepsilon, \epsilon \neq 0), (\varepsilon\epsilon \neq 0)$	[41]
Dual-hyperbolic Fibonacci	$F_n + F_{n+1}j + F_{n+2}\varepsilon + F_{n+3}j\varepsilon$	$J = j, j^2 = 1, (j \neq \pm 1)$	[38–40]
Dual-hyperbolic Lucas	$L_n + L_{n+1}j + L_{n+2}\varepsilon + L_{n+3}j\varepsilon$	$J = j, j^2 = 1, (j \neq \pm 1)$	[38–40]

- $\tilde{\mathcal{F}}_n^2 - \tilde{\mathcal{F}}_{n+1} \times \tilde{\mathcal{F}}_{n-1} = (-1)^{n-1}[(1-\mathfrak{p}) + J + 3(1-\mathfrak{p})\varepsilon + 3J\varepsilon]$,
- $\tilde{\mathcal{L}}_n^2 - \tilde{\mathcal{L}}_{n+1} \times \tilde{\mathcal{L}}_{n-1} = 5(-1)^n[(1-\mathfrak{p}) + J + 3(1-\mathfrak{p})\varepsilon + 3J\varepsilon]$.

Proof: By taking $r = 1$ in Catalan's identities given in the above theorem, the proof is completed.

Example 1. Let the above identities for the numbers in $\mathbb{DC}_\mathfrak{p}\mathbb{F}$ and $\mathbb{DC}_\mathfrak{p}\mathbb{L}$ for the given values of m, n, r be verified:

D'Ocagne's identities for $m = 3$, $n = 1$ and $\mathfrak{p} = -\frac{1}{3}$:

- $\tilde{\mathcal{F}}_3 \times \tilde{\mathcal{F}}_2 - \tilde{\mathcal{F}}_4 \times \tilde{\mathcal{F}}_1 = -\frac{4}{3} - J - 4\varepsilon - 3J\varepsilon$,
- $\tilde{\mathcal{L}}_3 \times \tilde{\mathcal{L}}_2 - \tilde{\mathcal{L}}_4 \times \tilde{\mathcal{L}}_1 = 5\left(\frac{4}{3} + J + 4\varepsilon + 3J\varepsilon\right)$.

Catalan's identities for $n = 2$, $r = 2$ and $\mathfrak{p} = 0$:

- $\tilde{\mathcal{F}}_2^2 - \tilde{\mathcal{F}}_4 \times \tilde{\mathcal{F}}_0 = 1 + J + 3\varepsilon + 3J\varepsilon$,
- $\tilde{\mathcal{L}}_2^2 - \tilde{\mathcal{L}}_4 \times \tilde{\mathcal{L}}_0 = -5(1 + J + 3\varepsilon + 3J\varepsilon)$.

Cassini's identities for $n = 2$ and $\mathfrak{p} = \frac{1}{5}$:

- $\tilde{\mathcal{F}}_2^2 - \tilde{\mathcal{F}}_3 \times \tilde{\mathcal{F}}_1 = -\frac{4}{5} - J - \frac{12}{5}\varepsilon - 3J\varepsilon$,
- $\tilde{\mathcal{L}}_2^2 - \tilde{\mathcal{L}}_3 \times \tilde{\mathcal{L}}_1 = 5\left(\frac{4}{5} + J + \frac{12}{5}\varepsilon + 3J\varepsilon\right)$.

CONCLUSION

The main objective of this study is to construct $\mathcal{DGC}$ Fibonacci and Lucas numbers by introducing their general recurrence relations for any real number $\mathfrak{p}$ in the light of the study [21]. The striking part of this paper is that one can reduce the calculations to dual complex, hyper-dual and dual-hyperbolic Fibonacci/Lucas numbers by taking the special real values $\mathfrak{p} = -1$, $\mathfrak{p} = 0$ and $\mathfrak{p} = 1$, respectively. Considering these values, the above mentioned special Fibonacci/Lucas numbers are generalized from the viewpoint of definition, algebraic properties, recurrence relations and well-known identities in Section 3. Hence, Section 3 is directly linked to the paper [36] for $\mathfrak{p} = -1$ (regarding dual-complex case) and the paper [38] for $\mathfrak{p} = 1$ (regarding dual-hyperbolic case). Additionally, Section 3 is closely associated with the papers [37], [41] and [39, 40] regarding dual-complex, hyper-dual and dual-hyperbolic situations as a special case. This classification can be seen in Table 4.

With a similar thought, our next goal is to present the $\mathcal{DGC}$ Oresme numbers which are given by the second-order relation $O_{n+2} = O_{n+1} - \frac{1}{4}O_n, n \geq 0$ with $O_0 = 0$ and $O_1 = O_2 = \frac{1}{2}$ (see in [43, 44]) and examine the linear relations and, nonlinear properties of them, and their connection with $\mathcal{DGC}$ Fibonacci and Pell numbers.

AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

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