Sigma Journal of Engineering and Natural Sciences, Vol. , No. , pp. , March,
Yildiz Technical University Press, Istanbul, Turkey

Novel analytical and approximate-analytical methods for solving the nonlinear fractional smoking mathematical model

Hatıra GÜNERHAN[1], Mohammed K. A. KAABAR[2], Ercan ÇELİK[3,*]

ABSTRACT

Smoking is globally a challenging issue that causes many fatal health problems. In this paper, a nonlinear fractional smoking mathematical model is proposed in the context of a modified form of the Caputo fractional-order derivative. The analytical and approximate-analytical solutions are obtained for the proposed mathematical model via the fractional differential transform method (FDTM) and Laplace Adomian decomposition method (LADM). The obtained solution is provided as a rapidly convergent series. Simulation results are provided in this paper to compare the obtained solutions by FDTM, LADM, Runge Kutta (RK) method, and reduced differential transforms method (RDTM) with the exact solution of the proposed problem. By comparing both FDTM and LADM solutions, the FDTM solution is closer to the exact solution than the LADM solution. All obtained solutions have been analyzed and compared graphically to validate the effiency and applicability of all results.

Keywords: *Mathematical Modelling; Fractional Calculus; Fractional Differential Transform Method (FDTM); Laplace Adomian Decomposition Method (LADM); Smoking Mathematical Model.*

INTRODUCTION

Smoking has a negative impact on both individuals and societies. [1, 5]. The smoking habit keeps spreading between all people of different genders and ages [1, 5]. Controlling the spread of smoking habit has been very challenging globally. A mathematical model that describes the smoking habit among various classifications of smokers is needed to be further studied in order to investigate this habit mathematically and provide some suggested solutions to this issue with a goal to control the spread of this habit or at least minimize its negative impact. This topic of research has attracted the interests of many researchers from various fields of science, engineering, and medicine to conduct further research studies concerning the smoking mathematical model. [1, 2, 3, 4, 5, 6]. Fractional calculus has recently attracted the interests of mathematicians and researchers due to the advantages of using fractional derivatives in modelling scientific and engineering phenomena. Fractional derivatives can provide a better understanding for the physical system and its dynamics than using the integer-order derivatives. Therefore, several research studies have been conducted on the mathematical analysis of fractional derivatives, and on solving fractional differential equations using analytical or approximate-analytical techniques. [7, 8, 9, 10, 11, 12, 17, 18]. Some other notable studies about other proposed techniques for solving integral and fuzzy integral equations such as fuzzy b-metric-like spaces for solving integral equation [19] and control fuzzy metric spaces with the help of orthogonality for solving fuzzy integral equation [20], respectively. Some novel fixed-point results of orthogonal neutrosophic metric spaces are provided in [21]. Two of the most interesting approaches for solving these equations of fractional order are FDTM and LADM. On one hand, the differential transform method was first created by Zhou [13] to obtain approximate-analytical solutions to ordinary differential equations using Taylor series formulation. [15]. Then, Arikoglu and Ozkol in [14] developed this approach by using the power series formulation for fractional-order differential equations (see [9] for more background information about the applicability of this technique for solving the second-order wave equation). On the other hand, the LADM is considered as a coupling Laplace transform method with Adomian decomposition which can provide a great help in solving nonlinear differential equations analytically [8]. For more examples

This paper was recommended for publication in revised form by Regional Editor Alper Yılmaz
[1] Department of Mathematics, Kafkas University, Kars, Türkiye
[2] Department of Mathematics and Statistics, Washington State University. Pullman, WA, USA
[3] Department of Mathematics,, Kyrgyz-Turkish Manas University, Kirgizistan
** E-mail address: ercan.celik@manas.edu.kg ; ercelik@atauni.edu.tr*
Orcid id: https://orcid.org 0000-0002-7802-477X Hatıra Günerhan, 0000-0003-2260-0341 Mohammed K.A. Kaabar, 0000-0001-5971-7653 Ercan Çelik
Manuscript Received 14 October 2021, Accepted 16 January 2022

about this method, we refer to [8] where the proposed HIV infection of $CD4^{+}T$ cells model is successfully solved via the Laplace Adomian decomposition method. An extension of the LADM was proposed by Kaabar et al. [10] to form a modified coupling method of double Laplace transform with Adomian decomposition to solve the nonlinear fractional-order Schrödinger equation with second-order spatio-temporal dispersion. Ali et al. [15] has applied LADM for the fractional-order immunology and AIDS model. Günerhan et al. [16] has also applied LADM for a fractional-order model of HIV infection. In this paper, the general form of the nonlinear fractional smoking model is written as follows:

$$
\begin{aligned}
D_*^{\alpha} P(t) &= a\big(1-P(t)\big) - bP(t)M(t),\\
D_*^{\alpha} W(t) &= -aW(t) + P(t)W(t) - cW(t)M(t),\\
D_*^{\alpha} M(t) &= -(a+d)M(t) + cW(t)M(t) + fQ(t),\\
D_*^{\alpha} Q(t) &= -(a+f)Q(t) + d(1-e)M(t),\\
D_*^{\alpha} R(t) &= -aR(t) + edM(t),
\end{aligned}
\tag{1}
$$

with initial conditions are written in the following form:

$$P(0) = P_0,\ W(0) = W_0,\ M(0) = M_0,\ Q(0) = Q_0,\ R(0) = R_0\,. \tag{2}$$

Let $N(t)$ be the total population with respect to time which consists of 5 smokers' classifications [1]: Potential smokers, occasional (light) smokers, heavy smokers, temporary quitters, and permanently quitters are denoted by $P(t)$, $W(t)$, $M(t)$, $Q(t)$, and $R(t)$, respectively. Therefore, $N(t)$ can be written as follows:

$$N(t) \ = \ P(t) + W(t) + M(t) + Q(t) + R(t).$$

Each parameter in (1) represents the following [1]:

a represents the natural death rate; b represents the connection index between potential and light smokers, c represents the connection index between light and heavy smokers; d represents the smoking quitting index; e represents the number of former smokers who quitted smoking permanently with a rate d; f represents the connection index between temporary quitters who might go back to the smoking habit.

While there are many methods for solving nonlinear models, our proposed nonlinear fractional smoking mathematical model in the context of a modified version of Caputo fractional derivative have not been investigated using the FDTM and LADM. Our numerical experiments and simulation make our study unique in comparison to many other related studies.

This paper is constructed as follows: In Section 2, some fundamental fractional calculus definitions and properties are introduced. In Section 3, the proposed smoking mathematical model will be solved using FDTM and LADM. In Section 4, all obtained results are compared and analyzed. In section 5, we conclude our research study.

2. BASIC DEFINITIONS

In this section, some basic fractional calculus definitions and properties are introduced which will be applied later.

DEFINITION 1. A real function $f(x),\ x > 0$ is said to be in the space $C_\mu,\ \mu\epsilon R$ if there exists a real number $P > \mu$ such that $f(x) = x^p f_1(x)$ where $f_1(x)\epsilon C[0,\infty)$. Clearly, we have the following: $C_\mu < C_\beta$ if $\mu < \beta$.

DEFINITION 2. A function $f(x),\ x > 0$ is said to be in the space C_μ^m, $m\epsilon N\cup\{0\}$ if $f^{(m)} \in C_\mu$.

DEFINITION 3. The Riemann-Liouville fractional integral operator of the order $\alpha > 0$ of a function, $f \in C_\mu, \mu \geq -1$ is defined as follows:

$$(J_a^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-\tau)^{\alpha-1} f(\tau) d\tau, x > a,$$

$$(J_a^0 f)(x) = f(x). \tag{3}$$

All the properties of the operator J^α are mentioned in [36] in which we will discuss only important properties as follows:

For $f \in C_\mu,\ \mu \geq -1, \alpha, \beta \geq 0,$ and $\gamma > -1$

a. $(J_a^\alpha J_a^\beta f)(x) = (J_a^{\alpha+\beta} f)(x),$ (4)

b. $(J_a^\alpha J_a^\beta f)(x) = (J_a^\beta J_a^\alpha f)(x),$ (5)

c. $J_a^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}.$ (6)

There are some advantages of using Riemann–Liouville fractional derivative over other fractional derivatives due to the fact that this type of fractional derivative can be efficiently applied in modelling scientific phenomena. A modified form of Caputo fractional operator, denoted by D^α, will be used in this research work.

3. METHODOLOGY

In this section, the proposed smoking mathematical model will be solved using the FDTM and LADM.

3.1. FRACTIONAL DIFFERENTIAL TRANSFORM METHOD

The fractional differential transform method (FDTM) is semi-numerical and analytical approach, and it is considered as a traditional differential transformed method reform. The differential transformation, $s(x)$, can be expressed as follows:

$$S_\varphi(k) = \frac{1}{\Gamma(\varphi k+1)} \left[\left(D_{x_0}^\varphi\right)^k s(x) \right]_{x=x_0}, \tag{7}$$

The differential transform inverse of $S_\varphi(k)$ is written as

$$s(x) = \sum_{k=0}^{\infty} S_\varphi(K)(x-x_0)^{\varphi k} \equiv D^{-1} S_\varphi(k), \tag{8}$$

By substituting Eq. (7) into Eq. (8), the following is obtaind:

$$\sum_{k=0}^{\infty} \frac{1}{\Gamma(\varphi k+1)} \left[\left(D_{x_0}^\varphi\right)^k s(x) \right]_{x=x_0} (x-x_0)^{\varphi k} = s(x), \tag{9}$$

Assume that $S_\varphi(k)$ is the fractional differential transform of $s(x)$. The approximate function $S(k)$ is written as:

$$s(x) = \sum_{k=0}^{n} S_\varphi(K)(x-x_0)^{\varphi k}. \tag{10}$$

The above equation (Eq. (10)) represents the differential transformation that is resulted from using Taylor series expansion where the derivatives' symbolic evaluation is not applicable for this technique. In addition, by applying the iterative procedure, comparative derivatives have been obtained. The orginal function is represented by a lower case letter, while the transformed function is represented by an upper case letter. According to Eq. (9) and Eq. (10), it is obvious to prove that the transformed functions have the basic mathematical values as mentioned in Table 2.

Table 1. The fractional differential transform method operations.

Given Function	Transformed Function
$w(x) = d(x) \pm b(x)$	$W(k) = D_{\varphi}(k) \pm B_{\varphi}(k)$
$w(x) = cb(x)$	$W(k) = cB_{\varphi}(k)$
$w(x) = q(cx)$	$W(k) = c^k Q_{\varphi}(k)$
$w(x) = q(\frac{x}{c})$	$W(k) = \frac{Q_{\varphi}(k)}{c^k}$
$w(x) = D_{x_0}^{m\varphi} q(x)$	$W(k) = \frac{\Gamma(\varphi k + m\varphi + 1)}{\Gamma(\varphi k + 1)} Q_{\varphi}(k+m)$
$w(x) = x^m$	$W(k) = \delta(k-m) = \begin{cases} 1 & k = m \\ 0 & k \neq m \end{cases}$
$w(x) = e^{x+c}$	$W(k) = \frac{e^c}{k!}$

By applying FDTM to the proposed smoking mathematical model, and by using with both Table 1 and the Eq. (10), a system of equations is obtained as follows:

$$P_{\alpha}(k+1) = \frac{\Gamma(1+\alpha k)}{\Gamma(\alpha(k+1)+1)} \left[a\delta(k) - aP_{\alpha}(k) - b\sum_{l=0}^{k} P_{\alpha}(l) M_{\alpha}(k-l) \right],$$

$$W_{\alpha}(k+1) = \frac{\Gamma(1+\alpha k)}{\Gamma(\alpha(k+1)+1)} \left[-aW_{\alpha}(k) + \sum_{l=0}^{k} P_{\alpha}(l) W_{\alpha}(k-l) - c\sum_{l=0}^{k} W_{\alpha}(l) M_{\alpha}(k-l) \right],$$

$$M_{\alpha}(k+1) = \frac{\Gamma(1+\alpha k)}{\Gamma(\alpha(k+1)+1)} \left[-aM_{\alpha}(k) - dM_{\alpha}(k) + c\sum_{l=0}^{k} W_{\alpha}(l) M_{\alpha}(k-l) + fQ_{\alpha}(k) \right],$$

$$Q_{\alpha}(k+1) = \frac{\Gamma(1+\alpha k)}{\Gamma(\alpha(k+1)+1)} \left[-aQ_{\alpha}(k) - fQ_{\alpha}(k) + dM_{\alpha}(k) - deM_{\alpha}(k) \right],$$

$$R_{\alpha}(k+1) = \frac{\Gamma(1+\alpha k)}{\Gamma(\alpha(k+1)+1)} \left[-aR_{\alpha}(k) + edM_{\alpha}(k) \right]. \tag{11}$$

with the following initial conditions:

$$P_{\alpha}(0) = P_0, \quad W_{\alpha}(0) = W_0, \quad M_{\alpha}(0) = M_0, \quad Q_{\alpha}(0) = Q_0, \quad R_{\alpha}(0) = R_0 .$$

From Eq. (11) and the above initail conditions, the numerical approximate values of $P(t), W(t), M(t), Q(t)$ and $R(t)$ for $k = 1,2,3,\ldots$ can be obtained for various values of α, and the numerical comparisons are shown in the comparison of results and discussion section.

By using the inverse reduced differential transform of $\mathrm{P}_\alpha(\mathrm{k}), \mathrm{W}_\alpha(\mathrm{k}), \mathrm{M}_\alpha(\mathrm{k}), \mathrm{Q}_\alpha(\mathrm{k})$ and $\mathrm{R}_\alpha(\mathrm{k})$, we get the following solution:

$$p(t) = \sum_{k=0}^{\infty} \mathrm{P}_\alpha(\mathrm{k}) t^{k\alpha} = \mathrm{P}_\alpha(0) + \mathrm{P}_\alpha(1) t^{\alpha} + \mathrm{P}_\alpha(2) t^{2\alpha} + \mathrm{P}_\alpha(3) t^{3\alpha} + \cdots$$

$$w(t) = \sum_{k=0}^{\infty} \mathrm{W}_\alpha(\mathrm{k}) t^{k\alpha} = \mathrm{W}_\alpha(0) + \mathrm{W}_\alpha(1) t^{\alpha} + \mathrm{W}_\alpha(2) t^{2\alpha} + \mathrm{W}_\alpha(3) t^{3\alpha} + \cdots$$

$$m(t) = \sum_{k=0}^{\infty} \mathrm{M}_\alpha(\mathrm{k}) t^{k\alpha} = \mathrm{M}_\alpha(0) + \mathrm{M}_\alpha(1) t^{\alpha} + \mathrm{M}_\alpha(2) t^{2\alpha} + \mathrm{M}_\alpha(3) t^{3\alpha} + \cdots$$

$$q(t) = \textstyle\sum_{k=0}^{\infty} \mathrm{Q}_\alpha(\mathrm{k}) t^{k\alpha} = \mathrm{Q}_\alpha(0) + \mathrm{Q}_\alpha(1) t^{\alpha} + \mathrm{Q}_\alpha(2) t^{2\alpha} + \mathrm{Q}_\alpha(3) t^{3\alpha} + \cdots$$

$$r(t) = \textstyle\sum_{k=0}^{\infty} \mathrm{R}_\alpha(\mathrm{k}) t^{k\alpha} = \mathrm{R}_\alpha(0) + \mathrm{R}_\alpha(1) t^{\alpha} + \mathrm{R}_\alpha(2) t^{2\alpha} + \mathrm{R}_\alpha(3) t^{3\alpha} + \cdots \qquad (12)$$

3.2. THE LAPLACE ADOMIAN DECOMPOSITION METHOD

In this section, we will illustrate the basic steps for LADM. We discuss the following important definitions or our research study:

DEFINITION 3.1 [8] A function f on $0 \le t < \infty$ is exponentially bounded of order $\sigma \in R$ if satisfies $\| f(t) \| \le Me^{\sigma t}$, for some real constant $M > 0$.

DEFINITION 3.2 [7,8, 17] The Caputo fractional derivative is defined as follows:

$$\boldsymbol{L\{D^{\sigma} f(t)\} = s^{\sigma} L\{f(t)\} - \sum_{k=0}^{m} s^{\sigma-k-1} f^{(k)}(0)}, \qquad (13)$$

where $m = \sigma + 1$, and $[\alpha]$ represents the integer part of σ. As a result, the following useful formula is obtained:

$$L(t^{\sigma}) = \frac{\Gamma(\sigma+1)}{s^{(\sigma+1)}}, \quad \sigma \in R^{+}. \qquad (14)$$

The last-mentioned definitions can be used in this section to discuss the general procedures for solving the proposed mathematical model (1). First of all, the Laplace transform is applied to both lift-hand and right-hand sides of Eq. (1) in the following form:

$$L\left(\frac{d^{\alpha}}{dt^{\alpha}} P\right) = \frac{a}{s} - aL(P) - bL(PM),$$

$$L\left(\frac{d^{\alpha}}{dt^{\alpha}} W\right) = -aL(W) + bL(PW) - cL(WM),$$

$$L\left(\frac{d^{\alpha}}{dt^{\alpha}} M\right) = -aL(M) - dL(M) + cL(WM) + fL(Q),$$

$$L\left(\frac{d^{\alpha}}{dt^{\alpha}} Q\right) = -aL(Q) - fL(Q) + dL(M) - deL(M),$$

$$L\left(\frac{d^{\alpha}}{dt^{\alpha}} R\right) = -aL(R) + edL(M). \qquad (15)$$

Then, by applying the formula (13) to Eq. (15), we reach the following:

$$s^{\alpha}L(P)-s^{\alpha-1}P(0)=\frac{a}{s}-aL(P)-bL(PM),$$
$$s^{\alpha}L(W)-s^{\alpha-1}W(0)=-aL(W)+bL(PW)-cL(WM),$$
$$s^{\alpha}L(M)-s^{\alpha-1}M(0)=-aL(M)-dL(M)+cL(WM)+fL(Q),$$
$$s^{\alpha}L(Q)-s^{\alpha-1}Q(0)=-aL(Q)-fL(Q)+dL(M)-deL(M),$$
$$s^{\alpha}L(R)-s^{\alpha-1}R(0)=-aL(R)+edL(M). \tag{16}$$

By applying the initial conditions, the following result is obtained:

$$L(P)=\frac{P_0}{s}+\frac{a}{s^{\alpha+1}}-\frac{a}{s^{\alpha}}L(P)-\frac{b}{s^{\alpha}}L(PM),$$
$$L(W)=\frac{W_0}{s}-\frac{a}{s^{\alpha}}L(W)+\frac{b}{s^{\alpha}}L(PW)-\frac{c}{s^{\alpha}}L(WM),$$
$$L(M)=\frac{M_0}{s}-\frac{a}{s^{\alpha}}L(M)-\frac{d}{s^{\alpha}}L(M)+\frac{c}{s^{\alpha}}L(WM)+\frac{f}{s^{\alpha}}L(Q),$$
$$L(Q)=\frac{Q_0}{s}-\frac{a}{s^{\alpha}}L(Q)-\frac{f}{s^{\alpha}}L(Q)+\frac{d}{s^{\alpha}}L(M)-\frac{de}{s^{\alpha}}L(M),$$
$$L(R)=\frac{R_0}{s}-\frac{a}{s^{\alpha}}L(R)+\frac{de}{s^{\alpha}}L(M). \tag{17}$$

By using this method, the solution is obtained as an infinite series. To apply the Adomian decomposition method, let the values of $A=PM, E=MW$ and $B=PW$. The solution is expressed as an infinite series in the following form:

$$P=\sum_{n=0}^{\infty}P_n,\qquad W=\sum_{n=0}^{\infty}W_n\ ,\qquad M=\sum_{n=0}^{\infty}M_n\ ,\qquad Q=\sum_{n=0}^{\infty}Q_n\ ,\qquad R=\sum_{n=0}^{\infty}R_n. \tag{18}$$

We decompose the two nonlinear parts, named A and C, in the following form:

$$\boldsymbol{A}=\sum_{\boldsymbol{n}=\boldsymbol{0}}^{\infty}\boldsymbol{A}_{\boldsymbol{n}},\qquad \boldsymbol{E}=\sum_{\boldsymbol{n}=\boldsymbol{0}}^{\infty}\boldsymbol{E}_{\boldsymbol{n}},\qquad \boldsymbol{B}=\sum_{\boldsymbol{n}=\boldsymbol{0}}^{\infty}\boldsymbol{B}_{\boldsymbol{n}}. \tag{19}$$

Here, A_n, $\mathrm{E_n}$, and $\mathrm{B_n}$ can be computed using the convolution operation as

$$A_n=\frac{1}{\Gamma(n+1)}\frac{d^n}{d\eta^n}\left[\sum_{i=0}^{n}\eta^iP_i\sum_{i=0}^{n}\eta^iM_i\right]_{\eta=0},$$
$$E_n=\frac{1}{\Gamma(n+1)}\frac{d^n}{d\eta^n}\left[\sum_{i=0}^{n}\eta^iM_i\sum_{i=0}^{n}\eta^iW_i\right]_{\eta=0},$$
$$B_n=\frac{1}{\Gamma(n+1)}\frac{d^n}{d\eta^n}\left[\sum_{i=0}^{n}\eta^iP_i\sum_{i=0}^{n}\eta^iW_i\right]_{\eta=0}. \tag{20}$$

Substituting Eq. (18) and Eq. (19) into Eq. (17), we obtain the following result:

$$L\left(\sum_{n=0}^{\infty}P_n\right)=\frac{P_0}{s}+\frac{a}{s^{\alpha+1}}-\frac{a}{s^{\alpha}}L\left(\sum_{n=0}^{\infty}P_n\right)-\frac{b}{s^{\alpha}}L\left(\sum_{n=0}^{\infty}A_n\right),$$
$$L\left(\sum_{n=0}^{\infty}W_n\right)=\frac{W_0}{s}-\frac{a}{s^{\alpha}}L\left(\sum_{n=0}^{\infty}W_n\right)+\frac{b}{s^{\alpha}}L\left(\sum_{n=0}^{\infty}B_n\right)-\frac{c}{s^{\alpha}}L\left(\sum_{n=0}^{\infty}E_n\right),$$

$$L\left(\sum_{n=0}^{\infty} M_n\right) = \frac{M_0}{s} - \frac{a}{s^\alpha} L\left(\sum_{n=0}^{\infty} M_n\right) - \frac{d}{s^\alpha} L\left(\sum_{n=0}^{\infty} M_n\right) + \frac{c}{s^\alpha} L\left(\sum_{n=0}^{\infty} E_n\right) + \frac{f}{s^\alpha} L\left(\sum_{n=0}^{\infty} Q_n\right),$$

$$L\left(\sum_{n=0}^{\infty} Q_n\right) = \frac{Q_0}{s} - \frac{a}{s^\alpha} L\left(\sum_{n=0}^{\infty} Q_n\right) - \frac{f}{s^\alpha} L\left(\sum_{n=0}^{\infty} Q_n\right) + \frac{d}{s^\alpha} L\left(\sum_{n=0}^{\infty} M_n\right) - \frac{de}{s^\alpha} L\left(\sum_{n=0}^{\infty} M_n\right),$$

$$L\left(\sum_{n=0}^{\infty} R_n\right) = \frac{R_0}{s} - \frac{a}{s^\alpha} L\left(\sum_{n=0}^{\infty} R_n\right) + \frac{de}{s^\alpha} L\left(\sum_{n=0}^{\infty} M_n\right). \tag{21}$$

By matching both left-hand and right-hand sides of Eq. (21), we get the following iterative algorithm:

$$L(P_0) = \frac{P_0}{s}, \quad L(W_0) = \frac{W_0}{s}, \quad L(M_0) = \frac{M_0}{s}, \quad L(Q_0) = \frac{Q_0}{s}, \quad L(R_0) = \frac{R_0}{s},$$

$$L(P_1) = \frac{a}{s^{\alpha+1}} - \frac{a}{s^\alpha} L(P_0) - \frac{b}{s^\alpha} L(A_0),$$

$$L(W_1) = -\frac{a}{s^\alpha} L(W_0) + \frac{b}{s^\alpha} L(B_0) - \frac{c}{s^\alpha} L(E_0),$$

$$L(M_1) = -\frac{a}{s^\alpha} L(M_0) - \frac{d}{s^\alpha} L(M_0) + \frac{c}{s^\alpha} L(E_0) + \frac{f}{s^\alpha} L(Q_0),$$

$$L(Q_1) = -\frac{a}{s^\alpha} L(Q_0) - \frac{f}{s^\alpha} L(Q_0) + \frac{d}{s^\alpha} L(M_0) - \frac{de}{s^\alpha} L(M_0),$$

$$L(R_1) = -\frac{a}{s^\alpha} L(R_0) + \frac{de}{s^\alpha} L(M_0), \dots$$

$$L(P_n) = -\frac{a}{s^\alpha} L(P_{n-1}) - \frac{b}{s^\alpha} L(A_{n-1}),$$

$$L(W_n) = -\frac{a}{s^\alpha} L(W_{n-1}) + \frac{b}{s^\alpha} L(B_{n-1}) - \frac{c}{s^\alpha} L(E_{n-1}),$$

$$L(M_n) = -\frac{a}{s^\alpha} L(M_{n-1}) - \frac{d}{s^\alpha} L(M_{n-1}) + \frac{c}{s^\alpha} L(E_{n-1}) + \frac{f}{s^\alpha} L(Q_{n-1}),$$

$$L(Q_n) = -\frac{a}{s^\alpha} L(Q_{n-1}) - \frac{f}{s^\alpha} L(Q_{n-1}) + \frac{d}{s^\alpha} L(M_{n-1}) - \frac{de}{s^\alpha} L(M_{n-1}),$$

$$L(R_n) = -\frac{a}{s^\alpha} L(R_{n-1}) + \frac{de}{s^\alpha} L(M_{n-1}). \tag{22}$$

Taking inverse transform of (22) we have:

$$P_0 = P_{00}, \quad W_0 = W_{00}, \quad M_0 = M_{00}, \quad Q_0 = Q_{00}, \quad R_0 = R_{00},$$

$$P_1 = [a - aP_{00} - bA_0] \frac{t^\alpha}{\Gamma(\alpha+1)},$$

$$W_1 = [-aW_{00} + bB_0 - E_0] \frac{t^\alpha}{\Gamma(\alpha+1)},$$

$$M_1 = [-M_0 - dM_{00} + cE_0 + fQ_{00}] \frac{t^\alpha}{\Gamma(\alpha+1)},$$

$$Q_1 = [-aQ_{00} - fQ_{00} + dM_{00} - deM_{00}] \frac{t^\alpha}{\Gamma(\alpha+1)},$$

$$R_1 = [-aR_{00} + deM_{00}] \frac{t^\alpha}{\Gamma(\alpha+1)}, \dots \tag{23}$$

The remaining terms can be obtained similarly. Then, the solution is obtained in the form of infinite series as follows:

$$\boldsymbol{P(t)} = \sum_{j=0}^{\infty} \boldsymbol{P_j}, \quad \boldsymbol{W(t)} = \sum_{j=0}^{\infty} \boldsymbol{W_j}, \quad \boldsymbol{M(t)} = \sum_{j=0}^{\infty} \boldsymbol{M_j}, \quad \boldsymbol{Q(t)} = \sum_{j=0}^{\infty} \boldsymbol{Q_j},$$

$$\boldsymbol{R(t)} = \sum_{j=0}^{\infty} \boldsymbol{R_j}. \tag{24}$$

The solution in Eq. (24) gives the results for the state variables for the SITR model of Eq. (1), and those results shall be illustrated in the next section.

4. COMPARİSON OF RESULTS AND DISCUSSION

In this section, the results for solving model (1) are investigated for various values of α to prove the effectiveness and validity of the proposed algorithm. The values of the parameters that have been used in numerical simulations are summarized in Table 2.

It is noticeable that FDTM and LADM are effective in producing approximate solutions of the proposed mathematical model. Numerical simulation of $P(t), W(t), M(t), Q(t)$ and $R(t)$ are shown in Figs. 1-10 over a interval of 0 < t < 1 for different values of $\alpha = 1, 0.8, 0.5$, and all results have been compared with the exact solution for the studied problem. The values of the parameters that have been used in numerical simulations are summarized in Table 2. Figs. 11-12 show the responses of the model investigated in this work at α= 1. In Figs. 13-17, we compared the obtained result from FDTM and LADM with the results from using other techniques such as Runge Kutta (RK) method and reduced differential transforms method (RDTM) [1] with the same values of the parameters that are shown in Table 2.

According to our comparative results, FDTM provides more reliable solutions than LADM. We conclude that the FDTM solution is closer to the exact solution than LADM solution. Therefore, our proposed technique is reliable and efficient. Numerical experiments have been conducted using the applied methods for various values of α which have successfully provided good results for the studied problem.

Table 2: Parameters values defined in Eq. (1).

Parameters	Values
P(0)	0.603
$W(0)$	0.24
$M(0)$	0.10628
$Q(0)$	0.0326
$R(0)$	0.01811
a	0.04
b	0.23
c	0.3
d	0.2
e	0.4
f	0.25

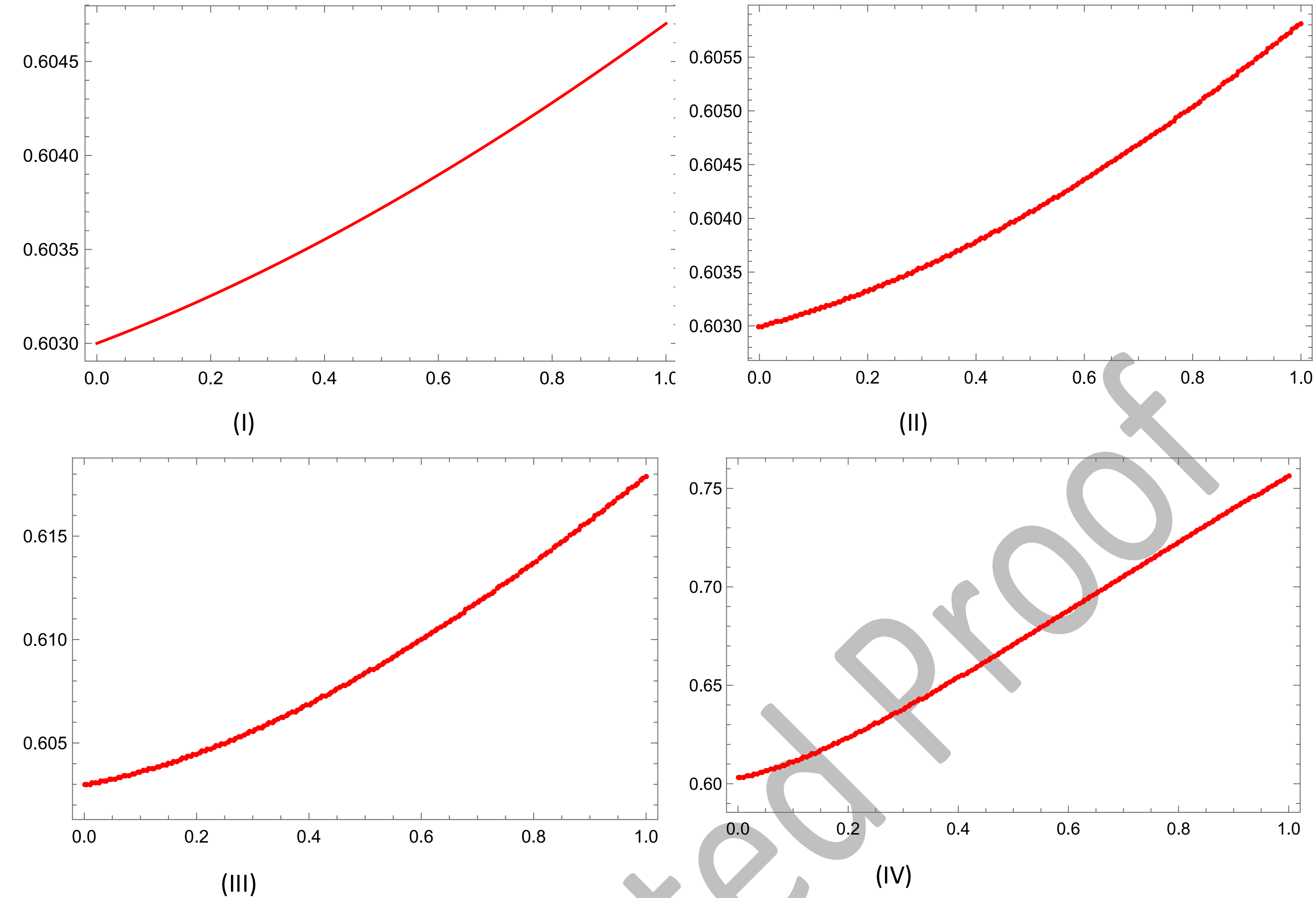
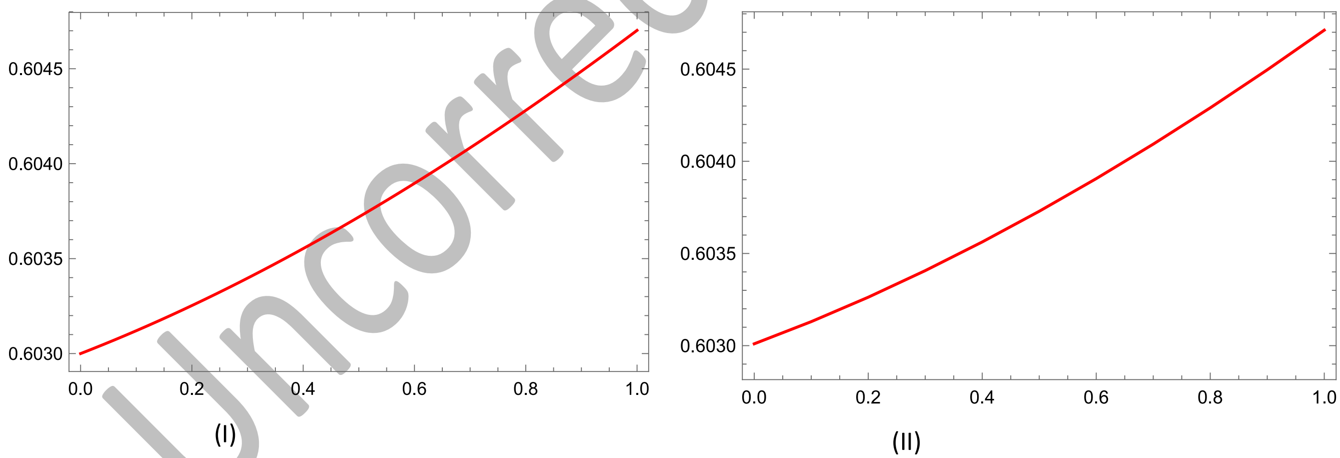


Figure 1. (I) Exact solution for $P(t)$, (II) Approximate solution by using LADM with $\alpha = 1$, (III) Approximate solution with $\alpha = 0.8$, (IV) Approximate solution with $\alpha = 0.5$ for $0 < t < 1$.



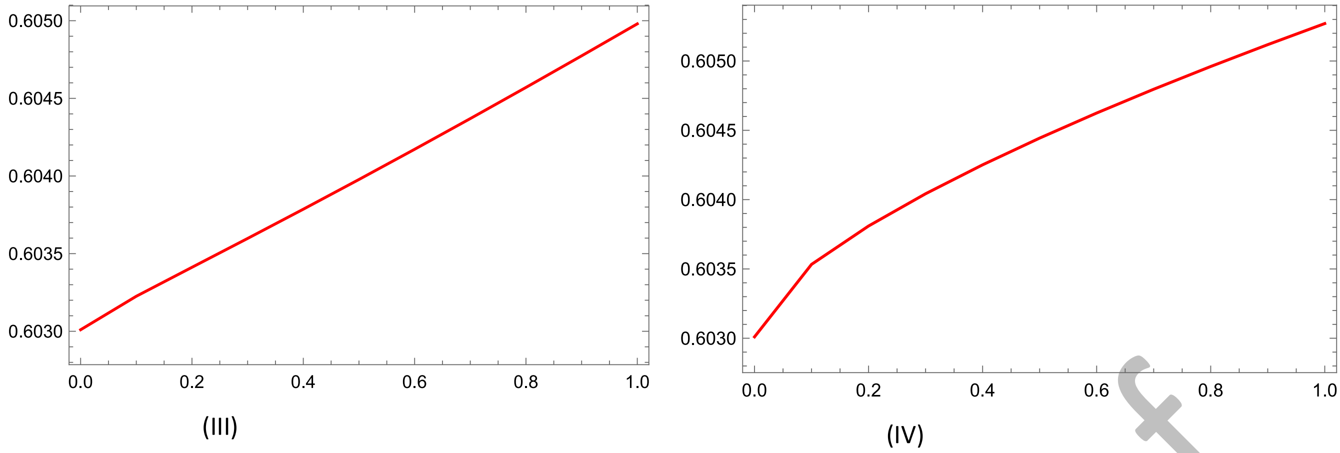


Figure 2. (I) Exact solution for $P(t)$, (II) Approximate solution by using FDTM with $\alpha = 1$, (III) Approximate solution with $\alpha = 0.8$, (IV) Approximate solution with $\alpha = 0.5$ for $0 < t < 1$.

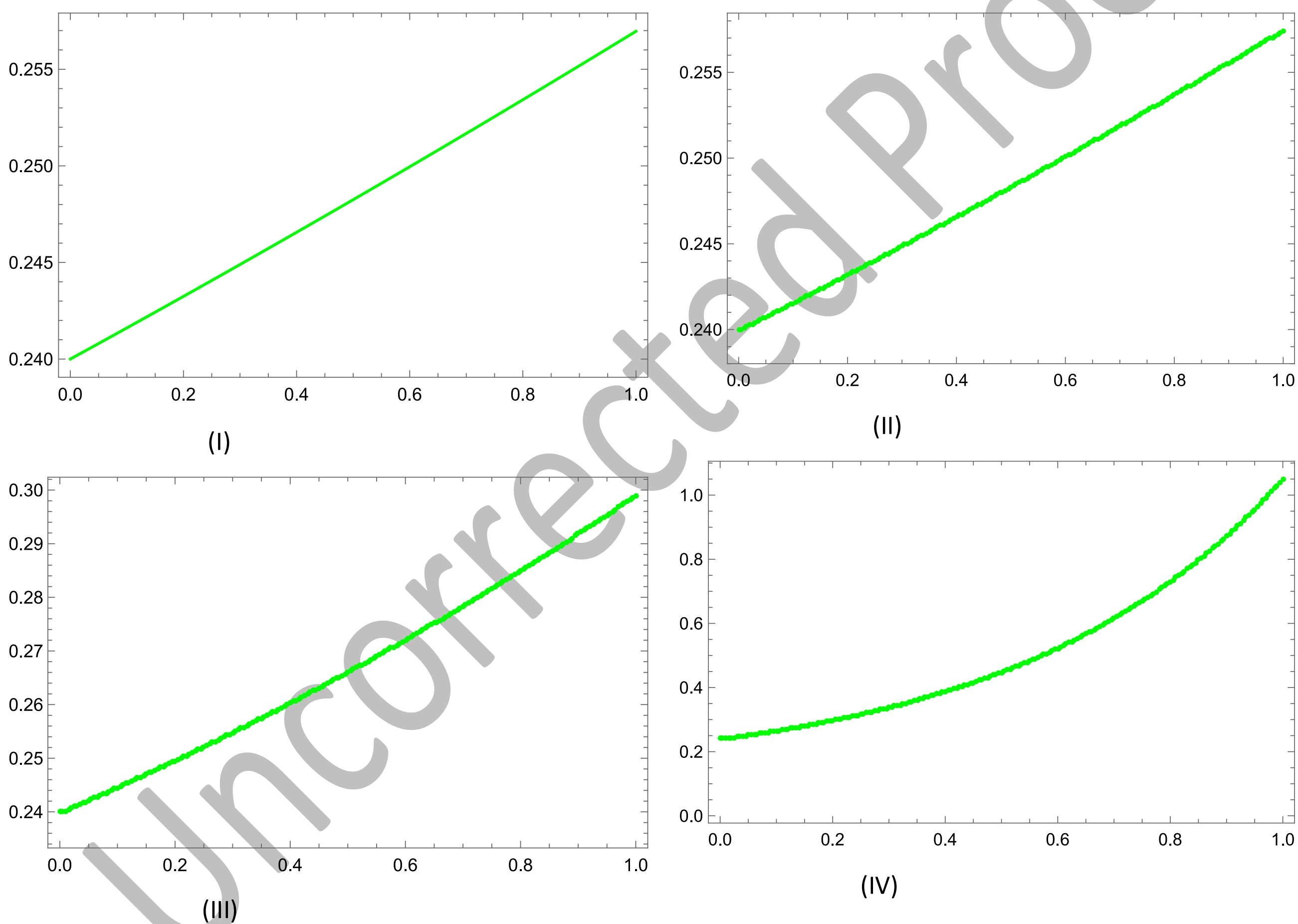


Figure 3. (I) Exact solution for $W(t)$, (II) Approximate solution by using LADM with $\alpha = 1$, (III) Approximate solution with $\alpha = 0.8$, (IV) Approximate solution with $\alpha = 0.5$ for $0 < t < 1$.

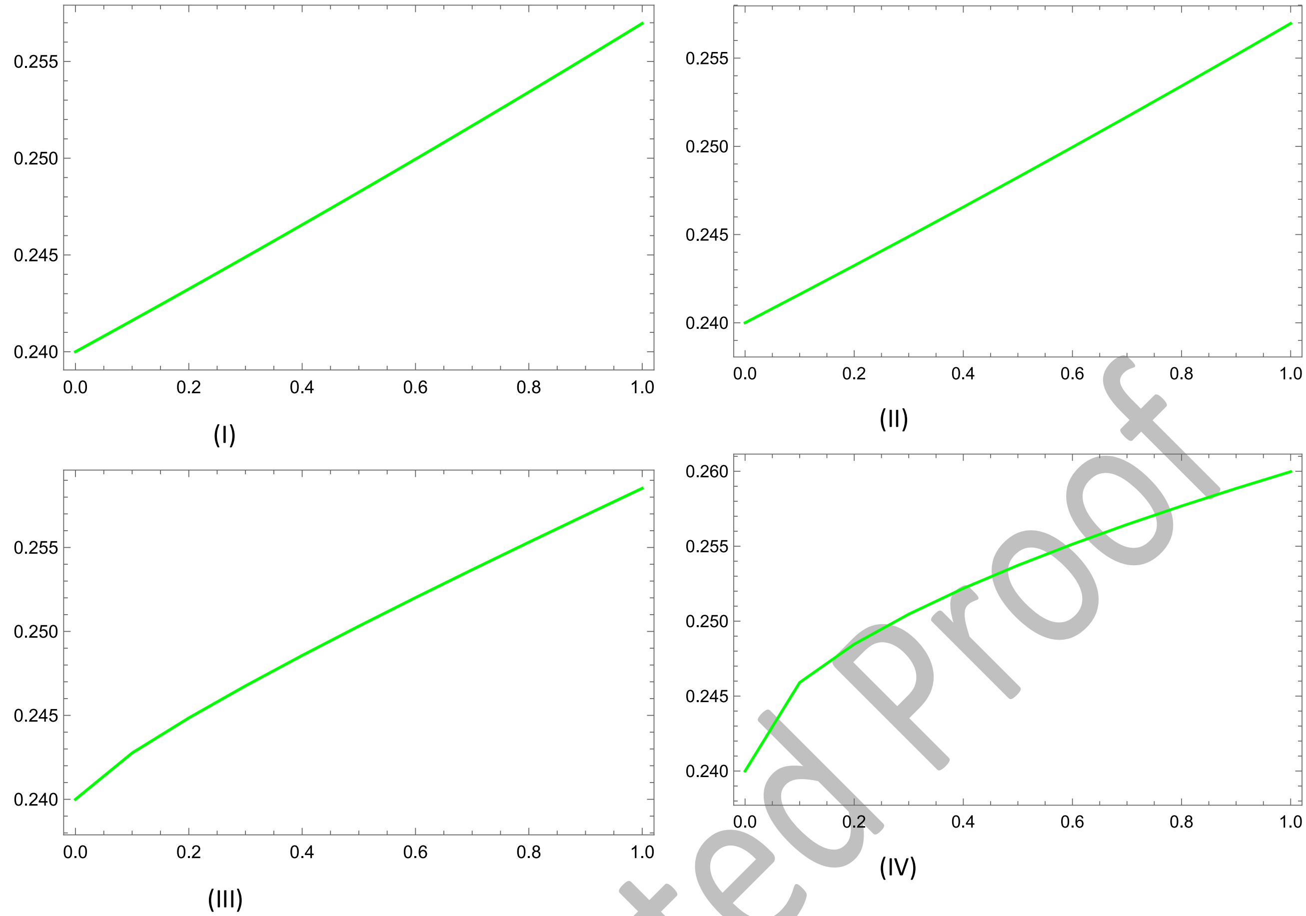
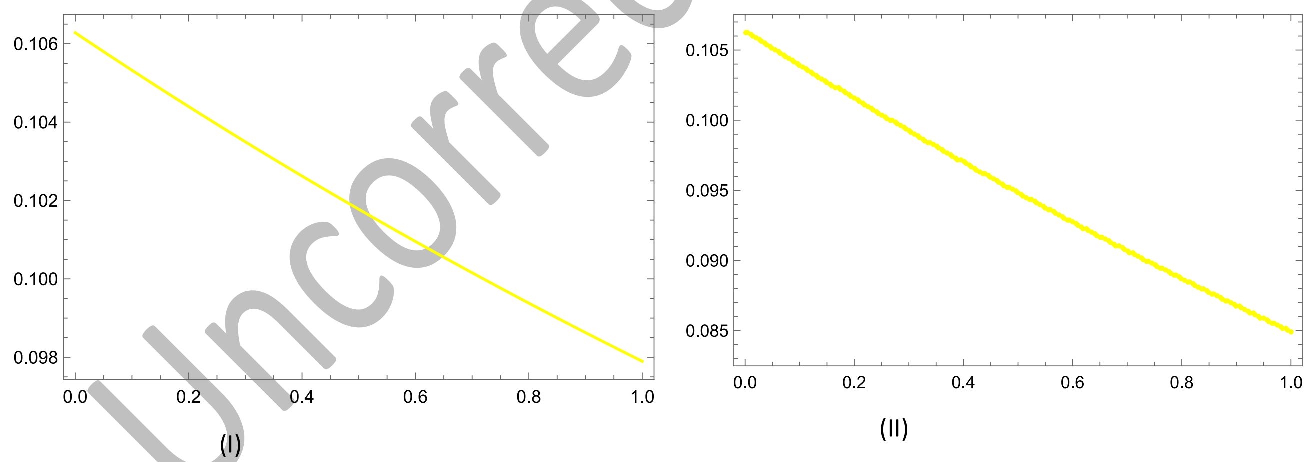


Figure 4. (I) Exact solution for $W(t)$, (II) Approximate solution by using FDTM with $\alpha = 1$, (III) Approximate solution with $\alpha = 0.8$, (IV) Approximate solution with $\alpha = 0.5$ for $0 < t < 1$.



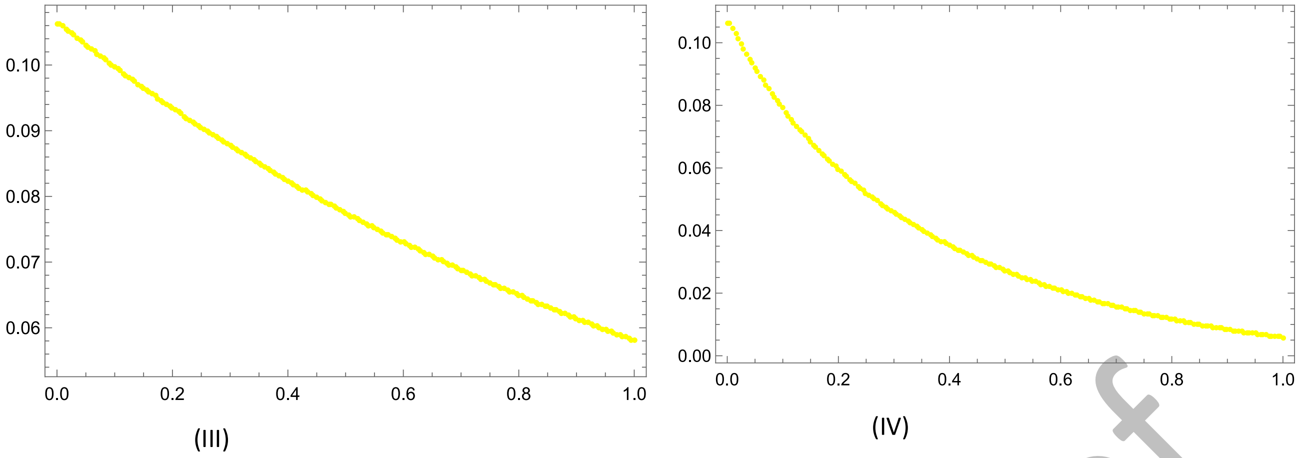


Figure 5. (I) Exact solution for $M(t)$, (II) Approximate solution by using LADM with $\alpha = 1,$ (III) Approximate solution with $\alpha = 0.8$, (IV) Approximate solution with $\alpha = 0.5$ for $0 < t < 1$.

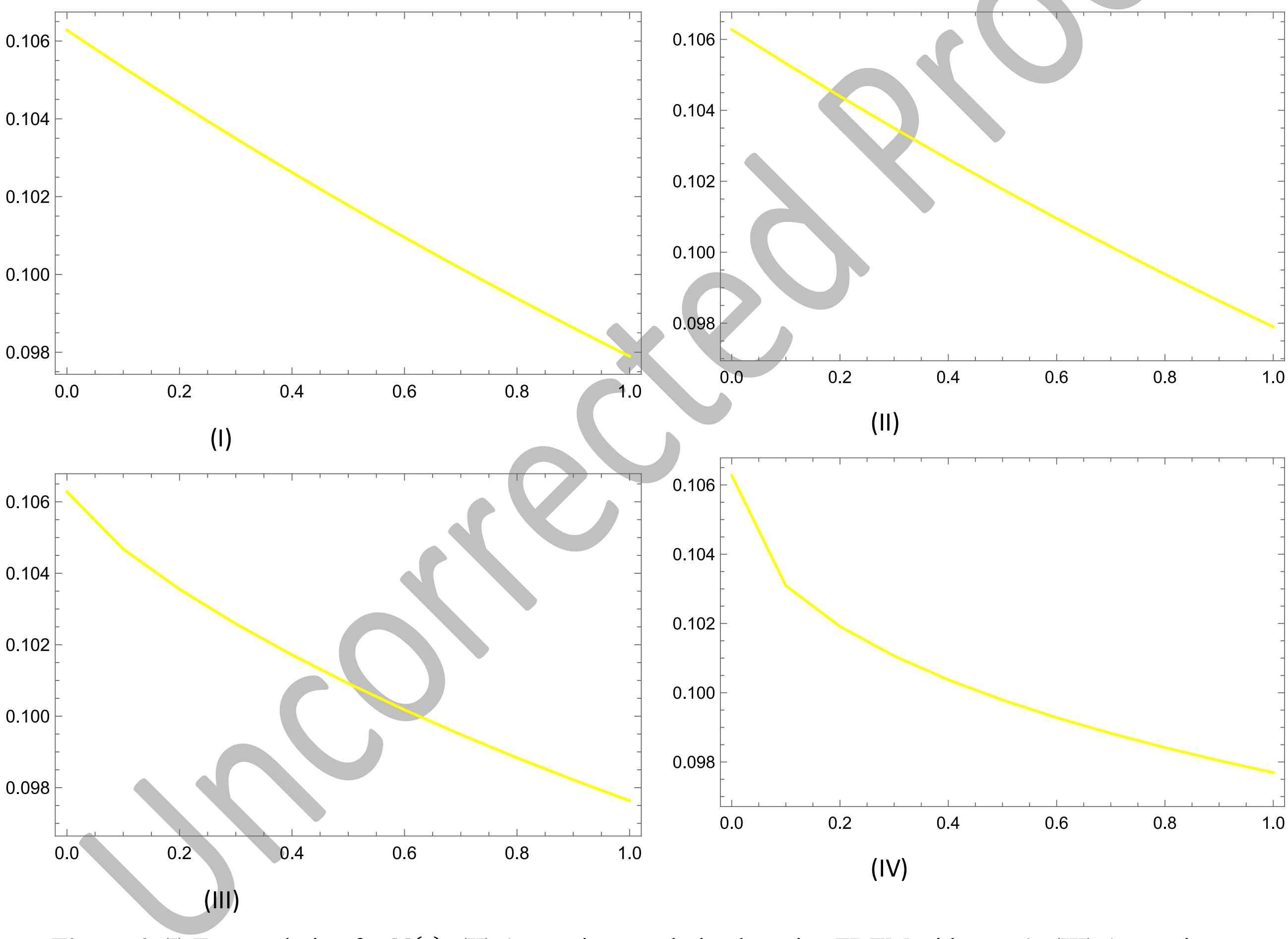


Figure 6. (I) Exact solution for $M(t)$, (II) Approximate solution by using FDTM with $\alpha = 1,$ (III) Approximate solution with $\alpha = 0.8$, (IV) Approximate solution with $\alpha = 0.5$ for $0 < t < 1$.

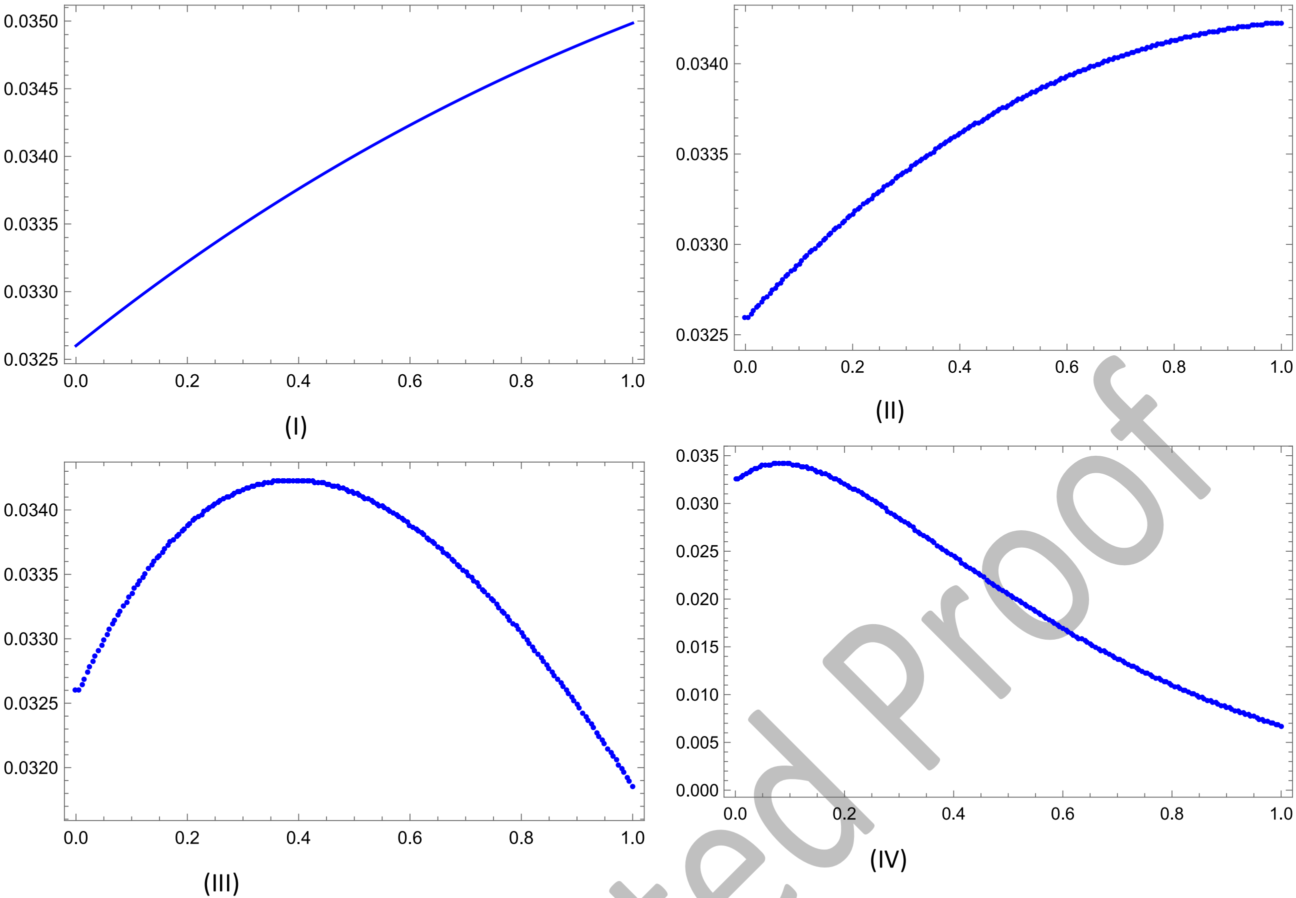
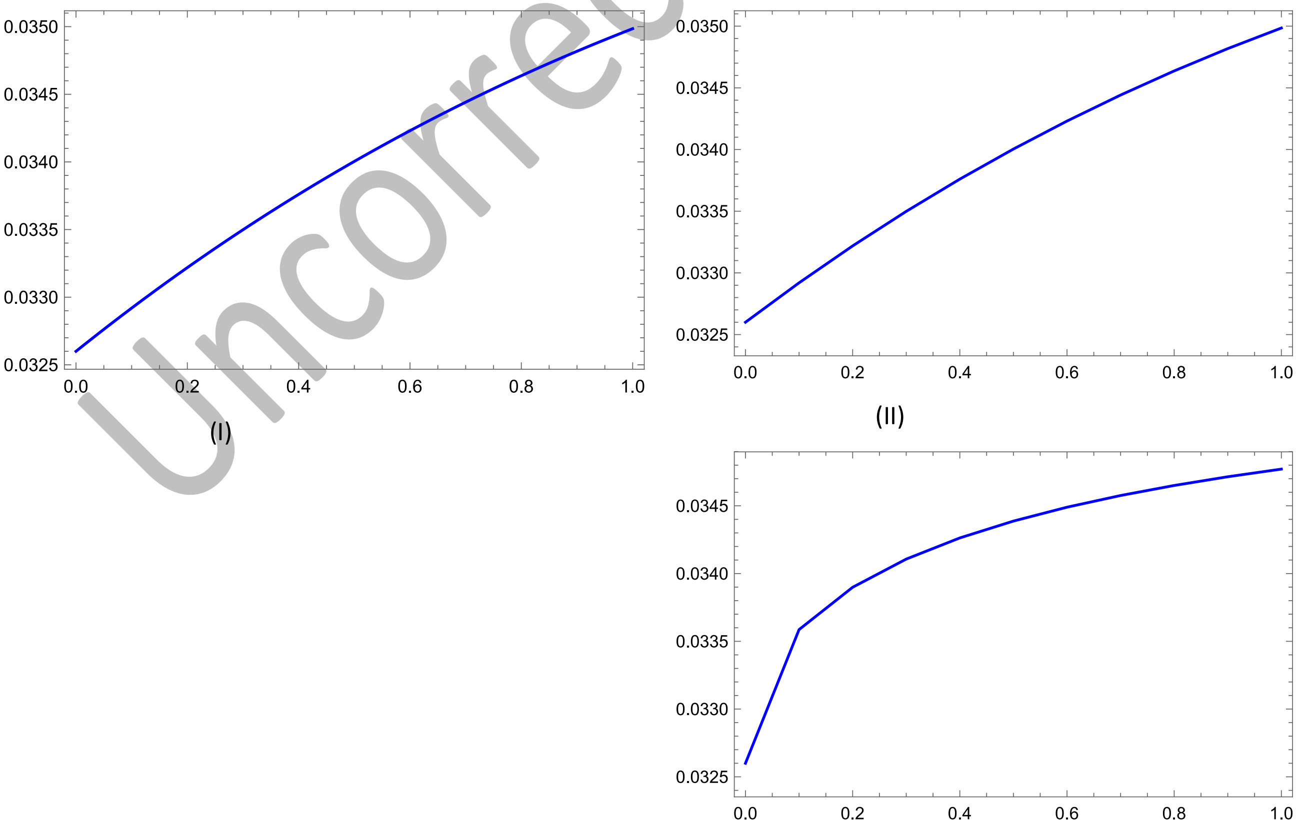
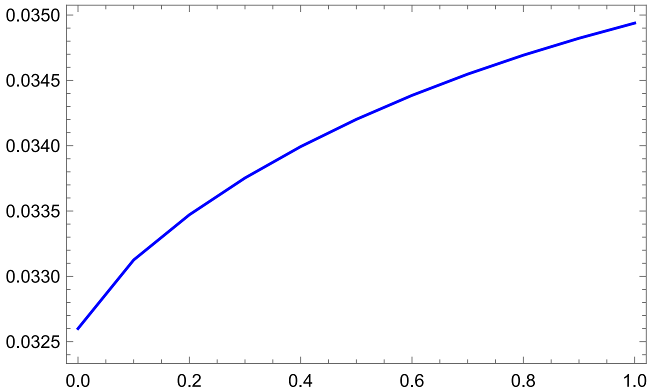


Figure 7. (I) Exact solution for $Q(t)$, (II) Approximate solution by using LADM with $\alpha = 1$, (III) Approximate solution with $\alpha = 0.8$, (IV) Approximate solution with $\alpha = 0.5$ for $0 < t < 1$.

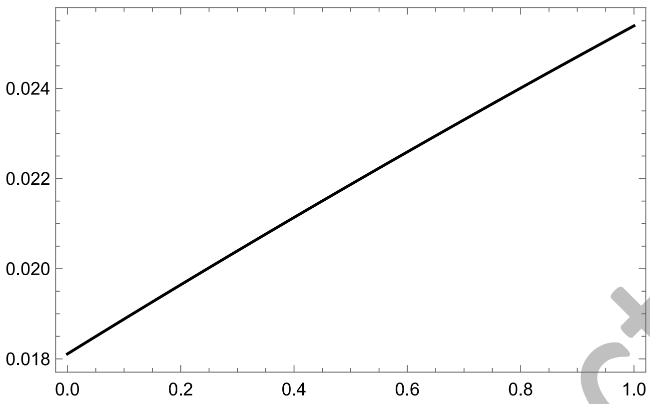




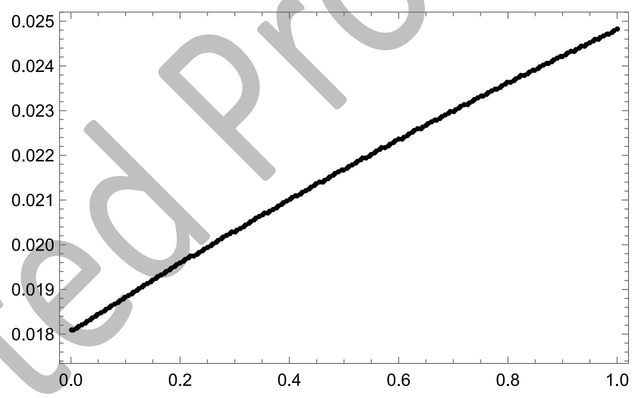
(III)

(IV)

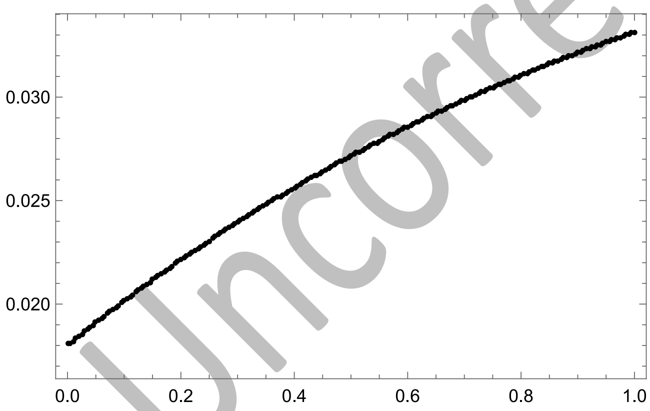
Figure 8. (I) Exact solution for $Q(t)$, (II) Approximate solution by using FDTM with $\alpha = 1,$ (III) Approximate solution with $\alpha = 0.8$, (IV) Approximate solution with $\alpha = 0.5$ for $0 < t < 1$.



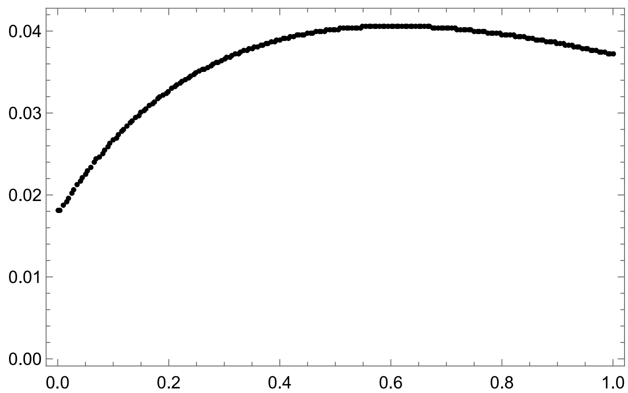
(I)



(II)



(II)



(IV)

Figure 9. (I) Exact solution for $R(t)$, (II) Approximate solution by using LADM with $\alpha = 1,$ (III) Approximate solution with $\alpha = 0.8$, (IV) Approximate solution with $\alpha = 0.5$ for $0 < t < 1$.

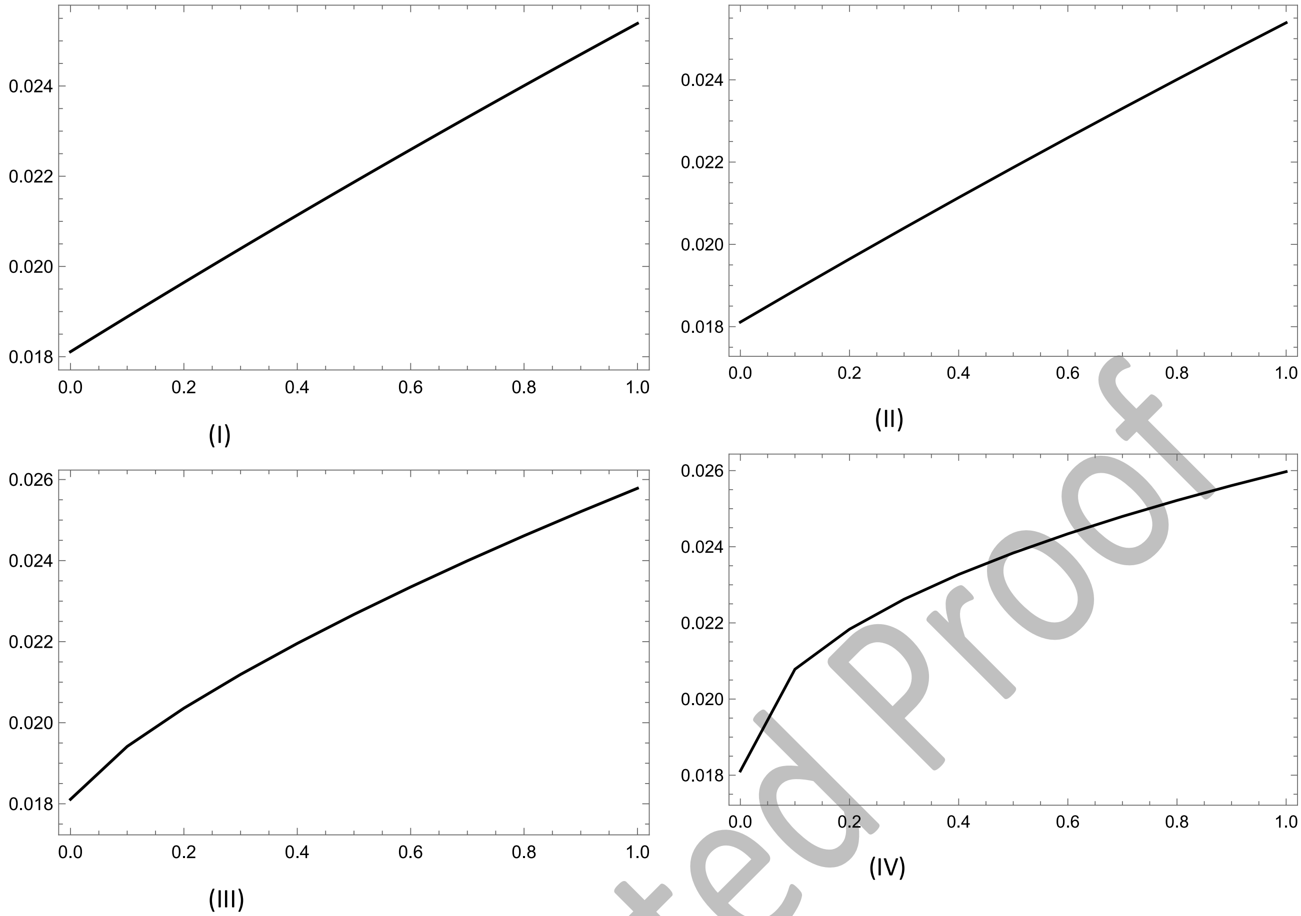


Figure 10. (I) Exact solution for $R(t)$, (II) Approximate solution by using FDTM with $\alpha = 1$, (III) Approximate solution with $\alpha = 0.8$, (IV) Approximate solution with $\alpha = 0.5$ for $0 < t < 1$.

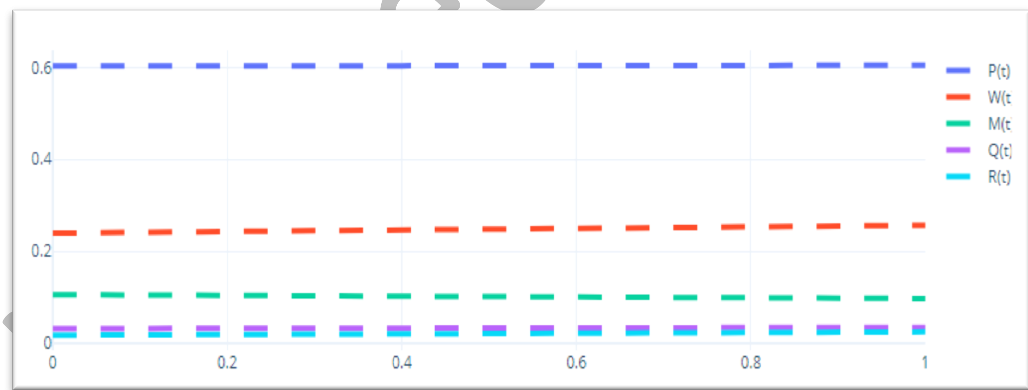


Figure 11. The relation between $P(t), W(t), M(t), Q(t)$ and $R(t)$ by using FDTM at $\alpha = 1$.

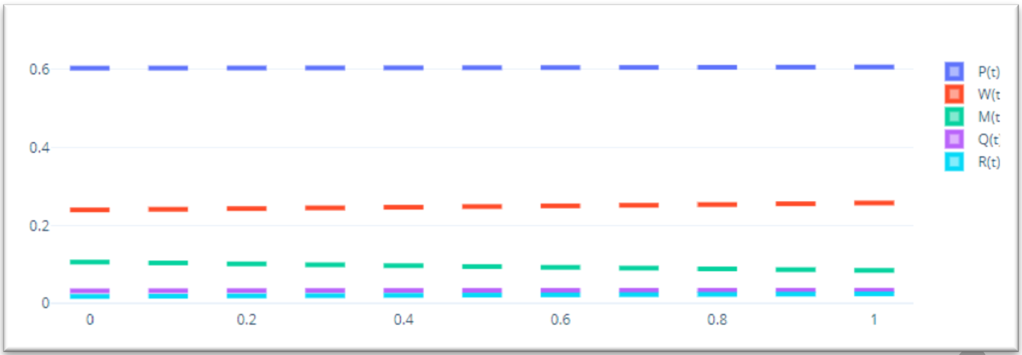


Figure 12. The relation between $P(t), W(t), M(t), Q(t)$ and $R(t)$ by using LADM at $\alpha = 1$.

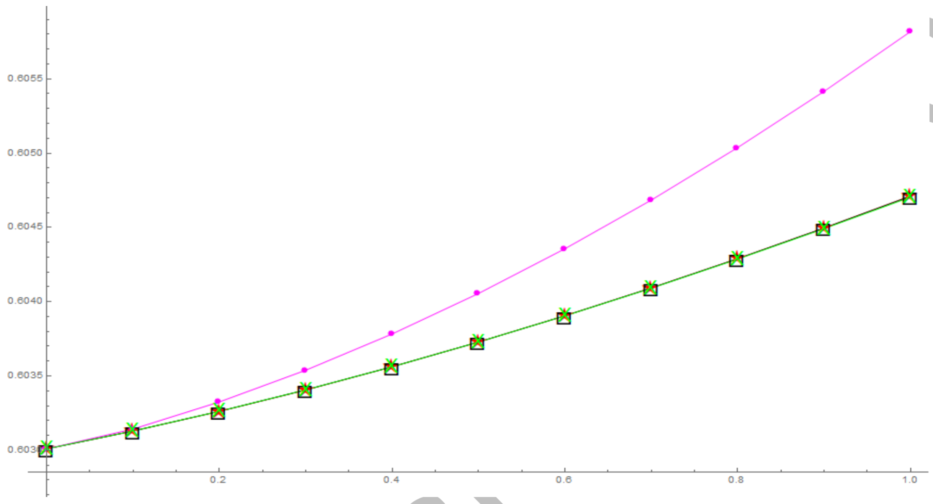


Figure 13. The solution of $P(t)$ obtained by FDTM (star), LADM(Triangle), RK method (Square), and RDTM(Circle) for $\alpha = 1$.

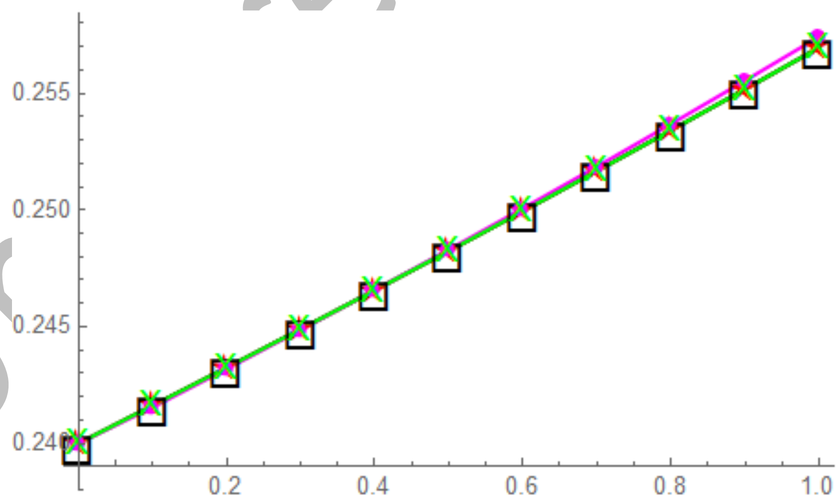


Figure 14. The solution of $W(t)$ obtained by FDTM (star), LADM(Triangle), RK method (Square), and RDTM(Circle) for $\alpha = 1$.

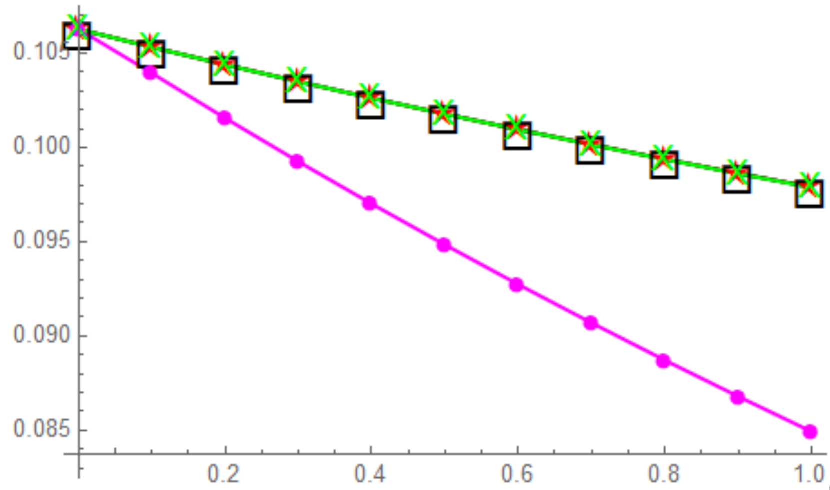


Figure 15. The solution of $M(t)$ obtained by FDTM (star), LADM(Triangle), RK method (Square), and RDTM(Circle) for $\alpha = 1$.

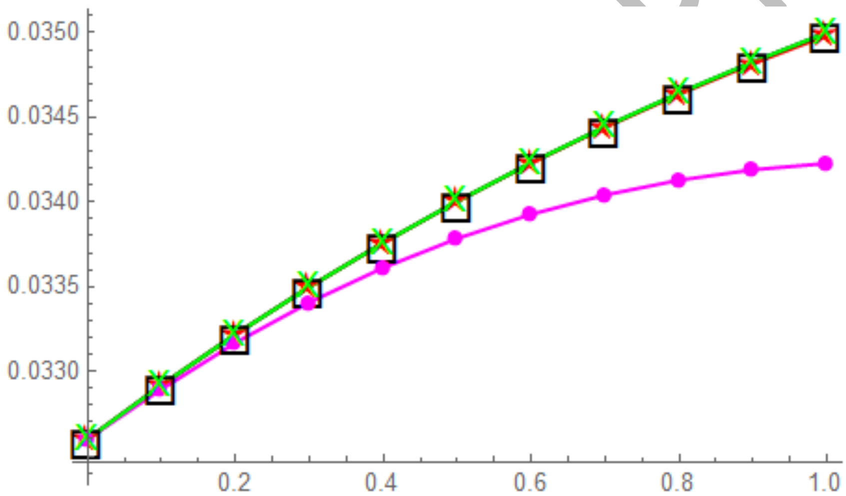


Figure 16. The solution of $Q(t)$ obtained by FDTM (star), LADM(Triangle), RK method (Square), and RDTM(Circle) for $\alpha = 1$.

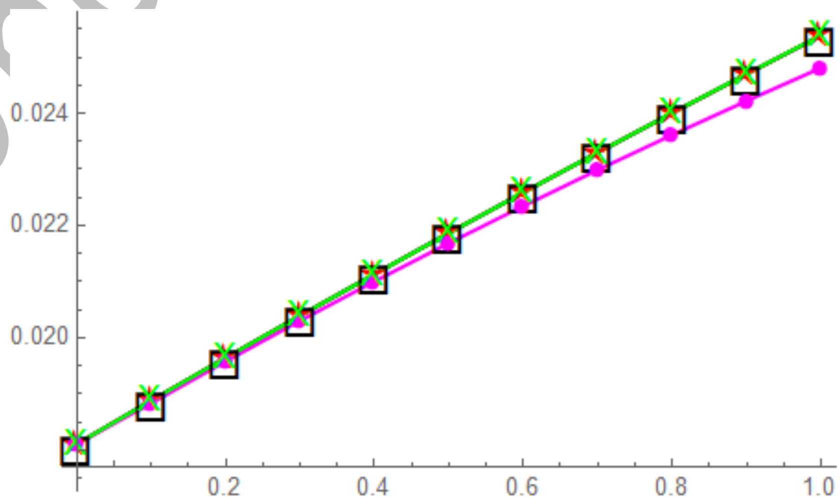


Figure 17. The solution of $R(t)$ obtained by FDTM (star), LADM(Triangle), RK method (Square), and RDTM(Circle) for $\alpha = 1$.

5. CONCLUSION

In this research work, the proposed nonlinear fractional smoking mathematical model in the context of a modified version of Caputo fractional derivative has been successfully solved using two different approaches: the fractional differential transform method and Laplace Adomian decomposition method. All obtained results have been analyzed and compared for various cases. Finally, all results prove the validity and efficiency of those methods in solving nonlinear fractional differential equations. Our results and methods in this work can be further extended or generalized in solving other interesting nonlinear models arising from some phenomena in physics and engineering. In addition, our results can also be applied for models formulated using other fractional derivatives.

REFERENCES

[1] Mahdy, A. M. S., Mohamed, M. S., Gepreel, K. A., AL-Amiri, A. and Higazy, M., (2020) Dynamical characteristics and signal flow graph of nonlinear fractional smoking mathematical model, *Chaos, Solitons & Fractals*, *141*, 110308.

[2] Ertürk, V. S., Zaman, G., Momani, S., (2012) A numeric–analytic method for approximating a giving up smoking model containing fractional derivatives, *Computers & Mathematics with Applications*, *64*(10), 3065-3074.

[3] Haq, F., Shah, K., ur Rahman, G., Shahzad, M., (2018) Numerical solution of fractional order smoking model via laplace Adomian decomposition method, *Alexandria Engineering Journal*, *57*(2), 1061-1069.

[4] Mahdy, A. M. S., Sweilam, N. H., Higazy, M. (2020) Approximate solution for solving nonlinear fractional order smoking model, *Alexandria Engineering Journal*, *59*(2), 739-752.

[5] Uçar, S., Uçar, E., Özdemir, N., Hammouch, Z. (2019) Mathematical analysis and numerical simulation for a smoking model with Atangana–Baleanu derivative, *Chaos, Solitons & Fractals*, *118*, 300-306.

[6] Veeresha, P., Prakasha, D. G., Baskonus, H. M., (2019) Solving smoking epidemic model of fractional order using a modified homotopy analysis transform method. *Mathematical Sciences*, *13*(2), 115-128.

[7] Ahmad, A., Farman, M., Ahmad, M. O., Raza, N., and Abdullah, M., (2018) Dynamical behavior of SIR epidemic model with non-integer time fractional derivatives: A mathematical analysis, *Int. J. Adv. Appl. Sci*, *5*(1), (2018), 123-129.

[8] Sefidgar, E., Celik, E., Shiri, B., (2017) Numerical Solution of Fractional Differential Equation in a Model of HIV Infection of CD4 (+) T Cells, *International Journal of Applied Mathematics and Statistics*, *56*, 23-32.

[9] Kaabar, M., (2020) Novel methods for solving the conformable wave equation, *Journal of New Theory*, (31), 56-85.

[10] Kaabar, M. K. A., Martínez, F., Gómez-Aguilar, J. F., Ghanbari, B., Kaplan, M., and Günerhan, H., (2021) New approximate analytical solutions for the nonlinear fractional Schrödinger equation with second-order spatio-temporal dispersion via double Laplace transform method, *Mathematical Methods in the Applied Sciences*, 44(14), 11138-11156.

[11] Martínez, F., Martínez, I., Kaabar, M. K. A., and Paredes, S., (2020) New results on complex conformable integral, *AIMS Mathematics*, *5*(6), 7695-7710.

[12] Martínez, F., Martínez, I., Kaabar, M. K. A., Ortiz-Munuera, R., and Paredes, S., (2020) Note on the conformable fractional derivatives and integrals of complex-valued functions of a real variable, *IAENG Int. J. Appl. Math*, *50*, 609-615.

[13] Zhou, J. K., (1986) Differential Transformation and Its Applications for Electronic Circuits, *Huazhong Science & Technology University Press,* China.

[14] Arikoglu, A., Ozkol, I., (2007) Solution of fractional differential equations by using differential transform method*, Chaos, Solitons & Fractals, 34(5), 1*473-1481.

[15] Ali A., Ertürk V.S, Zeb A., Khan R.A., (2019) Numerical solution of fractional order immunology and aids model via laplace transform Adomian decomposition method, *J Fract Calculus Appl,* 10 (1), 242–52.

[16] Günerhan, H., Dutta, H., Dokuyucu, M. A., and Adel, W., (2020) Analysis of a fractional HIV model with Caputo and constant proportional Caputo operators, *Chaos, Solitons & Fractals*, 139, 110053.

[17] Dokuyucu, M.A, Çelik, E, Bulut, H and Başkonuş, H.M., (2018) Cancer treatment model with the Caputo-Fabrizio fractional derivative, *The European Physical Journal Plus*, 133 (3), 1-6.

[18] Dokuyucu, M.A, Çelik, E., (2021) Analyzing a novel coronavirus model (COVID-19) in the sense of caputo-fabrizio fractional operator, *Applied and Computational Mathematics*, 20 (1), 49-69.

[19] Javed, K., Uddin, F., Aydi, H., Arshad, M., Ishtiaq, U., and Alsamir, H., (2021) On fuzzy b-metric-like spaces, *Journal of Function Spaces*, 2021, 1-9.

[20] Uddin, F., Javed, K., Aydi, H., Ishtiaq, U., and Arshad, M., (2021) Control Fuzzy Metric Spaces via Orthogonality with an Application, *Journal of Mathematics*, 2021, 1-12.

[21] Ishtiaq, U., Javed, K., Uddin, F., Sen, M. D. L., Ahmed, K., and Ali, M. U., (2021) Fixed point results in orthogonal neutrosophic metric spaces, *Complexity*, 2021, 1-18.