



Research Article

Cubic B-spline finite element method for generalized reaction-diffusion equation with delay

Gemeda Tolessa LUBO^{1,*}, Gemechis File DURESSA²

¹Department of Mathematics, Dambi Dollo University, Dambi Dollo, 381149, Ethiopia

²Department of Mathematics, Jimma University, Jimma, 1000, Ethiopia

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ABSTRACT

In this paper, a cubic B-spline finite element method is constructed based on redefined cubic B-spline basis functions for solving the generalized reaction-diffusion equations with delay. The time discretization process is based on Crank-Nicolson method. Examples are worked out to validate the theoretical convergence analysis. The numerical results given in graphs and tables demonstrate that the present method approximates the exact solution very well. The accurateness of the numerical scheme is confirmed by computing L_2 and L_∞ error norms.

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INTRODUCTION

In this paper, we consider the generalized reaction diffusion equation with delay of the following type

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = a \frac{\partial^2 u(x,t)}{\partial x^2} + b \frac{\partial^2 u(x,t-\tau)}{\partial x^2} + cu(x,t) + du(x,t-\tau), t > 0, 0 < x < \pi, \\ u(x,t) = \varphi(x,t), -\tau \leq t \leq 0, & 0 \leq x \leq \pi, \\ u(0,t) = u(\pi,t) = 0, t \geq -\tau, \end{cases}$$

where $a > 0$ and $b > 0$ represent the diffusion coefficients $c, d \in \mathbb{R}$ and $\tau > 0$ is a delay constant. The generalized reaction-diffusion equations with delay have attracted a significant interest in the last several decades due to their frequent occurrence in real life situations [1-7]. The governing equation has been studied widely in the case where $c = d = 0$ [8-13] and $b = c = 0$ [14-16]. Equation of the type (1) has been considered equation [17,18]. A delay term could not only make the numerical solutions difficult to be obtained

but also change the dynamical properties of a system [16]. This is the reason why such equations became the attention of researchers in numerical analysis and simulations. The accessibility of exact solutions, as well as efficient methods to obtain numerical approximations of the required precision, are very important. However, few delay generalized reaction-diffusion equations with a certain simple setting permit an analytic solution. Hence, computing approximate solutions essential are essential. Up to now, many of the numerical methods available to approximate the diffusion term are based on some classical numerical methods, for example, finite difference method [19], compact finite difference method [20,21], discontinuous Galerkin methods [22], spectral method [23], Haar Wavelet [24], variational method [25], waveform relation method [26], and so on.

In [27], an analytical technique for the nonlinear dynamic problem with delayed is solved. It is well known

*Corresponding author.

*E-mail address: gemedatolesa@gmail.com

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that the finite element method (FEM) can be easily designed for high order accuracy in space by taking higher-order polynomial as the basis (shape) function. The advantage of the FEM over the finite difference method is its ability to deal with the modeling of complex geometries and irregular shapes. The use of various degrees of B-spline functions to obtain the numerical solutions of some PDEs has been shown to provide easy and simple algorithms, for instance, quadratic B-spline least-square method [28], quadratic B-spline method [29,30], cubic B-spline finite element method [31-37], quartic B-spline method [38], quintic B-spline finite element method [39,40], and so on. To the best knowledge of the author, the redefined cubic B-spline finite element method is not considered for finding the approximate solution of delay PDEs. In this paper, we have applied a redefined cubic B-spline FEM to find numerical solutions of the reaction-diffusion with the delay of the type

(1). Our selection of redefined cubic B-spline basis functions improves accuracy to fourth-order. The proposed method give better convergence result than the linear θ -method without increasing the computational cost. Our best concern in this study is to formulate a numerical scheme with a higher order of accuracy by using the redefined cubic B-spline shape functions.

Notations: Denote $\|\cdot\|$ and $\|\cdot\|_r$ as the norm $L_2 = L_2(\Omega)$ and the sobolov space $H^r = H^r(\Omega) = W_2(\Omega)$ respectively, so that the real valued v ,

$$\|v\| = \|v\|_{L_2} := \left(\int_{\Omega} |v(x)|^2 dx \right)^{\frac{1}{2}}$$

and for a positive integer r

$$\|v\|_r = \|v\|_{H^r} := \left(\sum_{i \leq r} \left\| \frac{\partial^i v(x)}{\partial x^i} \right\|^2 \right)^{\frac{1}{2}}$$

Let $v(x), w(x)(x \in \Omega)$ be real-valued functions.

$$(v, w) := \int_{\Omega} v(x) w(x) dx, (\nabla v, \nabla w) := \int_{\Omega} \frac{\partial v(x)}{\partial x} \frac{\partial w(x)}{\partial x} dx,$$

and C denotes a positive not necessarily the same at different occurrences, which may depend on a, b, c, d and t of (1) but

independent of h and Δt (the stepsizes in t -direction). We denote $u(t, x)$ by u or $u(t)$.

Assumptions: In this paper, assume that $u(t) := u(t, \cdot)$, $u_t(t) := u_t(t, \cdot)$, $u_{tt}(t) := u_{tt}(t, \cdot)$, $u_{ttt}(t) := u_{ttt}(t, \cdot) \in H_0^1(\Omega)$, and at $t = n\tau (n = -1, 0, 1, \dots)$, the derivative of t is denoted as the left derivative.

Cubic B-spline Finite Element Method

Let $\Delta t = \tau / (m + 1)$ be a given step size with $m \geq 1$, the grid points $t_n = n\Delta t$ ($n = 0, 1, \dots$) and be a given be the

approximation in S_h^T of $u(t)$ at $t = t_n = n\Delta t$. For positive integer N , let $\{x_k\}_{k=0}^{N+1}$ be a uniform partition on $[0, \pi]$ in the x direction, such that $x_k = kh$, where $h = \pi / (N + 1)$ is the step size. Define the space

$$S_3 = \left\{ v : v \in C^2([0, \pi]), v|_{[x_{k-1}, x_k]} \in P_3, 1 \leq k \leq N + 1 \right\}$$

where P_3 is the space of all polynomial of degree ≤ 3 . Extending the partition $\{x_k\}_{k=0}^{N+1}$ using $x_k = kh, k = -3, -2, -1, N + 2, N + 3, N + 4$. As a basis for S_3 , we choose the B-splines, $\{Q_j\}_{j=-1}^{N+1}$, where

$$Q_j(x) = \begin{cases} h^{-3} f_1(x - x_{j-2}), & x \in [x_{j-2}, x_{j-1}], \\ f_2\left(\frac{x - x_{j-1}}{h}\right), & x \in [x_{j-1}, x_j], \\ f_2\left(\frac{x_{j+1} - x}{h}\right), & x \in [x_j, x_{j+1}], \\ h^{-3} f_1(x_{j+2} - x), & x \in [x_{j+1}, x_{j+2}], \\ 0, & x \notin [x_{j-2}, x_{j+2}], \end{cases} \quad (2)$$

and let $f_1(x) = x^3, f_2(x) = 1 + 3x + 3x^2 - 3x^3$. We want to construct a basis for the space

$$S_h^T = \{v : v \in S_3, v(0) = v(\pi) = 0\}. \quad (3)$$

The cubic B-spline $\{\overline{Q}_j\}_{j=-1}^{N+1}$ can be redefined by

$$\begin{cases} \overline{Q}_0 = Q_0 - 4Q_{-1}, & \overline{Q}_1 = Q_1 - Q_{-1}, \\ \overline{Q}_j = Q_j, & j = 2, 3, 4, \dots, N - 1. \\ \overline{Q}_N = Q_N - 4Q_{N+1}, & \overline{Q}_{N+1} = Q_{N+1} - 4Q_{N+2}. \end{cases} \quad (4)$$

As the cubic B-spline Q_j , the cubic \overline{Q}_j have support of at least 4 subintervals. We want to find an approximation $U^n \in S_h^T$ such that

$$a(U^n, v) = l(v), v \in S_h^T. \quad (5)$$

The application of Galerkin Crank - Nicolson method to (1) leads to a numerical process of the following type

$$\left(\frac{U^n - U^{n-1}}{\Delta t}, x \right) + a \left(\frac{U^n + U^{n-1}}{2}, \nabla x \right) + b \left(\frac{U^n - U^{n-1}}{2}, \nabla x \right) - c \left(\frac{U^n + U^{n-1}}{2}, x \right) - d \left(\frac{U^n - U^{n-1}}{2}, x \right) = 0, \text{ for } n > 0 \quad (6)$$

Where $U^n(\cdot) = \varphi(\cdot, t_n)$ for $-m \leq n \leq 0$.

Let

$$U^n(x) := \sum_{j=0}^{N+1} \overline{Q}_j(x) \alpha_j^n \quad (7)$$

Using (7) and choosing $\chi = \overline{Q}_j; j = 0, 1, \dots, N + 1$. Then (6) can be reduced to

$$\begin{aligned} \frac{1}{\Delta t} \sum_{j=0}^{N+1} (\alpha_j^n - \alpha_j^{n-1}) (\overline{Q}_j(x), \overline{Q}_j(x)) &= -\frac{a}{2} \sum_{j=0}^{N+1} (\alpha_j^n + \alpha_j^{n-1}) (\nabla \overline{Q}_j(x), \nabla \overline{Q}_j(x)) - \frac{b}{2} \sum_{j=0}^{N+1} (\alpha_j^{n-m} + \alpha_j^{n-m-1}) (\nabla \overline{Q}_j(x), \nabla \overline{Q}_j(x)) \\ &+ \frac{c}{2} \sum_{j=0}^{N+1} (\alpha_j^n + \alpha_j^{n-1}) (\overline{Q}_j(x), \overline{Q}_j(x)) + \frac{d}{2} \sum_{j=0}^{N+1} (\alpha_j^{n-m} + \alpha_j^{n-m-1}) (\overline{Q}_j(x), \overline{Q}_j(x)) \end{aligned} \quad (8)$$

Equation (8) can be written

$$\frac{1}{\Delta t} \sum_{j=0}^{n+1} (a_j^n - a_j^{n-1}) \int_0^\pi \bar{Q}_i(x) \bar{Q}_j(x) dx = -\frac{a}{2} \sum_{j=0}^{n+1} (a_j^n + a_j^{n-1}) \int_0^\pi \bar{Q}_i(x) \bar{Q}_j(x) dx - \frac{b}{2} \sum_{j=0}^{n+1} (a_j^{n-m} + a_j^{n-m-1}) \int_0^\pi \bar{Q}_i(x) \bar{Q}_j(x) dx + \frac{c}{2} \sum_{j=0}^{n+1} (a_j^n + a_j^{n-1}) \int_0^\pi \bar{Q}_i(x) \bar{Q}_j(x) dx + \frac{d}{2} \sum_{j=0}^{n+1} (a_j^{n-m} + a_j^{n-m-1}) \int_0^\pi \bar{Q}_i(x) \bar{Q}_j(x) dx \tag{9}$$

Defining the following matrices

$$E = \{e_{i,j}\}_{i,j=0}^{N+1} = \int_0^\pi \bar{Q}_i'(x) \bar{Q}_j'(x) dx \tag{10}$$

$$D = \{d_{i,j}\}_{i,j=0}^{N+1} = \int_0^\pi \bar{Q}_i(x) \bar{Q}_j(x) dx \tag{11}$$

Equation (9) is an $(N + 2) \times (N + 2)$ linear system

$$\left(\left(\frac{d}{2} + \frac{1}{2} a \Delta t E - \frac{1}{2} c \Delta t D \right) \alpha^n = \left(\frac{d}{2} - \frac{1}{2} a \Delta t E + \frac{1}{2} c \Delta t D \right) \alpha^{n-1} - \frac{1}{2} b \Delta t E (\alpha^{n-m} + \alpha^{n-m-1}) + \frac{1}{2} d \Delta t D (\alpha^{n-m} + \alpha^{n-m-1}) \right) \tag{12}$$

We can write the entries of the matrices E and D in (10) and (11) as stiffness matrix

$$E = \frac{3}{10h} \begin{bmatrix} 80 & 43 & -20 & -1 & 0 & \dots & \dots & \dots & 0 \\ 43 & 104 & -14 & -24 & -1 & \ddots & & & \vdots \\ -20 & -14 & 80 & -15 & -24 & -1 & \ddots & & \vdots \\ -1 & -24 & -15 & 80 & -15 & -24 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -1 & -24 & -15 & 80 & -15 & -24 & -1 \\ \vdots & & \ddots & -1 & -14 & -15 & 80 & -14 & -20 \\ \vdots & & & \ddots & -1 & -24 & -14 & 104 & 43 \\ 0 & \dots & \dots & \dots & 0 & -1 & -20 & 43 & 80 \end{bmatrix}$$

and the mass matrix

$$D = \frac{h}{140} \begin{bmatrix} 496 & 773 & 116 & 1 & 0 & \dots & \dots & \dots & 0 \\ 773 & 2296 & 1190 & 120 & 1 & \ddots & & & \vdots \\ 116 & 1190 & 2416 & 1191 & 120 & 1 & \ddots & & \vdots \\ 1 & 120 & 1191 & 2416 & 1191 & 120 & 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 1 & 120 & 1191 & 2416 & 1191 & 120 & 1 \\ \vdots & & \ddots & 1 & 120 & 1191 & 2416 & 1190 & 116 \\ \vdots & & & \ddots & 1 & 120 & 1190 & 2296 & 773 \\ 0 & \dots & \dots & \dots & 0 & 1 & 116 & 773 & 496 \end{bmatrix}$$

with γ^n is the initial approximation of $\varphi(t_n)$ and $\alpha^n := [\alpha_0, \alpha_1, \dots, \alpha_{N+1}]^T$.

Convergence Analysis

We shall prove the optimal convergence order for the numerical scheme.

Let u_h and u be the solutions of (1). Then error can be quantified by bounding the norm of the error

$u_h(t) - u(t)$ in terms of the mesh spacing h of the finite element mesh.

Remark 1. Using polynomial with degrees $p \geq 1$ as basis we expect an error bound of

$$\|u_h(t) - u(t)\| \leq Ch^{p+1},$$

where C is a problem-dependent constant independent of h and the constant $p + 1$ indicates the order of convergence of the FEM, as the mesh spacing h decreases.

Theorem 2 Let u and U^n be the exact and approximation solution of Eq.(1) respectively. Assume that $\|u(t) - R_h u(t)\| \leq Ch^4 \|u(t)\|_r$, $\|u_t(t) - R_h u_t(t)\|_i \leq Ch^4 \|u_t(t)\|_r$, $-\tau \leq i \leq 0$, and $\|\varphi_h(t) - \varphi(t)\|_i \leq Ch^4$. then

$$\|U^n - u(t_n)\| \leq C(h^4 + (\Delta t)^2), \text{ for any } n = 1, 2, \dots$$

Proof. Denote

$$U^n - u(t_n) = (U^n - D_h u(t_n)) + (D_h u(t_n) - u(t_n)) = \theta^n + \rho^n$$

and $\rho^n(t) = \rho(t_n)$ is bounded as in [26].

$$\left(\frac{\theta^n - \theta^{n-1}}{\Delta t}, \chi \right) + a \left(\frac{\nabla \theta^n + \nabla \theta^{n-1}}{2}, \nabla \chi \right) + b \left(\frac{\nabla \theta^{n-m} + \nabla \theta^{n-m-1}}{2}, \nabla \chi \right) - c \left(\frac{\theta^n + \theta^{n-1}}{2}, \chi \right) - d \left(\frac{\theta^{n-m} + \theta^{n-m-1}}{2}, \chi \right) = -(W^n, \chi), \forall \chi \in S_h^n \tag{15}$$

where

$$W^n = \frac{D_h u(t_n) - D_h u(t_{n-1})}{\Delta t} - \frac{u_t(t_n) + u_t(t_{n-1})}{2} = (D_h - I) \tilde{\delta} u(t_n) + \left(\tilde{\delta} u(t_n) - \frac{u_t(t_n) + u_t(t_{n-1})}{2} \right) =: W_1^n + W_2^n$$

Choosing $\chi = \frac{\theta^n + \theta^{n-1}}{2}$, gives

$$\left(\frac{\theta^n - \theta^{n-1}}{\Delta t}, \frac{\theta^n + \theta^{n-1}}{2} \right) + a \left\| \frac{\theta^n + \theta^{n-1}}{2} \right\|_1^2 + b \left(\frac{\nabla \theta^{n-m} + \nabla \theta^{n-m-1}}{2}, \frac{\nabla \theta^n + \nabla \theta^{n-1}}{2} \right) - c \left(\frac{\theta^n + \theta^{n-1}}{2}, \frac{\theta^n + \theta^{n-1}}{2} \right) - d \left(\frac{\theta^{n-m} + \theta^{n-m-1}}{2}, \frac{\theta^n + \theta^{n-1}}{2} \right) = - \left(W^n, \frac{\theta^n + \theta^{n-1}}{2} \right)$$

By using Schwartz inequality

$$\left(\frac{\theta^n - \theta^{n-1}}{\Delta t}, \frac{\theta^n + \theta^{n-1}}{2} \right) + \left\| \frac{\theta^n + \theta^{n-1}}{2} \right\|_1^2 \leq C \left(\left\| \frac{\theta^{n-m} + \theta^{n-m-1}}{2} \right\|_1^2 + \left\| \frac{\theta^n + \theta^{n-1}}{2} \right\|_1^2 + \left\| \frac{\theta^{n-m} + \theta^{n-m-1}}{2} \right\|_1^2 + \|W^n\| \left\| \frac{\theta^n + \theta^{n-1}}{2} \right\|_1 \right)$$

So

$$\| \theta^n \|^2 + \Delta t \left\| \frac{\theta^n + \theta^{n-1}}{2} \right\|_1^2 \leq C \left(\| \theta^{n-1} \|^2 + \Delta t \left\| \frac{\theta^{n-m} + \theta^{n-m-1}}{2} \right\|_1^2 + \Delta t \left\| \frac{\theta^n + \theta^{n-1}}{2} \right\|_1^2 + \Delta t \left\| \frac{\theta^{n-m} + \theta^{n-m-1}}{2} \right\|_1^2 + (\Delta t)^2 \|W^n\|^2 \right).$$

Without loss of generality, assume that $n \in ((k - 1)m, km]$, $k \in N$. Then,

$$\| \theta^n \|^2 \leq C \left(\| \theta^{n-1} \|^2 + \Delta t \left\| \frac{\theta^{n-m} + \theta^{n-m-1}}{2} \right\|_1^2 + \Delta t \left\| \frac{\theta^{n-m} + \theta^{n-m-1}}{2} \right\|_1^2 + (\Delta t)^2 \|W^n\|^2 \right) \leq C (\| \theta^{n-1} \|^2 + \| \theta^{n-m-1} \|^2 + \Delta t \| \theta^{n-2m} + \theta^{n-2m-1} \|^2 + (\Delta t)^2 (\|W^n\|^2 + \|W^{n-m}\|^2)) \leq \dots \leq C \left(\sum_{i=0}^{k-1} \| \theta^{n-im-1} \|^2 + \Delta t \| \theta^{n-km} + \theta^{n-km-1} \|^2 + (\Delta t)^2 \sum_{i=0}^{k-1} \|W^{n-im}\|^2 \right)$$

Therefore,

$$\|\theta^n\|^2 \leq C \left(\sum_{j=0}^{k-1} \|\theta^{n-im-1}\|^2 + \Delta t \|\theta^{n-km} + \theta^{n-km-1}\|_1^2 + (\Delta t)^2 \sum_{i=0}^{k-1} \|W^{n-im}\|^2 \right)$$

$$L_2 = \|u(t_n) - U^n\|_2 = \sqrt{h \sum_{n=0}^{N+1} |u(t_n) - U^n|^2}, \quad L_\infty = \|u(t_n) - U^n\|_\infty = \text{Max}|u(t_n) - U^n|$$

By the assumption of the theorem and using the discrete Gronwall inequality (see [42])

$$\|\theta^n\|^2 \leq C \left(\|\theta^0\|^2 + \Delta t \|\theta^{n-km} + \theta^{n-km-1}\|_1^2 + (\Delta t)^2 \sum_{i=0}^{k-1} \|W^{n-im}\|^2 \right) \quad (16)$$

We write

$$W_1^n = (D_h - I)\bar{\delta}u(t_n) = (\Delta t)^{-1} \int_{t_{n-1}}^{t_n} (D_h - I)u_t(t) dt$$

So

$$(\Delta t)^2 \sum_{i=1}^{k-1} \|W^{n-im}\|^2 \leq \sum_{i=1}^{k-1} \left(\int_{t_{n-im-1}}^{t_{n-im}} Ch^4 \|u_t(t)\|_4 dt \right)^2 \leq Ch^4 \quad (17)$$

Further,

$$\|\Delta t W_2^i\| = \left\| u(t_i) - u(t_{i-1}) - \Delta t \frac{u_t(t_i) + u_t(t_{i-1})}{2} \right\| \leq C(\Delta t)^2 \int_{t_{i-1}}^{t_i} \|u_{ttt}(t)\| dt$$

So that

$$(\Delta t)^2 \sum_{i=1}^{k-1} \|W^{n-im}\|^2 \leq C(\Delta t)^4 \sum_{i=1}^{k-1} \left(\int_{t_{i-1}}^{t_i} \|u_{ttt}(t)\| dt \right)^2 \leq C(\Delta t)^4, \quad (18)$$

$$\|U^n - u(t_n)\| \leq C(h^4 + (\Delta t)^2),$$

which together with Eq. (17) and Eq. (18) and the assumption of the theorem

Numerical Simulation

We carry out numerical experiments to illustrate the theoretical results. To evaluate errors, L_2 and L_∞ norms are applied as follows:

Order of convergence is denoted by

$$\text{Order}(\tau) = \frac{\log(\text{Error}(N)/\text{Error}(2N))}{\log(2 - \frac{1}{N+1})}$$

Example 1 We consider the following equation:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = a \frac{\partial^3 u(x,t)}{\partial x^2} + b \frac{\partial^3 u(x,t-\tau)}{\partial x^2} + cu(x,t) + du(x,t-\tau) \\ t > 0, 0 < x < \pi \\ u(x,t) = \sin(x), -\tau \leq t \leq 0, 0 \leq x \leq \pi \\ u(0,t) = \varphi(\pi,t) = 0, t \geq -\tau \end{cases} \quad (19)$$

We set the parameters $a = 2, b = 0.5, c = 1, d = 0.5, \tau = 1$ and $T = 2$ and solve problem Eq.(19) on $[0, \pi] \times [0,2]$ with different temporal and spatial step sizes ($\Delta t = \tau / (m + 1), h = \pi / (N + 1)$). The exact solution is $u(x, t) = \exp(-t)\sin(x)$.

Numerical errors and the corresponding orders are listed in Table 1 - 5. There is a noticeable decrease in both error norms when mesh sizes decrease. These results confirm that the numerical method is convergent. The results in Tables 3, 4 and 5 show that our result is more accurate than in comparison with those obtained by [18]. In Figure 1 and 5, the graph of approximation solutions for example 1 and 2 at different times respectively are given. In figures 2 and 6, the approximation and exact solution are drawn on the same coordinate axis. Approximation and exact solution are depicted in Figures 3, 4,7, and 8. It can be observed that the graphs are similar.

Table 1. Error norms and convergence orders for $T = 2, \tau = 1(\Delta t \approx \Delta x^2)$

N	L_2	Order	L_∞	Order
7	7.8685E-04	-	8.5506E-04	-
14	6.4117E-05	3.9887	6.9156E-05	4.006
28	4.4172E-06	4.0580	4.7832E-06	4.0520
56	2.7781E-07	4.0937	3.0115E-07	4.0921
112	1.7472E-08	4.0424	1.8946E-8	4.0419

Table 2. Error norms and convergence orders for $T = 2, \tau = 1(\Delta t \approx \Delta x^2)$

N	L_2	Order	L_∞	Order
10	2.3378E-04	-	2.5206E-04	-
20	1.6027E-05	4.1447	1.7332E-05	4.1340
40	1.0649E-06	4.0539	1.1530E-06	4.0508
80	6.7117E-8	4.0585	7.2770E-8	4.0577

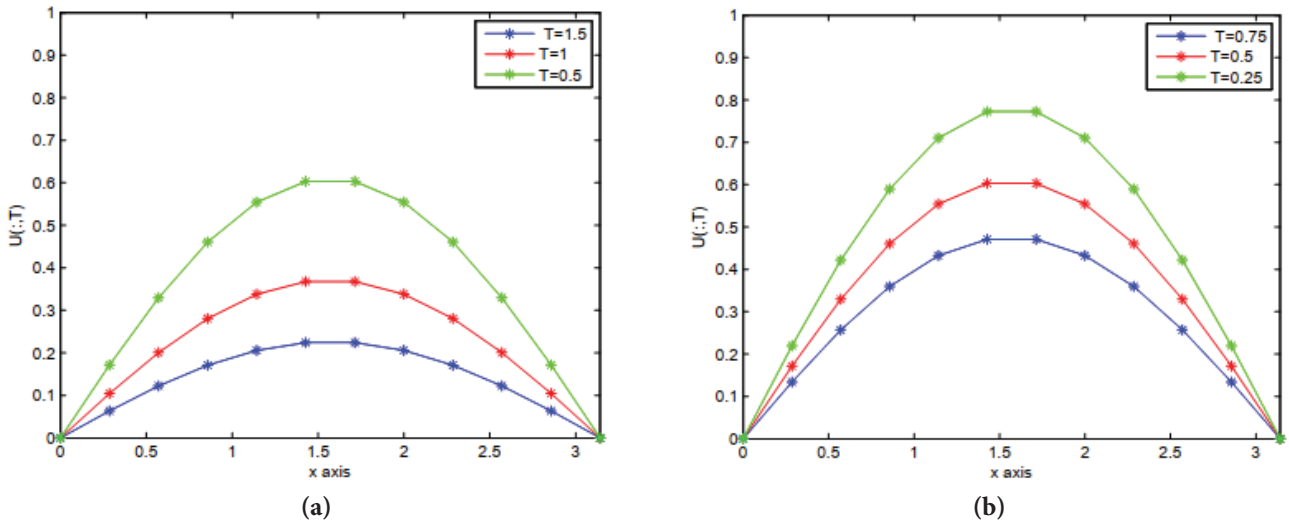


Figure 1. Approximate solutions for fixed time and step sizes ($N = 10, m = 100$) for example 1.

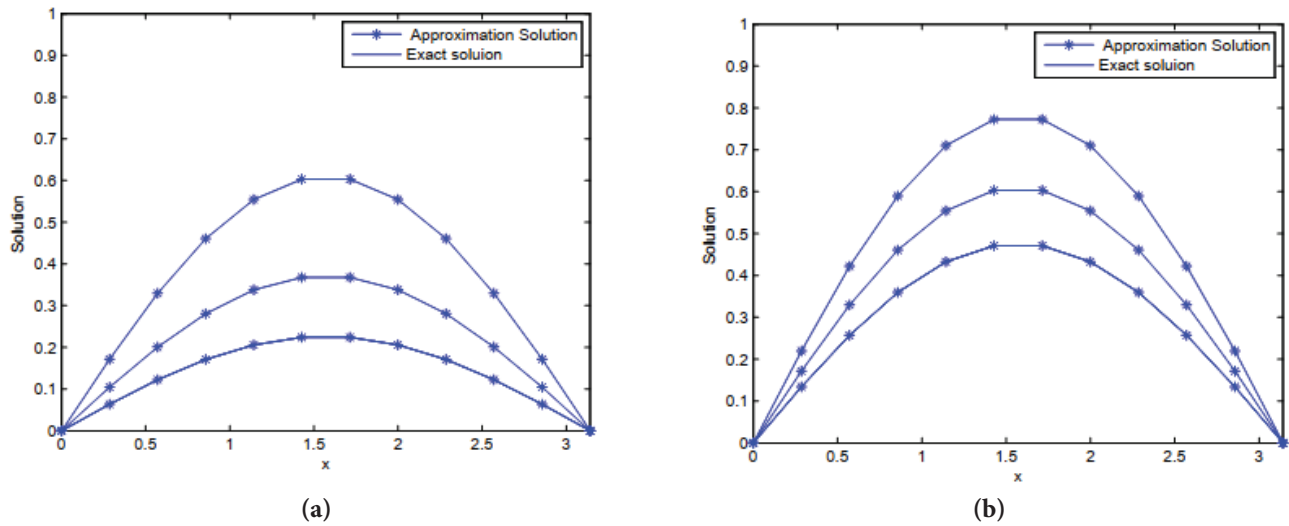


Figure 2. Comparison between approximate and exact solutions for example 1 where (a) $T = 1.5, T = 1, T = 0.5$ with step sizes ($N = 10, m = 100$). (b) $T = 0.75, T = 0.5, T = 0.25$ with step sizes ($N = 10, m = 100$).

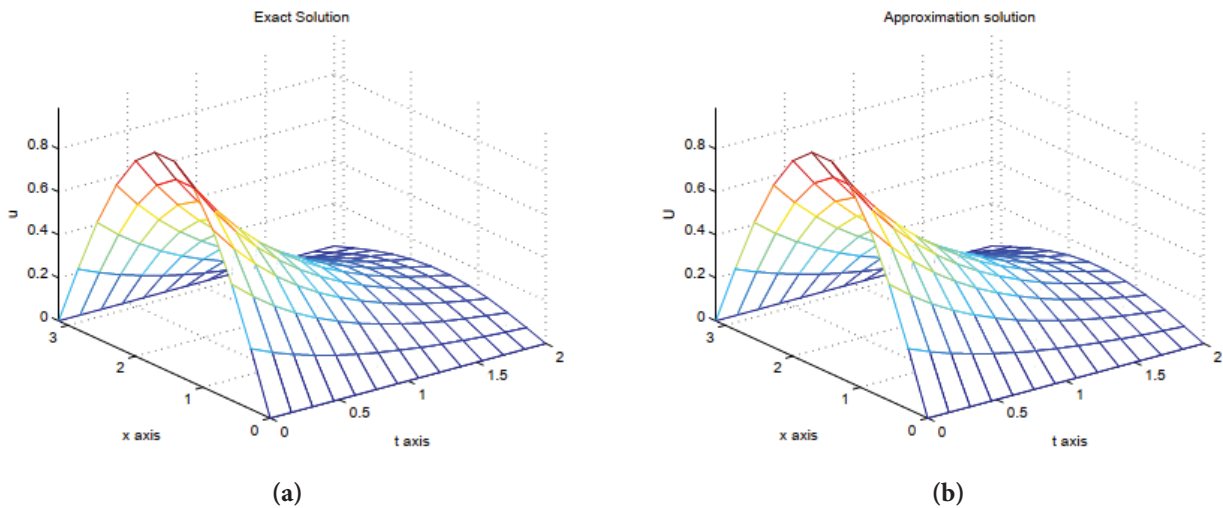


Figure 3. Approximation solution and analytical solution ($N = 10, m = 5, \text{ and } \tau = 1$).

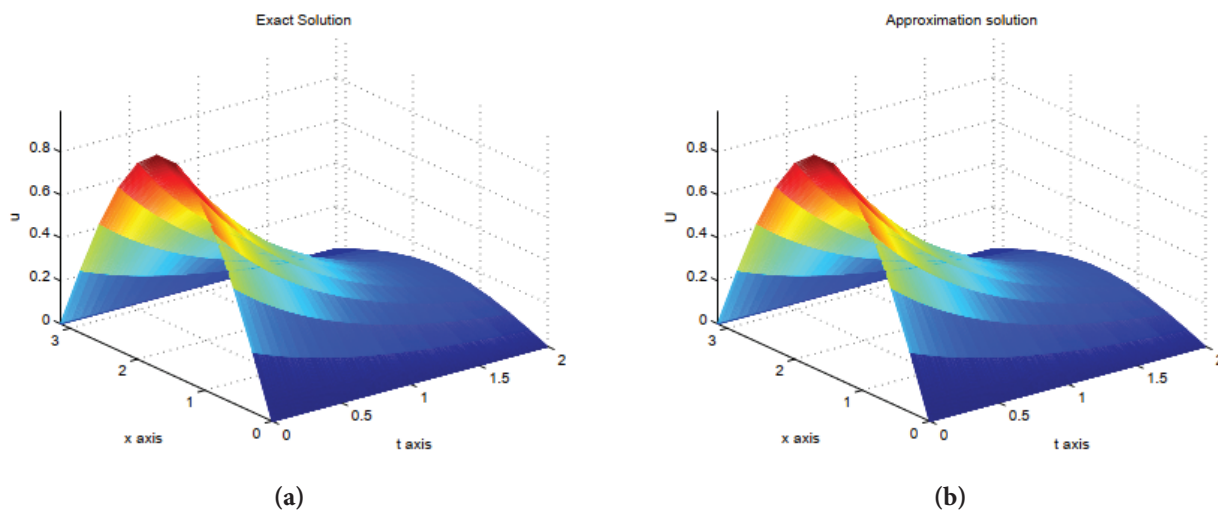


Figure 4. Comparisons between approximate and exact solutions of example 1 with step size ($N = 10, m = 200$).

Table 3. Error norms and convergence orders for $T = 2, \tau = 0.5 (\Delta t \approx 0.5\Delta x^2)$

N	Compact θ -method ($\theta=1$)[18]				Present method			
	L_2	Order	L_∞	Order	L_2	Order	L_∞	Order
10	2.96E-04	4.20	3.36E-04	4.13	6.11E-05	4.45	6.27E-05	4.31
20	1.82E-05	4.03	1.45E-05	4.03	4.05E-06	4.20	4.33E-06	4.14
40	1.13E-06	4.00	9.05E-07	4.00	2.67E-07	4.06	2.88E-07	4.05
80	7.09E-08	4.00	5.65E-08	4.00	1.68E-08	4.06	1.82E-08	4.06

Table 4. Error norms and convergence orders for $T = 2, \tau = 0.5 (\Delta t \approx \Delta x^2)$

N	Linear θ -method ($\theta=1$)[18]				Present method			
	L_2	Order	L_∞	Order	L_2	Order	L_∞	Order
10	4.34E-03	2.31	3.47E-03	2.24	2.13E-04	3.85	2.11E-04	3.65
20	9.83E-04	2.14	7.84E-04	2.14	1.64E-05	3.96	1.73E-05	3.86
40	2.38E-04	2.04	1.90E-04	2.04	1.06E-06	4.00	1.14E-06	4.07
80	5.94E-05	2.01	4.72E-05	2.01	6.70E-08	4.05	7.25E-08	4.04

Table 5. Error norms and convergence orders for $T = 2, \tau = 0.5 (\Delta t \approx 0.5\Delta x^2)$

N	Linear θ -method ($\theta=1/2$)[18]				Present method			
	L_2	Order	L_∞	Order	L_2	Order	L_∞	Order
5	1.40E-03	-	1.06E-03	-	9.17E-04	-	8.55E-04	-
10	3.81E-04	1.87	3.04E-04	1.80	6.11E-05	4.45	6.27E-05	4.31
20	9.67E-05	1.98	7.72E-05	1.98	4.05E-06	4.20	4.33E-06	4.13
40	2.61E-05	1.89	2.09E-05	1.89	2.67E-07	4.06	2.88E-07	4.05
80	6.15E-06	2.09	4.91E-06	2.09	1.68E-08	4.06	1.82E-08	4.06

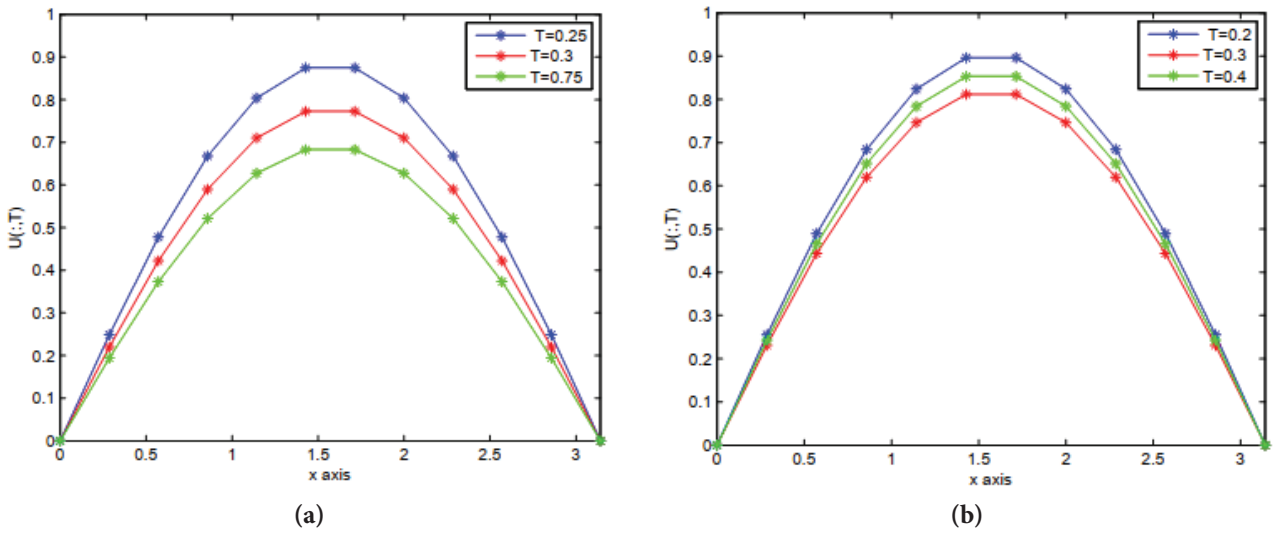


Figure 5. Approximate solutions for fixed time and step sizes ($N = 10, m = 100$) for example 2.

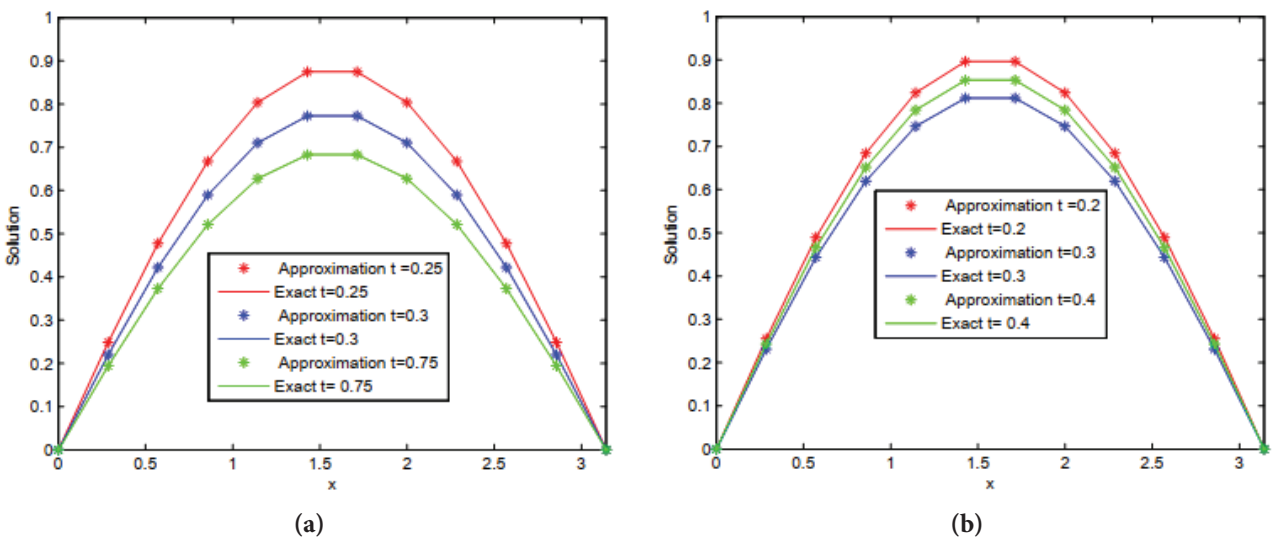


Figure 6. Comparisons between approximate and exact solutions of example 2 ($N = 10, m = 100$).

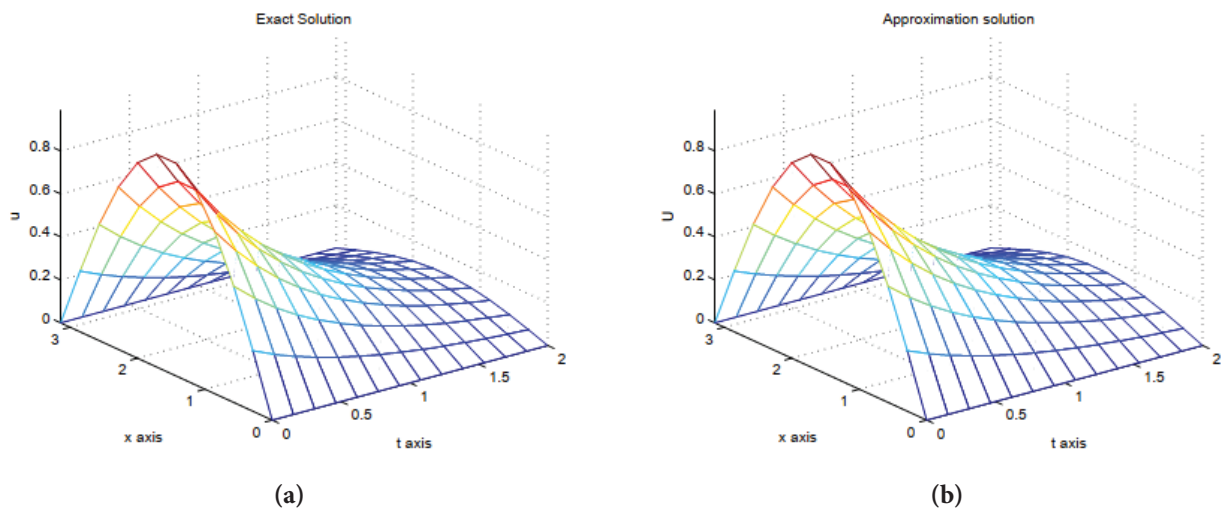


Figure 7. Approximation solution and analytical solution ($N = 10, m = 5$, and $\tau = 1$).

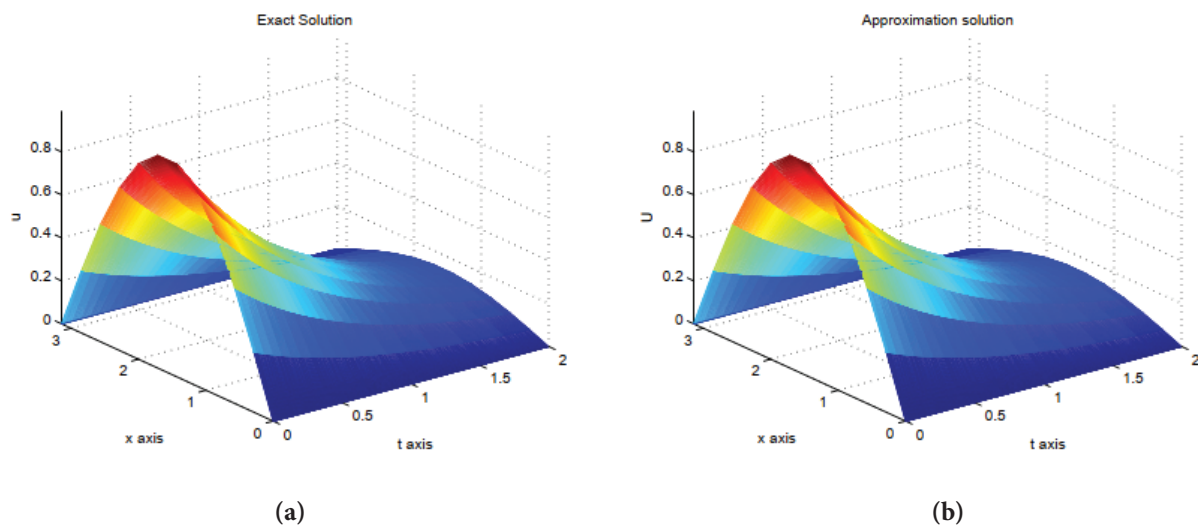


Figure 8. Comparison between approximate and exact solutions for example 2 ($N = 10$; $m = 100$).

CONCLUSION

A finite element method based on redefined cubic B-spline shape function has been developed to solve the generalized reaction-diffusion equation with delay. The convergence analysis is studied. From this result, we can conclude that the cubic B-spline finite element method is feasible. The approximation solutions are tested by comparing with analytical solutions. Our selection of redefined cubic B-spline basis functions improves accuracy to fourth-order. The advantage of the FEM over the finite difference method is its ability to deal with the modeling of complex geometries and irregular shapes. The proposed method gives a better convergence result than the linear θ -method without increasing the computational costs. The numerical results are obtained by applying MATLAB software. This method can be easily extended to tackle a broad class of PDEs.

AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

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