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Research Article

Soft Quasi-ideals of soft near-rings

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ABSTRACT

In this paper, we introduce the notions of soft quasi-ideal, soft minimal quasi-ideal, soft left (resp. right) N- subgroup and soft invariant subnear-ring of a soft near-ring with the help of soft set theory established by Molodtsov. We also introduce the concepts of soft zero-symmetric near-ring, soft constant near- ring, soft near-field and soft Q- simple near-ring over a near-ring. We investigate the properties of these notions with illustrative examples. We obtain the characterizations of soft quasi-ideals of a soft (zero-symmetric) near-ring and soft near-fields over a near-ring.

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INTRODUCTION

In 1965, Zadeh [1] initiated the notion of fuzzy set as an extension of the traditional crisp set. The theory of fuzzy sets is the most convenient approach to deal with uncertainties. Many notions of mathematics are extended to fuzzy sets and various properties of these notions in the context of fuzzy sets are established. Recently, Riaz et al. applied the fuzzy set theory in different disciplines such as linear diophantine fuzzy set [2], bipolar picture fuzzy operators [3], spherical linear diophantine fuzzy set [4] and linear diophantine fuzzy relations [5] in decision making theory, and Cubic M -polar fuzzy hybrid aggregation operator [6].

Many problems in different disciplines such as economics, social science, environmental science, engineering, artificial intelligence, medical science and some other fields are usually not precise. There are various types of uncertainties involved in the data. To describe the uncertainties, mathematical theories such as theory of probability, theory of fuzzy sets [1], theory of intuitionistic fuzzy sets [7], theory of rough sets [8] and many other theories were established by researchers. But each of these notions has its intrinsic problems. To overcome these kinds of difficulties, Molodtsov [9] initiated the innovative concept of soft set as a new mathematical theory for dealing with uncertainties. Soft set theory has numerous applications in different fields such as decision making [10, 11], topological structures

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[12], fractional calculus [13], coding theory [14], medical system [15, 16], prediction [17], investment[18], and so on. In theoretical view, many authors defined and studied several kinds of operations of soft sets and soft binary operations on soft sets such as Maji et al. [19], Ali et al. [20] and Sezgin and Atagün [21].

Algebraic structures play a prominent role in mathematics with wide ranging applications in many disciplines such as theoretical physics, computer sciences, control engineering, information sciences, coding theory, topological spaces and the like. Recent times, soft set theory attained much attention of the researchers since its appearance and start studying soft algebraic structures. In [22], Aktaş and Çağman defined the notions of soft groups and derived some of its basic properties. Since then, the study of soft set algebraic structures has been pursued in many directions. Sezgin and Atagün [23] introduced the concepts of normalistic soft group and normalistic soft group homomorphism, and studied their related properties. Feng et al. [24] introduced the notions of soft semirings and soft ideals on semiring. Acar et al. [25] introduced the initial concepts of soft rings. Onar et al. studied the notion of vague soft module in [26].

Atagün [27] and Sezgin introduced and studied the notions of soft subrings, soft ideals of a ring and soft subfields over a field. They studied the notions of maximal and principal soft ideals of soft rings in [28]. Abdullah

[29] applied soft intersection sets in Γ-near-rings. In [30-32], Sezgin et al. introduced the concepts of soft near-ring, soft subnear-ring, soft (left, right) ideals, (left, right) idealistic soft near-ring, soft ideal, soft N-subgroup and idealistic soft near-ring with related examples.

Ideal theory plays an important role in advanced studies of algebraic structures. Many mathematicians proved important results and characterizations of different algebraic structures by using the concepts of ideals. For example, Narayanan introduced the notion of fuzzy quasi-ideals of near-rings [33], Manikantan [34] proposed the concepts of fuzzy bi-ideals of near-rings, Abdullah et al. [35] defined bi-Γ-hyperideals of Γ-hypersemigroups, Ersoy et al. [36] studied the structure of idealistic fuzzy soft Γ-near-rings, Tang et al. [37] studied the concepts of fuzzy interior Γ- hyperideals in ordered Γ-hypersemigroups and Hila et al. [38] introduced the notions of bi-Γ-hyperideals in quasi-Γ- hypersemigroups.

Generalization of ideals in algebraic structures is necessary for further study of algebraic structures. The quasi-ideals are generalization of left ideal and right ideal. In [39, 40, 41], Yakabe introduced the notions of quasi-ideal and minimal quasi-ideal in a near-ring and characterize these near-rings which are near-fields in terms of quasi-ideals. Motivated by the above theories, in this paper, we apply the theory of soft sets to quasi-ideals of near-rings.

The first objective of this paper is to introduce and study the concepts of soft quasi-ideal, soft left (resp. right) *N*-subgroup and soft invariant subnear-ring of a soft near-ring. The second objective is to introduce the notions of soft zero-symmetric near-ring and soft constant near-ring over a near-ring, and investigate the properties of these notions with illustrative examples. The third objective is to introduce and study the concepts of soft minimal quasi-ideal of a soft near-ring, soft near-field and soft *Q*-simple near-ring over a near-ring.

This paper is composed in the following order: Section 2 contains some basic definitions which are used in the subsequent sections. In section 3, we introduce the notions of soft quasi-ideal of a soft near-ring and soft quasi- ideal over a near- ring. We also introduce the notions of soft left (resp. right) N-subgroup and soft invariant subnear- ring of a soft near-ring. In section 4, we introduce the notions of soft zero-symmetric near-ring and soft constant near- ring over a near-ring. We obtain the characterization of soft quasi-ideal of a soft (zero-symmetric) near-ring and discuss some of its properties. In section 5, we define the notions of soft near-field over a near-ring, soft *Q*-simple near-ring over a near-ring and soft minimal quasi-ideal of a soft near-ring. We discuss the characterization of soft near-rings which are soft near-fields. Finally, in section 6, we present the conclusion of this research paper.

PRELIMINARIES

A near-ring [42] is an algebraic structure (N, +,·) such that (N, +) is a group (not necessarily an abelian) with zero element 0, (N, ·) is a semigroup and the right distributive holds: $(u + v) \cdot n = u \cdot n + v \cdot n$ for all u, v, n ∈ N. In other words, it is a right near-ring. We will use the word "near-ring" to mean "right near-ring". Throughout this article, N stands for near-ring. We write uv for $u \cdot v$. Note that 0u = 0 for all u ∈ N, while it may exists u ∈ N such that u0 ≠ 0. The set $N_0 = \{n \in N/n0 = 0\}$ is called the zero-symmetric part of N; $N_c = \{n \in N/n0 = n\}$ is called the constant part of N. N is called zero-symmetric near-ring if $N = N_0$; N is called constant near-ring if $N = N_c$. An element d of N is called distributive if $d(n + n') = dn + dn'$ ∀ n, n′ ∈ N. N_d stands for the set of all distributive elements of N. If A and B are two non-empty subsets of N, then $A * B = \{a(a' + b) - aa'/a, a' \in A, b \in B\}$.

A subgroup M of (N, +) with MM ⊆ M (resp. NM ⊆ M, MN ⊆ M) is called a subnear-ring (resp. left N- subgroup, right N-subgroup) of N. A subnear-ring M of N with MN ⊆ M and NM ⊆ M is called an invariant subnear- ring of N [42]. A normal subgroup G of (N, +) is called an ideal of N, if it satisfies:(i) GN ⊆ G and (ii) N ∗ G ⊆ G. A normal subgroup G of (N, +) with (i) is called a right ideal of N and with (ii) is called a left ideal of N [42]. A subgroup M of (N, +) is called a quasi-ideal, denoted by $M \lhd_q N$, of N if $MN \cap NM \cap N * M \subseteq M$. In case of zero-symmetric near-ring a subgroup M of (N, +) is called a quasi-ideal of N if $MN \cap NM \subseteq M$ [39]. Clearly {0} and N are quasi-ideals of N. If N has no quasi-ideals except {0} and N, we say that N is Q-simple [39].

A non-zero quasi-ideal M of a near-ring N is named as minimal quasi-ideal (MQI, for short) if M does not properly contain any non-zero quasi-ideal of N. A near-ring N is called a near-field if it has at least two elements and its non-zero elements form a group with respect to the multiplication defined in N [40].

In what follows U is a basic universal set and E is a set of parameters, P(U) is the power set of U and $A \subseteq E$.

Definition 2.1. [9] A pair $(\tilde{H}, A)$ is called a *soft set* (briefly, SS) over U, where $\tilde{H}$ is a mapping given by $\tilde{H} : A \rightarrow P(U)$.

In othe words, a SS over U is a parameterized family of subsets of the universe U. For $e \in A$, $\tilde{H}(e)$ may be considered as the set of e-approximate elements of the SS $(\tilde{H}, A)$.

Definition 2.2. [19] Let $(\tilde{H}_1, A_1)$ and $(\tilde{H}_2, A_2)$ be SSs over U. Then $(\tilde{H}_2, A_2)$ is called a *soft subset* of $(\tilde{H}_1, A_1)$ if $A_2 \subseteq A_1$ and $\tilde{H}_2(\rho) \subseteq \tilde{H}_1(\rho)$ for all $\rho \in A_2$.

Definition 2.3. [24] For a SS $(\tilde{H}, A)$ over U, the set $Supp(\tilde{H}, A) = \{\rho \in A/\tilde{H}(\rho) \neq \emptyset\}$ is called a *support* of

$(\tilde{H}, A)$. If $Supp(\tilde{H}, A) \neq \emptyset$, then the SS $(\tilde{H}, A)$ is called *non-null*.

Definition 2.4. [43] The SS $(\tilde{H}, A)$ over U is called a *relative whole* SS (with respect to A) if $\tilde{H}(\rho) = U$ for all $\rho \in A$. The relative whole SS with respect to E is called the *absolute* SS over U and is denoted by A_u.

Definition 2.5. [20] Let $(\tilde{H}_1, A_1)$ and $(\tilde{H}_2, A_2)$ be SSs over U. Then,

(i) the *restricted intersection* of these SSs, denoted by $(\tilde{H}_1, A_1) \tilde{\cap}_R (\tilde{H}_2, A_2)$, is defined as $(\tilde{H}_1, A_1) \tilde{\cap}_R (\tilde{H}_2, A_2) = (\tilde{H}, A)$, where $A = A_1 \cap A_2 \neq \emptyset$ and $\tilde{H}(\rho) = \tilde{H}_1(\rho) \cap \tilde{H}_2(\rho)$ for all $\rho \in A$.

(ii) the *extended intersection* of these SSs, denoted by $(\tilde{H}_1, A_1) \tilde{\cap}_E (\tilde{H}_2, A_2)$, is defined as $(\tilde{H}_1, A_1) \tilde{\cap}_E (\tilde{H}_2, A_2)$, where $A = A_1 \cup A_2$ and for all $\rho \in A$,

$$\tilde{H}(\rho) = \begin{cases} \tilde{H}_1(\rho), & \text{if } \rho \in A_1 \backslash A_2 \\ \tilde{H}_2(\rho), & \text{if } \rho \in A_2 \backslash A_1 \\ \tilde{H}_1(\rho) \cap \tilde{H}_2(\rho), & \text{if } \rho \in A_1 \cap A_2. \end{cases}$$

Definition 2.6. [21] Let $(\tilde{H}_i, A_i)i\in\Omega \neq \emptyset$ be a family of SSs over U. The *restricted intersection* of these SSs, denoted by $(\tilde{\cap}_R)_{i\in\Omega}(\tilde{H}_i, A_i)$, is defined to be the SS $(\tilde{H}, A)$ such that $A = \cap_{i\in\Omega} A_i \neq \emptyset$ and $\tilde{H}(\rho) = \cap_{i\in\Omega} \tilde{H}_i(\rho)$ for all $\rho \in A$.

Definition 2.7. [44] Let $((\tilde{H}_1, A_1)$ and $(\tilde{H}_2, A_2)$ be SSs over U. The *sum* of $(\tilde{H}_1, A_1)$ and $(\tilde{H}_2, A_2)$ is denoted by $(\tilde{H}_1, A_1) \tilde{+} (\tilde{H}_2, A_2)$, and is defined as the SS $(\tilde{H}, A)$, where $A = A_1 \cap A_2 \neq \emptyset$ and $\tilde{H}(\rho) = \tilde{H}_1(\rho) + \tilde{H}_2(\rho)$ for all $\rho \in A$ and $\tilde{H}_1(\rho) + \tilde{H}_2(\rho) = \{x_1 + x_2/x_1 \in \tilde{H}_1(\rho), x_2 \in \tilde{H}_2(\rho)\}$.

Definition 2.8. [44] Let $(\tilde{H}_1, A_1)$ and $(\tilde{H}_2, A_2)$ be SSs over U. The *product* of $(\tilde{H}_1, A_1)$ and $(\tilde{H}_2, A_2)$ is denoted by $(\tilde{H}_1, A_1) \tilde{\circ} (\tilde{H}_2, A_2)$, and is defined as the SS $(\tilde{H}, A)$, where $A = A_1 \cap A_2 \neq \emptyset$ and $\tilde{H}(\rho) = \tilde{H}_1(\rho)\, \tilde{H}_2(\rho)$ for all $\rho \in A$ and $\tilde{H}_1(\rho)\, \tilde{H}_2(\rho) = \{x_1 x_2/x_1 \in \tilde{H}_1(\rho), x_2 \in \tilde{H}_2(\rho)\}$.

In a same manner, we define the product of an element $x \in N$ with the SS $(\tilde{H}_1, A_1)$ as the SSs $x \tilde{\circ} (\tilde{H}_1, A_1) = \{x \tilde{H}_1(\rho)/\rho \in A_1\}$ and $(\tilde{H}_1, A_1) \tilde{\circ} x = \{\tilde{H}_1(\rho) x/\rho \in A_1\}$.

From now on, let P be a nonempty subset of E and R be an arbitrary binary relation between an element of P and an element of N, that is R is a subset of $P \times N$. A set-valued function $\tilde{H} : P \rightarrow P(N)$ can be defined as $\tilde{H}(\rho) = \{\sigma \in N/(\rho, \sigma) \in R\}$ for all $\rho \in P$. Then the pair $(\tilde{H}, P)$ is a SS over N, which is obtained from the relation R.

Definition 2.9. [22] Let $(\tilde{H}, P)$ be a non-null SS over a group G. Then $(\tilde{H}, P)$ is called a *soft group* (briefly, SG) over G if and only if $\tilde{H}(\rho)$ is a subgroup of G for all $\rho \in P$.

Definition 2.10. [22] Let $(\tilde{H}_1, P_1)$ and $(\tilde{H}_2, P_2)$ be SGs over a group G. Then the SG $(\tilde{H}_2, P_2)$ is called a *soft subgroup* (briefly, SSG) of $(\tilde{H}_1, P_1)$ if $P_2 \subseteq P_1$ and $\tilde{H}_2(\rho)$ is a subgroup of $\tilde{H}_1(\rho)$ for all $\rho \in P$.

Definition 2.11. [32] Let $(\tilde{H}, P)$ be a non-null SS over N. Then $(\tilde{H}, P)$ is called a *soft near-ring* (SN, for short) over N if $\tilde{H}(\rho)$ is a subnear-ring of N for all $\rho \in P$.

Definition 2.12. [32] Let $(\tilde{H}_1, P_1)$ and $(\tilde{H}_2, P_2)$ be SNs over a group N. Then the SN $(\tilde{H}_2, P_2)$ is called a *soft subnear- ring* (briefly, SSN) of $(\tilde{H}_1, P_1)$ if $P_2 \subseteq P_1$ and $\tilde{H}_2(\rho)$ is a subnear-ring of $\tilde{H}_1(\rho)$ for all $\rho \in P_2$.

Definition 2.13. [32] Let $(\tilde{H}, P)$ be a SN over N. A non-null SS $(\tilde{K}, I)$ over N is called a *soft left ideal* (briefly, SLI) (resp. *soft right ideal* (SRI, in short)) of $(\tilde{H}, P)$ if $I \subseteq P$ and $\tilde{K}(\rho)$ is a left (resp. right) ideal of $\tilde{H}(\rho)$ for all $\rho \in I$.

If $(\tilde{K}, I)$ is both SLI and SRI of $(\tilde{H}, P)$, then it is said that $(\tilde{K}, I)$ is a *soft ideal* (briefly, SI) of $(\tilde{H}, P)$.

SOFT QUASI-IDEALS OF SOFT NEAR-RINGS

In this section, we introduce the notion of $\tilde{*}$-product of two soft sets of a near-ring which is used to study the soft ideal structures of soft near-rings. We introduce the notions of soft quasi-ideal of a soft near-ring and soft quasi- ideal over a near-ring. We also introduce the notions of soft left (resp. right) N-subgroup and soft invariant subnear- ring of a soft near-ring. We discuss the properties of these notions with illustrative examples. We prove that the restricted intersection of a soft quasi-ideal and a soft subnear-ring of a soft near- ring is a soft quasi-ideal of the soft subnear-ring.

Definition 3.1. Let $(\tilde{H}_1, P_1)$ and $(\tilde{H}_2, P_2)$ be SSs over U. The $\tilde{*}$*– product* of $(\tilde{H}_1, P_1)$ and $(\tilde{H}_2, P_2)$ is denoted by $(\tilde{H}_1, P_1) \tilde{*} (\tilde{H}_2, P_2)$, and is defined as the SS $(\tilde{H}, P)$, where $P = P_1 \cap P_2 \neq \emptyset$ and $\tilde{H}(\rho) = \tilde{H}_1(\rho) * \tilde{H}_2(\rho)$ for all $\rho \in P$ and $\tilde{H}_1(\rho) * \tilde{H}_2(\rho) = \{n_1(n_2 + n_3) - n_1 n_2/n_1, n_2 \in \tilde{H}_1(\rho), n_3 \in \tilde{H}_2(\rho)\}$.

Example 3.2. Consider a near-ring N = {0, 1, 2, 3} with the operations Table 1 and Table 2 (Scheme 20: (7,8,2,1) see [42], p.408).

Table 1. Addition table of near-ring N in Example 3.2

+	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Table 2. Multiplication table of near-ring N in Example 3.2

.	0	1	2	3
0	0	0	0	0
1	1	1	1	1
2	0	1	2	3
3	1	0	3	2

Let A = N and $(\tilde{H}, A)$ be a SS over N, where $\tilde{H}: A \to P(N)$ is a set-valued function defined by $\tilde{H}(\rho) = \{\sigma \in N / \rho R \sigma \Leftrightarrow \rho\sigma \in \{0, 1, 2\}\}$ for all $\rho \in A$. Then, $\tilde{H}(0) = \tilde{H}(1) = N$, $\tilde{H}(2) = \{0, 1, 2\}$ and $\tilde{H}(3) = \{0, 1, 3\}$. Let $(\tilde{G}, B)$ be the SS over N defined by $\tilde{G}(\rho) = \{0\} \cup \{\sigma \in N / \rho R \sigma \Leftrightarrow \rho + \sigma = 0\}$, where $B = \{1, 2, 3\}$. Then, $\tilde{G}(1) = \{0, 1\}$, $\tilde{G}(2) = \{0, 2\}$ and $\tilde{G}(3) = \{0, 3\}$. It is easy to verify that $(\tilde{H}, A) \tilde{\ast} (\tilde{G}, B)$ is a SS over N, where $A \cap B = \{1, 2, 3\} \neq \emptyset$ and $\tilde{H}(1) \ast \tilde{G}(1) = \{0, 1\}$, $\tilde{H}(2) \ast \tilde{G}(2) = \{0, 2\}$ and $\tilde{H}(3) \ast \tilde{G}(3) = \{0, 3\}$.

Definition 3.3. Let $(\tilde{H}, P)$ be a SN over N. A non-null SS $(\tilde{K}, Q)$ over N is called a *soft quasi-ideal* (briefly, SQI) of $(\tilde{H}, P)$, denoted by $(\tilde{K}, Q) \tilde{\lhd}_q (\tilde{H}, P)$, if $Q \subseteq P$ and $\tilde{K}(\rho) \lhd_q \tilde{H}(\rho)$ for all $\rho \in Supp(\tilde{K}, Q)$.

Theorem 3.4. Let $(\tilde{H}, P)$ be a SN over N. Then, a SSG $(\tilde{K}, Q)$ of $(\tilde{H}, P)$ is a SQI of $(\tilde{H}, P)$ if and only if $((\tilde{K}, Q) \tilde{\circ} (\tilde{H}, P)) \Cap_R ((\tilde{H}, P) \tilde{\circ} (\tilde{K}, Q)) \Cap_R ((\tilde{H}, P) \tilde{\ast} (\tilde{K}, Q)) \subseteq (\tilde{K}, Q)$.

Proof. $(\tilde{K}, Q)$ is a SQI of $(\tilde{H}, P)$

$\Leftrightarrow Q \subseteq P$ and each $\tilde{K}(\rho) \lhd_q \tilde{H}(\rho)$ for all $\rho \in Supp(\tilde{K}, Q)$, as $P \cap Q = Q \neq \emptyset$

$\Leftrightarrow Q \subseteq P$ and $\tilde{K}(\rho)\, \tilde{H}(\rho) \cap \tilde{H}(\rho)\tilde{K}(\rho) \cap \tilde{H}(\rho) \ast \tilde{K}(\rho) \subseteq \tilde{K}(\rho)$ for all $\rho \in Supp(\tilde{K}, Q)$

$\Leftrightarrow ((\tilde{K}, Q) \tilde{\circ} (\tilde{H}, P)) \Cap_R ((\tilde{H}, P) \tilde{\circ} (\tilde{K}, Q)) \Cap_R ((\tilde{H}, P) \tilde{\ast} (\tilde{K}, Q)) \subseteq (\tilde{K}, Q)$.

Definition 3.5. A non-null SS $(\tilde{K}, Q)$ over N is called a *soft quasi-ideal* over N, denoted by $(\tilde{K}, Q) \tilde{\lhd}_q N$, if $\tilde{K}(\rho) \lhd_q N$ for all $\rho \in Supp(\tilde{K}, Q)$.

Theorem 3.6. Let N be a near-ring. Then, a SSG $(\tilde{K}, Q)$ over N is a SQI over N if and only if $((\tilde{K}, Q) \tilde{\circ} \tilde{A}_N) \Cap_R (\tilde{A}_N \tilde{\circ} (\tilde{K}, Q)) \Cap_R (\tilde{A}_N \tilde{\ast} (\tilde{K}, Q)) \subseteq (\tilde{K}, Q)$.

Proof. $(\tilde{K}, Q)$ is a SQI over N

$\Leftrightarrow$ each $\tilde{K}(\rho) \lhd_q N$ for all $\rho \in Supp(\tilde{K}, Q)$, as $N \cap Q = Q \neq \emptyset$

$\Leftrightarrow \tilde{K}(\rho)N \cap N\tilde{K}(\rho) \cap N \ast \tilde{K}(\rho) \subseteq \tilde{K}(\rho)$ for all $\rho \in Supp(\tilde{K}, Q)$

$\Leftrightarrow ((\tilde{K}, Q) \tilde{\circ} \tilde{A}_N) \Cap_R (\tilde{A}_N \tilde{\circ} (\tilde{K}, Q)) \Cap_R (\tilde{A}_N \tilde{\ast} (\tilde{K}, Q)) \subseteq (\tilde{K}, Q)$.

Example 3.7. Consider a near-ring N over the Dihedral group $D_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$ with the operations Table 3 and Table 4 (Scheme 76:(15,1,35,5,15,1,35,5) see [42], p. 415).

Let P = N and $(\tilde{H}, P)$ be a SS over N, where $\tilde{H}: P \to P(N)$ is given by $\tilde{H}(\rho) = \{\sigma \in N / \rho R \sigma \Leftrightarrow \rho\sigma \in \{0, 2, 4, 6\}\}$ for all $\rho \in P$. Then, $\tilde{H}(0) = \tilde{H}(2) = \tilde{H}(4) = \tilde{H}(6) = N$ and $\tilde{H}(1) = \tilde{H}(3) = \tilde{H}(5) = \tilde{H}(7) = \{0, 2, 4, 6\}$ which are all subnear-rings of N. Hence, $(\tilde{H}, P)$ is a SN over N.

Table 3. Addition table of near-ring N in Example 3.7

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	0	5	6	7	4
2	2	3	0	1	6	7	4	5
3	3	0	1	2	7	4	5	6
4	4	7	6	5	0	3	2	1
5	5	4	7	6	1	0	3	2
6	6	5	4	7	2	1	0	3
7	7	6	5	4	3	2	1	0

Table 4. Multiplication table of near-ring N in Example 3.7

·	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	4	1	6	3	4	1	6	3
2	0	2	0	2	0	2	0	2
3	4	3	6	1	4	3	6	1
4	4	4	4	4	4	4	4	4
5	0	5	2	7	0	5	2	7
6	4	6	4	6	4	6	4	6
7	0	7	2	5	0	7	2	5

Let $(\tilde{K}, Q)$ be a SS defined by $\tilde{K}(\rho) = \{\sigma \in N / \rho R \sigma \Leftrightarrow \rho\sigma \in \{0, 4\}\}$, where $Q = \{1, 2, 3\}$. Then, $\tilde{K}(1) = \{0, 4\}$, $\tilde{K}(2) = \{0, 2, 4, 6\}$ and $\tilde{K}(3) = \{0, 4\}$ are QIs of $\tilde{H}(1)$, $\tilde{H}(2)$ and $\tilde{H}(3)$, respectively. So, $\tilde{K}(\rho) \lhd_q \tilde{H}(\rho)$ for all $\rho \in Supp(\tilde{K}, Q)$. Hence, $(\tilde{K}, Q) \tilde{\lhd}_q (\tilde{H}, P)$ over N.

Example 3.8. Consider a near-ring $N = M_2(Z)$, where $M_2(Z)$ is the set of 2×2 matrices with integer terms, with addition and multiplication operations of matrices.

Let P = Z and Q = 3Z. Let $(\tilde{H}, P)$ be SS over N, where $\tilde{H}: P \to P(N)$ is a set-valued function defined by $\tilde{H}(x) = \left\{\begin{bmatrix} nx & nx \\ 0 & mx \end{bmatrix} / n, m \in \mathbb{Z}\right\}$forallx∈P.Then,$(\tilde{H}, P)$isaSNover N. Let $(\tilde{K}, Q)$ be SS over N, given by $\tilde{K}(x) = \left\{\begin{bmatrix} nx & nx \\ 0 & 0 \end{bmatrix} / n \in \mathbb{Z}\right\}$. Since $n_1, n_2 \in Z$, we have $n_1 - n_2 \in Z$ and so $\begin{bmatrix} n_1x & n_1x \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} n_2x & n_2x \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} (n_1 - n_2)x & (n_1 - n_2)x \\ 0 & 0 \end{bmatrix} \in \tilde{K}(x)$. Therefore, $\tilde{K}(x)$ is a subgroup of $\tilde{H}(x)$ for all $x \in Supp(\tilde{K}, Q)$. Hence, $(\tilde{K}, Q)$ is a SSG of $(\tilde{H}, P)$. Now for each $x \in Supp(\tilde{K}, Q)$, let $\begin{bmatrix} nx & nx \\ 0 & mx \end{bmatrix}, \begin{bmatrix} n'x & n'x \\ 0 & m'x \end{bmatrix} \in \tilde{H}(x)$ and $\begin{bmatrix} nx & nx \\ 0 & 0 \end{bmatrix} \in \tilde{K}(x)$, where n, n', m, m' ∈ Z.

Since $\begin{bmatrix} nx & nx \\ 0 & 0 \end{bmatrix}\begin{bmatrix} nx & nx \\ 0 & mx \end{bmatrix} = \begin{bmatrix} n^2x^2 & n^2x^2 + nmx^2 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} nx & nx \\ 0 & mx \end{bmatrix}\begin{bmatrix} nx & nx \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} n^2x^2 & n^2x^2 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} nx & nx \\ 0 & mx \end{bmatrix}\left(\begin{bmatrix} n'x & n'x \\ 0 & m'x \end{bmatrix} + \begin{bmatrix} nx & nx \\ 0 & 0 \end{bmatrix}\right) - \begin{bmatrix} nx & nx \\ 0 & mx \end{bmatrix}\begin{bmatrix} n'x & n'x \\ 0 & m'x \end{bmatrix} = \begin{bmatrix} n^2x^2 & n^2x^2 \\ 0 & 0 \end{bmatrix}$, we have, $\tilde{K}(x)\tilde{H}(x) \cap \tilde{H}(x)\tilde{K}(x) \cap \tilde{H}(x) * \tilde{K}(x) = \emptyset \subseteq \tilde{K}(x)$, when $m \neq 0$, $m' \neq 0$ and $\tilde{K}(x)\tilde{H}(x) \cap \tilde{H}(x)\tilde{K}(x) \cap \tilde{H}(x) * \tilde{K}(x) \subseteq \tilde{K}(x)$, when $m = 0$, $m' = 0$ for all $x \in Supp(\tilde{K}, Q)$. Hence, $(\tilde{K}, Q) \tilde{\lhd}_q (\tilde{H}, P)$ over N.

Remark 3.9. The SQI of a SN as in Definition 3.3 is different from the SQI over a near-ring N as in Definition 3.5. The next example explain this situation.

Example 3.10. Consider a near-ring N over the Dihedral group $D_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$ with the operations Table 5 and Table 6 (Scheme 132: (15,35,15,35,15,1,15,1) see [42], p. 415).

Let $(\tilde{H}, P)$ be a SS over N, where P = N and $\tilde{H}$ is defined as $\tilde{H}(0) = \tilde{H}(2) = \tilde{H}(4) = N$, $\tilde{H}(1) = \tilde{H}(3) = \tilde{H}(6) = \{0, 2, 4, 6\}$ and $\tilde{H}(5) = \tilde{H}(7) = \{0, 2, 5, 7\}$. Then $(\tilde{H}, P)$ is a SN over N.

Table 5. Addition table of near-ring N in Example 3.10

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	0	5	6	7	4
2	2	3	0	1	6	7	4	5
3	3	0	1	2	7	4	5	6
4	4	7	6	5	0	3	2	1
5	5	4	7	6	1	0	3	2
6	6	5	4	7	2	1	0	3
7	7	6	5	4	3	2	1	0

Table 6. Multiplication table of near-ring N in Example 3.10

·	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	4	6	4	6	4	1	4	1
2	0	0	0	0	0	2	0	2
3	4	6	4	6	4	3	4	3
4	4	4	4	4	4	4	4	4
5	0	2	0	2	0	5	0	5
6	4	4	4	4	4	6	4	6
7	0	2	0	2	0	7	0	7

Let $(\tilde{K}, Q)$ be a SS defined by $\tilde{K}(\rho) = \{0\} \cup \{\sigma \in N/\rho R\sigma \Leftrightarrow \rho + \sigma = 0\}$, where $Q = \{5, 6, 7\}$. Then, $\tilde{K}(5) = \{0, 5\}$, $\tilde{K}(6) = \{0, 6\}$ and $\tilde{K}(7) = \{0, 7\}$ are QIs of $\tilde{H}(5)$, $\tilde{H}(6)$ and $\tilde{H}(7)$, respectively. Hence, $(\tilde{K}, Q) \tilde{\lhd}_q (\tilde{H}, P)$ over N. Whereas $\tilde{K}(5) = \{0, 5\}$ and $\tilde{K}(7) = \{0, 7\}$ are not QI of N, $(\tilde{K}, Q)$ is not a SQI of N.

Theorem 3.11. The restricted intersection of family of SQIs of a SN $(\tilde{H}, P)$ over N is a SQI of $(\tilde{H}, P)$ when it is non- null.

Proof. Let $(\tilde{K}_i, Q_i)_{i\in\Omega}$ be a family of SQIs of a SN $(\tilde{H}, P)$ over N an let $(\tilde{K}, Q) = (\tilde{\cap}_R)_{i\in\Omega} (\tilde{K}_i, Q_i)$, where $Q = \cap_{i\in\Omega} Q_i \neq \emptyset$ and $\tilde{K}(\rho) = \cap_{i\in\Omega} \tilde{K}_i(\rho)$ for $\rho \in Supp(\tilde{K}, Q)$. Suppose that $(\tilde{K}, Q)$ is a non-null SS. Since each $(\tilde{K}_i, Q_i)$ is a SSG of $(\tilde{H}, P)$, by Theorem 24 of [22], $(\tilde{K}, Q)$ is also a SSG of $(\tilde{H}, P)$. Since $\tilde{K}(\rho) = \cap_{i\in\Omega} \tilde{K}_i(\rho) \subseteq \tilde{K}_i(\rho)$ for all $i \in \Omega$ and $\rho \in Supp(\tilde{K}, Q)$, we have $\tilde{K}(\rho)\tilde{H}(\rho) \cap \tilde{H}(\rho)\tilde{K}(\rho) \cap \tilde{H}(\rho) * \tilde{K}(\rho) \subseteq \tilde{K}_i(\rho)\tilde{H}(\rho) \cap \tilde{H}(\rho)\tilde{K}_i(\rho) \cap \tilde{H}(\rho) * \tilde{K}_i(\rho) \subseteq \tilde{K}_i(\rho)$ for all $i \in \Omega$. It follows that $\tilde{K}(\rho)\tilde{H}(\rho) \cap \tilde{H}(\rho)\tilde{K}(\rho) \cap \tilde{H}(\rho) * \tilde{K}(\rho) \subseteq \cap_{i\in\Omega} \tilde{K}_i(\rho) = \tilde{K}(\rho)$ for all $\rho \in Supp(\tilde{K}, Q)$. Hence, $(\tilde{K}, Q) \tilde{\lhd}_q (\tilde{H}, P)$ over N.

Corollary 3.12. The restricted intersection of two SQIs of a SN $(\tilde{H}, P)$ over N is a SQI of $(\tilde{H}, P)$ when it is non-null.

Proof. The proof is straightforward from Theorem 3.11.

Theorem 3.13. Let $(\tilde{K}_1, Q)$ be a SQI and $(\tilde{K}_2, S)$ be a SSN of a SN $(\tilde{H}, P)$ over N. Then, $(\tilde{K}_1, Q) \tilde{\cap}_R (\tilde{K}_2, S) \tilde{\lhd}_q (\tilde{K}_2, S)$ when it is non-null.

Proof. Using Definition 2.5, we can write $(\tilde{K}_1, Q) \tilde{\cap}_R (\tilde{K}_2, S) = (\tilde{J}, C)$ where $C = Q \cap S \neq \emptyset$ and $\tilde{J}(\rho) = \tilde{K}_1(\rho) \cap \tilde{K}_2(\rho)$ for all $\rho \in Supp(\tilde{J}, C)$. Since $Q \subseteq P$ and $S \subseteq P$, $Q \cap S = C \subseteq P$. Suppose that $(\tilde{J}, C)$ is non-null. If $\rho \in Supp(\tilde{J}, C)$, then $\tilde{J}(\rho) = \tilde{K}_1(\rho) \cap \tilde{K}_2(\rho) \neq \emptyset$. Since $\tilde{K}_1(\rho)$ and $\tilde{K}_2(\rho)$ are subgroups of $\tilde{H}(\rho)$, $\tilde{K}_2(\rho)$ is also a subgroup of $\tilde{K}_2(\rho)$ for all $\rho \in Supp(\tilde{J}, C)$. For each $\rho \in Supp(\tilde{J}, C)$,

$\tilde{J}(\rho)\tilde{K}(\rho) \cap \tilde{K}_2(\rho)\tilde{J}(\rho) \cap \tilde{K}_2(\rho) * \tilde{J}(\rho)$
$= (\tilde{K}_1(\rho) \cap \tilde{K}_2(\rho))\, \tilde{K}_2(\rho) \cap \tilde{K}_2(\rho)\, (\tilde{K}_1(\rho) \cap \tilde{K}_2(\rho)) \cap \tilde{K}_2(\rho) * (\tilde{K}_1(\rho) \cap \tilde{K}_2(\rho))$
$\subseteq (\tilde{K}_1(\rho) \cap \tilde{K}_2(\rho))\, \tilde{K}_2(\rho) \cap \tilde{K}_2(\rho)\, (\tilde{K}_1(\rho) \cap \tilde{K}_2(\rho))$
$\subseteq \tilde{K}_2(\rho)\tilde{K}_2(\rho) \cap \tilde{K}_2(\rho)\tilde{K}_2(\rho)$, as $\tilde{K}_1(\rho) \cap \tilde{K}_2(\rho) \subseteq \tilde{K}_2(\rho)$
$\subseteq \tilde{K}_2(\rho)\tilde{K}_2(\rho) \subseteq \tilde{K}_2(\rho)$
and
$\tilde{J}(\rho)\tilde{K}_2(\rho) \cap \tilde{K}_2(\rho)\tilde{J}(\rho) \cap \tilde{K}_2(\rho) * \tilde{J}(\rho)$
$= (\tilde{K}_1(\rho) \cap \tilde{K}_2(\rho))\, \tilde{K}_2(\rho) \cap \tilde{K}_2(\rho)\, (\tilde{K}_1(\rho) \cap \tilde{K}_2(\rho)) \cap \tilde{K}_2(\rho) * (\tilde{K}_1(\rho) \cap \tilde{K}_2(\rho))$
$\subseteq \tilde{K}_1(\rho)\tilde{H}(\rho) \cap \tilde{H}(\rho)\tilde{K}_1(\rho) \cap \tilde{H}(\rho) * \tilde{K}_1(\rho)$
$\subseteq \tilde{K}_1(\rho)$, as $\tilde{K}_2(\rho) \subseteq \tilde{H}(\rho)$, $\tilde{K}_1(\rho) \cap \tilde{K}_2(\rho) \subseteq \tilde{K}_1(\rho)$ and $\tilde{K}_1(\rho) \lhd_q \tilde{H}(\rho)$.

Therefore,

$\tilde{J}(\rho)\tilde{K}_2(\rho) \cap \tilde{K}_2(\rho)\tilde{J}(\rho) \cap \tilde{K}_2(\rho) * \tilde{J}(\rho) \subseteq \tilde{K}_1(\rho) \cap \tilde{K}_2(\rho) = \tilde{J}(\rho)$. Consequently, $\tilde{J}(\rho) = \tilde{K}_1(\rho) \cap \tilde{K}_2(\rho) \lhd_q \tilde{K}_2(\rho)$ for all $\rho \in Supp(\tilde{J}, C)$. Hence, $(\tilde{J}, C) \tilde{\lhd}_q (\tilde{K}_2, S)$.

Remark 3.14. Let $(\tilde{K}_1, Q)$ be a SQI and $(\tilde{K}_2, S)$ be a SSN of a SN $(\tilde{H}, P)$ over N. By Definition 2.5, we can write $(\tilde{K}_1, Q) \tilde{\cap}_E (\tilde{K}_2, S) = (\tilde{K}, A)$, where $A = Q \cup S$. Suppose that $\rho \in Supp(\tilde{K}, A)$. If $\rho \in Q\backslash S$, then $\tilde{K}_1(\rho) = \tilde{K}(\rho)$ is a quasi-ideal of $\tilde{H}(\rho)$ but need not be a quasi-ideal of $\tilde{K}_2(\rho)$. Hence, $(\tilde{K}_1, Q) \tilde{\cap}_E (\tilde{K}_2, S)$ need not be a SQI of $(\tilde{K}_2, S)$.

Theorem 3.15. Let $(\tilde{H}, P)$ be a SN over N. Then, $m' \tilde{\circ} (\tilde{H}, P)$ and $(\tilde{H}, P) \tilde{\circ}\, m$ are SQIs of $(\tilde{H}, P)$, where $m, m' \in \tilde{H}(\rho)$ and m' is a distributive element in $\tilde{H}(\rho)$ for all $\rho \in supp(\tilde{H}, P)$.

Proof. First we prove that $m' \tilde{\circ} (\tilde{H}, P) \tilde{\lhd}_q (\tilde{H}, P)$. Since m' is a distributive element in $\tilde{H}(\rho)$, $m'\tilde{H}(\rho)$ is a subgroup of $\tilde{H}(\rho)$ for all $\rho \in Supp(\tilde{H}, P)$. Then, $m'\tilde{H}(\rho)\tilde{H}(\rho) \cap \tilde{H}(\rho) m'\tilde{H}(\rho) \cap \tilde{H}(\rho) * m'\tilde{H}(\rho) \subseteq m'\tilde{H}(\rho) \cap \tilde{H}(\rho)m'\tilde{H}(\rho) \cap \tilde{H}(\rho) * m'\tilde{H}(\rho) \subseteq m'\tilde{H}(\rho)$ for all $\rho \in Supp(\tilde{H}, P)$. Hence, $m' \tilde{\circ} (\tilde{H},$

P) $\tilde{\lhd}_q$ (H̃, P). In a similar manner, we can prove (H̃, P) $\tilde{\circ}$ m $\tilde{\lhd}_q$ (H̃, P).

Theorem 3.16. The restricted intersection of SLI and SRI of a SN (H̃, P) over N is a SQI of (H̃, P) when it is non-null.

Proof. Let $(\tilde{K}_1, I_1)$ and $(\tilde{K}_2, I_2)$ be the SLI and SRI of the SN (H̃, P) over N, respectively. Then, $\tilde{H}(\rho) * \tilde{K}_1(\rho) \subseteq \tilde{K}_1(\rho)$ for all $\rho \in Supp(\tilde{K}_1, I_1)$ and $\tilde{K}_2(\rho)\tilde{H}(\rho) \subseteq \tilde{K}_2(\rho)$ for all $\rho \in Supp(\tilde{K}_2, I_2)$. Let $(\tilde{K}, Q) = (\tilde{K}_1, I_1) \tilde{\cap}_R (\tilde{K}_2, I_2)$, where $Q = I_1 \cap I_2 \neq \emptyset$ and $\tilde{K}(\rho) = \tilde{K}_1(\rho) \cap \tilde{K}_2(\rho)$ for all $\rho \in Supp(\tilde{K}, Q)$. Since $I_1 \subseteq P$ and $I_2 \subseteq P$, $Q = I_1 \cap I_2 \subseteq P$, and since $\tilde{K}_1(\rho)$ and $\tilde{K}_2(\rho)$ are subgroups of $\tilde{H}(\rho)$, $\tilde{K}(\rho)$ is also a subgroup of $\tilde{H}(\rho)$ for all $\rho \in Supp(\tilde{K}, Q)$. Then for all $\rho \in Supp(\tilde{K}, Q)$,

$\tilde{K}(\rho)\tilde{H}(\rho) \cap \tilde{H}(\rho)\tilde{K}(\rho) \cap \tilde{H}(\rho) * \tilde{K}(\rho)$

$= (\tilde{K}_1(\rho) \cap \tilde{K}_2(\rho))\tilde{H}(\rho) \cap \tilde{H}(\rho)(\tilde{K}_1(\rho) \cap \tilde{K}_2(\rho)) \cap \tilde{H}(\rho) * (\tilde{K}_1(\rho) \cap \tilde{K}_2(\rho))$

$\subseteq \tilde{K}_1(\rho)\tilde{H}(\rho) \cap \tilde{H}(\rho)\tilde{K}_1(\rho) \cap \tilde{H}(\rho) * \tilde{K}_1(\rho)$, as $\tilde{K}_1(\rho) \cap \tilde{K}_2(\rho) \subseteq \tilde{K}_1(\rho)$

$\subseteq \tilde{K}_1(\rho)\tilde{H}(\rho) \cap \tilde{H}(\rho)\tilde{K}_1(\rho) \cap \tilde{K}_1(\rho)$, as $\tilde{H}(\rho) * \tilde{K}_1(\rho) \subseteq \tilde{K}_1(\rho)$

$\subseteq \tilde{K}_1(\rho)$.

Similarly, we can prove $\tilde{K}(\rho)\tilde{H}(\rho) \cap \tilde{H}(\rho)\tilde{K}(\rho) \cap \tilde{H}(\rho) * \tilde{K}(\rho) \subseteq \tilde{K}_2(\rho)$. Thus, $\tilde{K}(\rho)\tilde{H}(\rho) \cap \tilde{H}(\rho)\tilde{K}(\rho) \cap \tilde{H}(\rho) * \tilde{K}(\rho) \subseteq \tilde{K}_1(\rho) \cap \tilde{K}_2(\rho) = \tilde{K}(\rho)$ for all $\rho \in Supp(\tilde{K}, Q)$. Hence, (K̃, Q) $\tilde{\lhd}_q$ (H̃, P) over N.

Theorem 3.17. Every SLI (resp. SRI, SI) of a SN (H̃, P) over N is a SQI of (H̃, P).

Proof. The proof follows from the proof of Theorem 3.16.

Definition 3.18. Let (H̃, P) be a SN over N. A non-null SS (K̃, B) over N is called a *soft left N-subgroup* (resp. *soft right N-subgroup*) of (H̃, P) if B ⊆ P and K̃(ρ) is a left N-subgroup (resp. right N-subgroup) of H̃(ρ) for all ρ ∈ Supp(K̃, B).

Theorem 3.19. Every soft left N-subgroup (resp. right N-subgroup) of a SN (H̃, P) over N is a SQI of (H̃, P).

Proof. Let (K̃, Q) be a soft left N-subgroup of (H̃, P) over N. Then, $\tilde{H}(\rho)\tilde{K}(\rho) \subseteq \tilde{K}(\rho)$ for all $\rho \in Supp(\tilde{K}, Q)$. Since $\tilde{K}(\rho)\tilde{H}(\rho) \cap \tilde{H}(\rho)\tilde{K}(\rho) \cap \tilde{H}(\rho) * \tilde{K}(\rho) \subseteq \tilde{K}(\rho)\tilde{H}(\rho) \cap \tilde{K}(\rho) \cap \tilde{H}(\rho) * \tilde{K}(\rho) \subseteq \tilde{K}(\rho)$, $K(\rho) \lhd_q \tilde{H}(\rho)$ for all $\rho \in Supp(\tilde{K}, Q)$. Hence, (K̃, Q) $\tilde{\lhd}_q$ (H̃, P). Similarly, we can prove every soft right N-subgroup of (H̃, P) is also a SQI of (H̃, P).

Theorem 3.20. Let (K̃, Q) be a soft left N-subgroup of a SN (H̃, P) over N. Then e $\tilde{\circ}$ (K̃, Q) $\tilde{\lhd}_q$ (H̃, P), where e is a distributive idempotent element of H̃(ρ) for all ρ ∈ Supp(K̃, Q).

Proof. Since each K̃(ρ) is a left N-subgroup of H̃(ρ), we have $\tilde{H}(\rho)\tilde{K}(\rho) \subseteq \tilde{K}(\rho)$ for all $\rho \in Supp(\tilde{K}, Q)$. Now, since e $\tilde{K}(\rho) \subseteq \tilde{H}(\rho)\tilde{K}(\rho) \subseteq \tilde{K}(\rho)$ and $e\tilde{K}(\rho) \subseteq e\tilde{H}(\rho)$, we have $e\tilde{K}(\rho) \subseteq \tilde{K}(\rho) \cap e\tilde{H}(\rho)$ for all $\rho \in Supp(\tilde{K}, Q)$. Let u be any element of $\tilde{K}(\rho) \cap e\tilde{H}(\rho)$, then we write u = v = en, where $v \in \tilde{K}(\rho)$ and $n \in \tilde{H}(\rho)$ for all $\rho \in Supp(\tilde{K}, Q)$. From where $u = en = een = ev \in e\tilde{K}(\rho)$, $\tilde{K}(\rho) \cap e\tilde{H}(\rho) \subseteq e\tilde{K}(\rho)$ for all $\rho \in Supp(\tilde{K}, Q)$. Therefore, e $\tilde{\circ}$ (K̃, Q) = (K̃, Q) $\tilde{\cap}_R$ (e $\tilde{\circ}$ (H̃, P)). By Theorem 3.15 and Corollary 3.12, e $\tilde{\circ}$ (K̃, Q) $\tilde{\lhd}_q$ (H̃, P).

Definition 3.21. Let (K̃, B) be a SSN of a SN (H̃, P) over N. Then (K̃, B) is called a *soft invariant subnear-ring* of (H̃, P) if K̃(ρ) is a invariant subnear-ring of H̃(ρ) for all ρ ∈ Supp(K̃, B).

Theorem 3.22. Every soft invariant subnear-ring of a SN (H̃, P) over N is a SQI of (H̃, P)over N.

Proof. The proof is similar to that of Theorem 3.20.

CHARACTERIZATIONS OF SOFT QUASI-IDEALS OF SOFT NEAR-RINGS

In this section, we study the concepts of soft quasi-ideals on zero-symmetric near-ring and constant near- ring. We introduce the notions of soft zero-symmetric near-ring and soft constant near-ring over a near-ring. We obtain the characterization of soft quasi-ideal of a soft (zero-symmetric) near-ring and discuss some of its properties. We provide the condition for a soft subgroup of a soft near-ring to be a soft quasi-ideal.

Definition 4.1. Let (H̃, P) be a SN over N. Then, a non-null SS $(\tilde{H}, P)_0$ over N is called a *soft zero-symmetric part* of (H̃, P) if $\tilde{H}(\rho)_0$ is a zero-symmetric part of H̃(ρ) for all ρ ∈ Supp(H̃, P) , where $\tilde{H}(\rho)_0 = \{n \in \tilde{H}(\rho)/n0 = 0\}$. The SN (H̃, P) is called a *soft zero-symmetric near-ring* (SZN, for short) over N if $\tilde{H}(\rho) = \tilde{H}(\rho)_0$, that is, H̃(ρ) is a zero- symmetric subnear-ring of N for all ρ ∈ Supp(H̃, P).

Note that a soft zero-symmetric near-ring over a near-ring is same as a soft near-ring over a zero-symmetric near-ring.

Definition 4.2. Let (H̃, P) be a SN over N. Then, a non-null SS $(\tilde{H}, P)_c$ over N is called a *soft constant part* of (H̃, P) if $\tilde{H}(\rho)_c$ is a constant part of H̃(ρ) for all ρ ∈ Supp(H̃, P), where $\tilde{H}(\rho)_c = \{n \in \tilde{H}(\rho)/n0 = n\}$. The SN (H̃, P) is called a *soft constant near-ring* (SZN, for short) over N if $\tilde{H}(\rho) = \tilde{H}(\rho)_c$, that is, H̃(ρ) is a constant subnear-ring of N for all ρ ∈ Supp(H̃, P).

Example 4.3. Consider the SN (H̃, P) over N as defined in Example 3.7.

Let P = N and (H̃, P) be a SS over N defined by $\tilde{H}(0)_0 = \tilde{H}(2)_0 = \tilde{H}(4)_0 = \tilde{H}(6)_0 = \{0, 2, 5, 7\}$ and $\tilde{H}(1)_0 = \tilde{H}(3)_0 = \tilde{H}(5)_0 = \tilde{H}(7)_0 = \{0, 2\}$. Then, $\tilde{H}(\rho)_0$ is a zero-symmetric part of H̃(ρ) for all ρ ∈ P. Hence, $(\tilde{H}, P)_0$ is a soft zero-symmetric part of (H̃, P).

Let P = N and (H̃, P) be a SS over N defined by $\tilde{H}(\rho)_c = \{0, 4\}$ for all ρ ∈ P. Then, $\tilde{H}(\rho)_c$ is a constant part of H̃(ρ) for all ρ ∈ P. Therefore, $(\tilde{H}, P)_c$ is a soft constant part of (H̃, P).

Suppose we define H̃(ρ) = {0, 2, 5, 7} for all ρ ∈ Supp(H̃, P), then (H̃, P) is a SZN over N. If we define H̃(ρ) = {0, 4} for all ρ ∈ Supp(H̃, P), then (H̃, P) is a SCN over N.

Remark 4.4. In Definition 4.1 and Definition 4.2, since the zero-symmetric part and the constant part of a near-ring N are subnear-rings of N, $\tilde{H}(\rho)_0$ and $\tilde{H}(\rho)_c$ are subnear-rings of H̃(ρ) for all ρ ∈ Supp(H̃, P). Therefore, $(\tilde{H}, P)_0$ and $(\tilde{H}, P)_c$ are SSNs of (H̃, P) over N. Hence, for a given SN (H̃, P) over N we can obtain at least two SSNs $(\tilde{H}, P)_0$ and $(\tilde{H}, P)_c$ of (H̃, P).

Remark 4.5. Let (H̃, P) be a SZN over N and (K̃, B) be a non-null SS of (H̃, P) over N. Since, for any $a \in \tilde{H}(\rho)$ and

$b \in \tilde{K}(\rho)$, $ab = a(0 + b) - a0$, we have $\tilde{H}(\rho)\tilde{K}(\rho) \subseteq \tilde{H}(\rho) * \tilde{K}(\rho)$ for all $\rho \in B$. That is, $(\tilde{H}, P)\ \tilde{\circ}\ (\tilde{K}, B) \subseteq (\tilde{H}, P)\ \tilde{*}\ (\tilde{K}, B)$.

Theorem 4.6. The soft zero-symmetric part $(\tilde{H}, P)_0$ of a SN $(\tilde{H}, P)$ over N is a SQI of $(\tilde{H}, P)$.

Proof. By Remark 4.4, $(\tilde{H}, P)_0$ is a SSN of $(\tilde{H}, P)$ and so $\tilde{H}(\rho)$ is a subgroup of $\tilde{H}(\rho)$ for all $\rho \in \mathrm{Supp}(\tilde{H}, P)$. For all $m, m' \in \tilde{H}(\rho)$ and $m_0 \in \tilde{H}(\rho)_0$, we have $(m(m' + m_0) - mm')0 = m(m' + m_0)0 - mm'0 = m(m'0 + m_00) - mm'0 = 0$.Therefore, $(m(m' + m_0) - mm') \in \tilde{H}(\rho)_0$. Consequently, $\tilde{H}(\rho) * \tilde{H}(\rho)_0 \subseteq \tilde{H}(\rho)_0$ for all $\rho \in \mathrm{Supp}(\tilde{H}, P)$. Now $\tilde{H}(\rho)_0\tilde{H}(\rho) \cap \tilde{H}(\rho)\tilde{H}(\rho)_0 \cap \tilde{H}(\rho) * \tilde{H}(\rho)_0 \subseteq \tilde{H}(\rho)_0$. Thus, $\tilde{H}(\rho)_0 \lhd_q H(\rho)$ for all $\rho \in \mathrm{Supp}(\tilde{H}, P)_0$. Hence, $(\tilde{H}, P)_0\ \tilde{\lhd}\ (\tilde{H}, P)$.

Theorem 4.7. The soft constant part $(\tilde{H}, P)_c$ of a SN $(\tilde{H}, P)$ over N is a SQI of $(\tilde{H}, P)$.

Proof. By Remark 4.4, $(\tilde{H}, P)_c$ is a SSN of (H, P). So $\tilde{H}(\rho)_c$ is a SSG of $\tilde{H}(\rho)\ \forall\ \rho \in \mathrm{Supp}(\tilde{H}, P)_c$. For all $m \in \tilde{H}(\rho)$ and $m_c \in \tilde{H}(\rho)_c$, we have $(mm_c)0 = m(m_c0) = mm_c$. Thus $mm_c \in \tilde{H}(\rho)_c$ and so $\tilde{H}(\rho)\tilde{H}(\rho)_c \subseteq \tilde{H}(\rho)_c$ for all $\rho \in (\tilde{H}, P)_c$. Now, $\tilde{H}(\rho)_c\tilde{H}(\rho) \cap \tilde{H}(\rho)\tilde{H}(\rho)_c \cap \tilde{H}(\rho) * \tilde{H}(\rho)_c \subseteq \tilde{H}(\rho)_c$ for all $\rho \in \mathrm{Supp}(\tilde{H}, P)_c$. This implies that $\tilde{H}(\rho)_c \lhd_q \tilde{H}(\rho)$ for all $\rho \in \mathrm{Supp}(\tilde{H}, P)_c$. Hence, $(\tilde{H}, P)_c\ \tilde{\lhd}_q\ (\tilde{H}, P)$.

Theorem 4.8. A SSG $(\tilde{K}, Q)$ of a SZN $(\tilde{H}, P)$ over N is a SQI of $(\tilde{H}, P)$ if and only if $((\tilde{K}, P)\ \tilde{\circ}\ (\tilde{H}, Q))\ \tilde{\cap}_R\ ((\tilde{H}, P)\ \tilde{\circ}\ (\tilde{K}, Q)) \subseteq (\tilde{K}, Q)$.

Proof. Assume that $(\tilde{K}, Q)\ \tilde{\lhd}_q\ (\tilde{H}, P)$. Since $(\tilde{H}, P)$ is a SZN over N, by Remark 4.5, $\tilde{H}(\rho)\tilde{K}(\rho) \subseteq \tilde{H}(\rho) * \tilde{K}(\rho)$ for all $\rho \in \mathrm{Supp}(\tilde{K}, Q)$. Then, by Theorem 3.4, we have $\tilde{K}(\rho)\tilde{H}(\rho) \cap \tilde{H}(\rho)\tilde{K}(\rho) = \tilde{K}(\rho)\tilde{H}(\rho) \cap \tilde{H}(\rho)\tilde{K}(\rho) \cap \tilde{H}(\rho)\tilde{K}(\rho) \subseteq \tilde{K}(\rho)\tilde{H}(\rho) \cap \tilde{H}(\rho)\tilde{K}(\rho) \cap \tilde{H}(\rho) * \tilde{K}(\rho) \subseteq \tilde{K}(\rho)$. Thus, $\tilde{K}(\rho)\tilde{H}(\rho) \cap \tilde{H}(\rho)\tilde{K}(\rho) \subseteq \tilde{K}(\rho)$ for all $\rho \in \mathrm{Supp}(\tilde{K}, Q)$. Hence, $((\tilde{K}, P)\ \tilde{\circ}\ (\tilde{H}, Q))\ \tilde{\cap}_R\ ((\tilde{H}, P)\ \tilde{\circ}\ (\tilde{K}, Q)) \subseteq (\tilde{K}, Q)$.

Conversely, assume that $((\tilde{K}, P)\ \tilde{\circ}\ (\tilde{H}, Q))\ \tilde{\cap}_R\ ((\tilde{H}, P)\ \tilde{\circ}\ (\tilde{K}, Q)) \subseteq (\tilde{K}, Q)$. That is, $\tilde{K}(\rho)\tilde{H}(\rho) \cap \tilde{H}(\rho)\tilde{K}(\rho) \subseteq \tilde{K}(\rho)$ for all $\rho \in \mathrm{Supp}(\tilde{K}, Q)$. Then, $\tilde{K}(\rho)\tilde{H}(\rho) \cap \tilde{H}(\rho)\tilde{K}(\rho) \cap \tilde{H}(\rho) * \tilde{K}(\rho) \subseteq \tilde{K}(\rho) \cap \tilde{H}(\rho) * \tilde{K}(\rho) \subseteq \tilde{K}(\rho)$ for all $\rho \in \mathrm{Supp}(\tilde{K}, Q)$. consequently, $(\tilde{K}, Q)\ \tilde{\lhd}_q\ (\tilde{H}, P)$.

Theorem 4.9. Let N be a zero-symmetric near-ring. Then, a SSG $(\tilde{K}, Q)$ over N is a SQI over N if and only if $(\tilde{K}, Q)\ \tilde{\circ}\ \tilde{A}_N\ \tilde{\cap}_R\ \tilde{A}_N\ \tilde{\circ}\ (\tilde{K}, Q) \subseteq (\tilde{K}, Q)$.

Proof. The proof is similar to that of Theorem 4.8.

Theorem 4.10. Let $(\tilde{K}, Q)$ be a SQI of a SN $(\tilde{H}, P)$ over N. Then,

(i) $(\tilde{K}, Q) = (\tilde{K}, Q)\ \tilde{\cap}_R\ (\tilde{H}, P) \lhd_q (\tilde{H}, P)$ over N when it is non-null.

(ii) $(\tilde{K}, Q)_c = (\tilde{K}, Q)\ \tilde{\cap}_R\ (\tilde{H}, P)\ \tilde{\lhd}_q\ (\tilde{H}, P)$ over N when it is non-null.

Proof. The proof is straightforward from Corollary 3.12.

Remark 4.11. In general, every SQI of a SN over N is not a SSN of the SN as given in the following example.

Example 4.12. Consider the SN $(\tilde{H}, P)$ over N as defined in Example 3.7.

Suppose that $(\tilde{K}, Q)$ is a SS over N defined by $\tilde{K}(1) = \{0, 6\}$ where $Q=\{1\}$. Then $\tilde{K}(1) \lhd_q \tilde{H}(1)$. Hence, $(\tilde{K}, Q)\ \tilde{\lhd}_q\ (\tilde{H}, P)$. Since $\tilde{K}(1)\tilde{K}(1) = \{0, 6\}\{0, 6\} = \{0, 4\} \not\subseteq \{0, 6\} = \tilde{K}(1)$, $(\tilde{K}, Q)$ is not a SSN of $(\tilde{H}, P)$. To prove that a SQI of a SN is a SSN we need some additional conditions and so we have the following theorem.

Theorem 4.13. Let $(\tilde{K}, Q)$ be a SQI of a SN $(\tilde{H}, P)$ over N and 0 is the zero element of N. Then, the following conditions are equivalent:

(i) $(\tilde{K}, Q)$ is a SSN of $(\tilde{H}, P)$,

(ii) $(\tilde{K}, Q)\ \tilde{\circ}\ 0 \subseteq (\tilde{K}, Q)$,

(iii) $(\tilde{K}, Q) = (\tilde{K}, Q)_0 + (\tilde{K}, Q)_c$.

Proof. $(i) \Rightarrow (ii)$ Since $\tilde{K}(\rho)\tilde{K}(\rho) \subseteq \tilde{K}(\rho)$ and $0 \in N$, the zero element of $\tilde{K}(\rho)$, we have $\tilde{K}(\rho)0 \subseteq \tilde{K}(\rho)\tilde{K}(\rho) \subseteq \tilde{K}(\rho)$ and so $\tilde{K}(\rho)0 \subseteq \tilde{K}(\rho)$ for all $\rho \in Supp(\tilde{K}, Q)$. Hence, $(\tilde{K}, Q)\ \tilde{\circ}0 \subseteq (\tilde{K}, Q)$.

$(ii) \Rightarrow (iii)$ Since $\tilde{K}(\rho) \subseteq \tilde{H}(\rho)$, by Theorem 1.13 of [42] for any element $y \in \tilde{K}(\rho)$, we have $y = (y - y0) + y0$, where $y - y0 \in \tilde{H}(\rho)_0$ and $y0 \in \tilde{H}(\rho)_c$ for all $\rho \in \mathrm{Supp}(\tilde{K}, Q)$. By assumption $y0 \in \tilde{K}(\rho)0 \subseteq \tilde{K}(\rho)$ and so $y - y0 \in \tilde{K}(\rho)$ for all $\rho \in \mathrm{Supp}(\tilde{K}, Q)$. Since $y - y0 \in \tilde{K}(\rho)$ and $(y - y0)0 = 0$, we have $y - y0 \in \tilde{K}(\rho)_0$. Similarly, since $y0 \in \tilde{K}(\rho)$ and $(y0)0 = y0$, we have $y0 \in \tilde{K}(\rho)_c$ for all $\rho \in \mathrm{Supp}(\tilde{K}, Q)$. This implies that $y = (y - y0) + y0 \in \tilde{K}(\rho)_0 + \tilde{K}(\rho)_c$. So $\tilde{K}(\rho) \subseteq \tilde{K}(\rho)_0 + \tilde{K}(\rho)_c \subseteq \tilde{K}(\rho)$ for all $\rho \in \mathrm{Supp}(\tilde{K}, Q)$. This shows that $\tilde{K}(\rho) = \tilde{K}(\rho)_0 + \tilde{K}(\rho)_c$ for all $\rho \in \mathrm{Supp}(\tilde{K}, Q)$. Hence, $(\tilde{K}, Q) = (\tilde{K}, Q)_0 + (\tilde{K}, Q)_c$.

(iii) $\Rightarrow$ (i) For any element $y \in \tilde{K}(\rho)$, we can write $y = y_0 + y_c$ with $y_0 \in \tilde{K}(\rho)_0$ and $y_c \in \tilde{K}(\rho)_c$. Then for any elements $y, y' \in \tilde{K}(\rho)$, we have $yy' = (y_0 + y_c)y' = y_0y' + y_cy' = y_0y' + y_c$. Moreover, we have $y_0y' = y_0(0 + y') - y_00 \in \tilde{H}(\rho) * \tilde{K}(\rho)$ for all $\rho \in \mathrm{Supp}(\tilde{K}, Q)$. From where $y_0y' \in \tilde{K}(\rho)\tilde{H}(\rho) \cap \tilde{H}(\rho)\tilde{K}(\rho) \cap \tilde{H}(\rho) * \tilde{K}(\rho) \subseteq \tilde{K}(\rho)$ implies that $yy' = y_0y' + y_c \in \tilde{K}(\rho)$. So $\tilde{K}(\rho)\tilde{K}(\rho) \subseteq \tilde{K}(\rho)$ for all $\rho \in \mathrm{Supp}(\tilde{K}, Q)$. Therefore, $(\tilde{K}, Q)\ \tilde{\circ}\ (\tilde{K}, Q) \subseteq (\tilde{K}, Q)$. Hence, $(\tilde{K}, Q)$ is a SSN of $(\tilde{H}, P)$ over N.

Theorem 4.14. Let $(\tilde{H}, P)$ be either a SZN or a SCN over N. Then, each SQI of $(\tilde{H}, P)$ is a SSN of $(\tilde{H}, P)$.

Proof. Let $(\tilde{K}, Q)$ be a SQI of $(\tilde{H}, P)$. Suppose $(\tilde{H}, P)$ is a SZN. Then, $\tilde{K}(\rho)0 = \{0\} \subseteq \tilde{K}(\rho)$ for all $\rho \in Supp(\tilde{K}, Q)$ and so $(\tilde{K}, Q)\ \tilde{\circ}\ 0 \subseteq (\tilde{K}, Q)$ Suppose $(\tilde{H}, P)$ is a SCN. Then, $\tilde{K}(\rho)0 = \tilde{K}(\rho)$ for all $\rho \in Supp(\tilde{K}, Q)$ and so $(\tilde{K}, Q)\ \tilde{\circ}\ 0 = (\tilde{K}, Q)$. Hence, by Theorem 4.13, $(\tilde{K}, Q)$ is a SSN of $(\tilde{H}, P)$ over N.

Theorem 4.15. Let N be either a zero-symmetric or a constant near-ring. Then, each SQI over N is a SSN over N.

Proof. The proof is similar to that of Theorem 4.14.

Remark 4.16. The converse of Theorem 4.15 is not true in general as given in the following example.

Example 4.17. Consider a zero-symmetric near-ring $N = \{0, 1, 2, 3\}$ with the addition operation Table 7 and the multiplication operation Table 8 (Scheme 4: (0,14,2,1) see [42], p.408).

Table 7. Addition table of near-ring N in Example 4.17

+	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Table 8. Multiplication table of near-ring N in Example 4.17

.	0	1	2	3
0	0	0	0	0
1	0	0	1	1
2	0	1	3	2
3	0	1	2	3

Let $Q = \{1, 3\}$ and $(\tilde{K}, Q)$ be the SS over N defined by $\tilde{K}(\rho) = \{\sigma \in N/\rho R\sigma \Leftrightarrow \rho\sigma = \{0, 3\}\}$.

Then, $\tilde{K}(1) = \{0, 1\}$ and $\tilde{K}(3) = \{0, 3\}$ are subnear-rings of N. Therefore, $(\tilde{K}, Q)$ is a SSN of N. Since $\tilde{K}(3)N \cap N\tilde{K}(3) = N \not\subseteq \{0, 3\} = \tilde{K}(3)$, $(\tilde{K}, Q)$ is not a SQI over N. Hence, a SSN over N need not be a SQI over N.

CHARACTERIZATIONS OF SOFT NEAR-FIELDS

In this section, we define the notions of soft near-field over a near-ring, soft Q-simple near-ring over a near- ring and soft minimal quasi-ideal of a soft near-ring. We discuss the characterization of soft near-rings which are soft near-fields. Throughout this section, we consider the near-fields which are zero-symmetric near-rings.

Definition 5.1. A non-null SS $(\tilde{H}, P)$ over N is called a *soft near-field* (briefly, SNF) over N if $\tilde{H}(\rho)$ is a subnear-field of N for all $\rho \in Supp(\tilde{H}, P)$.

Definition 5.2. Let $(\tilde{H}_1, P_1)$ and $(\tilde{H}_2, P_2)$ be two SNFs over N. Then $(\tilde{H}_2, P_2)$ is called a *soft subnear-field* of $(\tilde{H}_1, P_1)$ if it satisfies:

(i) $P_2 \subseteq P_1$ and

(ii) $\tilde{H}_2(\rho)$ is a subnear-field of $\tilde{H}_1(\rho)$ for all $\rho \in Supp(\tilde{H}_2, P_2)$.

Definition 5.3. A SN $(\tilde{H}, P)$ is called a *soft Q-simple* over N if $\tilde{H}(\rho)$ is Q-simple for all $\rho \in Supp(\tilde{H}, P)$.

Theorem 5.4. If a SN $(\tilde{H}, P)$ over N is soft Q-simple, then $(\tilde{H}, P)$ is either a SZN or a SCN over N.

Proof. Let $(\tilde{H}, P)_0$ be a soft zero-symmetric part of $(\tilde{H}, P)$. By Theorem 4.6, $(\tilde{H}, P)_0 \tilde{\triangleleft}_q (\tilde{H}, P)$ and so $\tilde{H}(\rho)_0 \triangleleft_q \tilde{H}(\rho)$ and since $(\tilde{H}, P)$ is soft Q-simple, we have either $\tilde{H}(\rho) = \tilde{H}(\rho)_0$ or $\tilde{H}(\rho) = \{0\}$ for all $\rho \in Supp(\tilde{H}, P)$. This implies that $\tilde{H}(\rho)$ is either a zero-symmetric near-ring or a constant near-ring for all $\rho \in Supp(\tilde{H}, P)$. Hence, $(\tilde{H}, P)$ is either a SZN or a SCN over N .

Theorem 5.5. Let $(\tilde{H}, P)$ be a SN over N such that each $\tilde{H}(\rho)$ has more than one element for all $\rho \in Supp(\tilde{H}, P)$. Then the following conditions are equivalent:

(i) $(\tilde{H}, P)$ is a SNF over N,

(ii) $(\tilde{H}, P)$ is soft Q-simple and each $\tilde{H}(\rho)$ has a left identity for all $\rho \in Supp(\tilde{H}, P)$,

(iii) $(\tilde{H}, P)$ is soft Q-simple, $\tilde{H}(\rho)_d \neq \{0\}$ and for each non-zero element y of $\tilde{H}(\rho)$ there exists an element y' of $\tilde{H}(\rho)$ such that $y'y \neq 0$, where $\tilde{H}(\rho)_d$ is the set of all distributive element of $\tilde{H}(\rho)$ for all $\rho \in Supp(\tilde{H}, P)$.

(iv) $(\tilde{H}, P)$ is soft Q-simple and each $\tilde{H}(\rho)$ has a left cancellable element for all $\rho \in Supp(\tilde{H}, P)$.

Proof. $(i) \Rightarrow (ii)$ Since $\tilde{H}(\rho)$ is a near-field, $\tilde{H}(\rho)$ is zero-symmetric and $\tilde{H}(\rho)$ has a left identity for all $\rho \in Supp(\tilde{H}, P)$. Let $(\tilde{K}, P)$ be a SQI of $(\tilde{H}, P)$. Then, $\tilde{K}(\rho)\tilde{H}(\rho) \cap \tilde{H}(\rho)\tilde{K}(\rho) \subseteq \tilde{K}(\rho)$ for all $\rho \in Supp(\tilde{H}, P)$. Let m, m′ be non-zero elements of $\tilde{K}(\rho)$, where m′ is a distributive element. Then $\tilde{H}(\rho) = m\tilde{H}(\rho) = \tilde{H}(\rho)m$ implies that $\tilde{H}(\rho) = m\tilde{H}(\rho) \cap \tilde{H}(\rho)m \subseteq \tilde{K}(\rho)\tilde{H}(\rho) \cap \tilde{H}(\rho)\tilde{K}(\rho) \subseteq \tilde{K}(\rho)$ and since $\tilde{K}(\rho) \subseteq \tilde{H}(\rho)$, we have $\tilde{K}(\rho) = \tilde{H}(\rho)$ for all $\rho \in Supp(\tilde{H}, P)$. Thus, $\tilde{H}(\rho)$ is a Q-simple for all $\rho \in Supp(\tilde{H}, P)$. Hence, $(\tilde{H}, P)$ is soft Q-simple.

$(ii) \Rightarrow (iii)$ Suppose $\tilde{H}(\rho)$ has a left identity e. Then e is a non-zero distributive element. Hence, $\tilde{H}(\rho)_d \neq \{0\}$ and $ey = y \neq 0$ for every non-zero element y of $\tilde{H}(\rho)$ for all $\rho \in Supp(\tilde{H}, P)$.

(iii) $\Rightarrow$ (iv) It is enough to show that each $\tilde{H}(\rho)$ has a left cancellable element for all $\rho \in Supp(\tilde{H}, P)$. Let $m \in \tilde{H}(\rho)_d \neq \{0\}$ such that $my_1 = my_2$, where y_1 and y_2 are any two elements of $\tilde{H}(\rho)$. Then $my_1 - my_2 = m(y_1 - y_2) = 0$. By assumption $y_1 - y_2 = 0$ which implies $y_1 = y_2$. Therefore, $\tilde{H}(\rho)$ has a left cancellable element m for all $\rho \in Supp(\tilde{H}, P)$.

(iv) $\Rightarrow$ (i) Let $m \in \tilde{H}(\rho)_d \neq \{0\}$ be a left cancellable element of $\tilde{H}(\rho)$. Then for any two elements $y_1, y_2 \in \tilde{H}(\rho)$ such that $y_1 = y_2 \Rightarrow my_1 = my_2 \Rightarrow m(y_1 - y_2) = 0$. This shows that $\tilde{H}(\rho)$ is zero-symmetric for all $\rho \in Supp(\tilde{H}, P)$. By Theorem 3.15, $m\tilde{H}(\rho)$ is QI of $\tilde{H}(\rho)$. Since $\tilde{H}(\rho)$ is Q-simple, we have $m\tilde{H}(\rho) = \tilde{H}(\rho)$ for all $\rho \in Supp(\tilde{H}, P)$. It follows from the Theorem 8.3 of [42], $\tilde{H}(\rho)$ is a near-field for all $\rho \in Supp(\tilde{H}, P)$. Therefore, $(\tilde{H}, P)$ is a SNF.

Definition 5.6. Let $(\tilde{K}, Q)$ be a SQI of a SN $(\tilde{H}, P)$ over N. Then $(\tilde{K}, Q)$is called a soft minimal quasi-ideal (SMQI, for short) of $(\tilde{H}, P)$ if $\tilde{K}(\rho)$ is a MQI of $\tilde{H}(\rho)$ for all $\rho \in Supp(\tilde{K}, Q)$.

Example 5.7. Consider the SN $(\tilde{H}, P)$ over N as defined in Example 3.7.

Let Q = {1, 3} and $\tilde{K}: Q \to P(N)$ be defined by $\tilde{K}(1) = \{0, 4\}$ and $\tilde{K}(3) = \{0, 2\}$. Then, $\tilde{K}(1)$ and $\tilde{K}(3)$ are QIs of $\tilde{H}(1)$ and $\tilde{H}(3)$, respectively. Since $\tilde{K}(1)$ and $\tilde{K}(3)$ do not properly contain any non-zero quasi-ideal of $\tilde{H}(1)$ and $\tilde{H}(3)$, we have $\tilde{K}(1)$ and $\tilde{K}(3)$ are MQIs of $\tilde{H}(1)$ and $\tilde{H}(3)$, respectively. So $\tilde{K}(\rho)$ is a MQI of $\tilde{H}(\rho)$ for all $\rho \in Supp(\tilde{K}, Q)$. Hence, $(\tilde{K}, Q)$ is a SMQI of $(\tilde{H}, P)$ over N.

Now consider the SS $(\tilde{K}, B)$ defined by $\tilde{K}_1(\rho) = \{\sigma \in N/\rho R\sigma \Leftrightarrow \rho\sigma \in \{0, 4\}\}$, where B = {1, 2, 3}.

Then $\tilde{K}_1(1) = \{0, 4\}$, $\tilde{K}_1(2) = \{0, 2, 4, 6\}$ and $\tilde{K}_1(3) = \{0, 4\}$ are QIs of $\tilde{H}(1)$, $\tilde{H}(2)$ and $\tilde{H}(3)$ respectively.

Since$\tilde{K}_1(2)$ contains the proper quasi-ideal {0,4} of $\tilde{H}(2)$, we have $\tilde{K}_1(2)$ is not a MQI of $\tilde{H}(2)$. Thus, $\tilde{K}_1(\rho)$ is not a MQI of $\tilde{H}(\rho)$ for all $\rho \in Supp(\tilde{K}_1, B)$. Hence, $(\tilde{K}_1, B)$ is not a SMQI of $(\tilde{H}, P)$ over N.

Theorem 5.8. Let $(\tilde{K}, Q)$ be a SQI of a SN $(\tilde{H}, P)$ over N such that $\tilde{K}(\rho) \neq \{0\}$ for all $\rho \in Supp(\tilde{K}, Q)$. If $(\tilde{K}, Q)$ is a SNF of $(\tilde{H}, P)$, then $(\tilde{K}, Q)$ is a SMQI of a soft zero-symmetric part $(\tilde{H}, P)_0$ of $(\tilde{H}, P)$.

Proof. Let $(\tilde{K}, Q)$ be a soft subnear-field of $(\tilde{H}, P)$. Since each $\tilde{K}(\rho)$ is a subnear-field of $\tilde{H}(\rho)$ and $\tilde{K}(\rho) \neq \{0\}$, $\tilde{K}(\rho)$ is a zero-symmetric subnear-ring of $\tilde{H}(\rho)$ and since $\tilde{H}(\rho)_0$ is a zero-symmtric part of $\tilde{H}(\rho)$, we have $\tilde{K}(\rho) \subseteq \tilde{H}(\rho)_0$ for all $\rho \in Supp(\tilde{K}, Q)$. By Remark 4.4 and Theorem 3.13, we have $(\tilde{K}, Q) = (\tilde{K}, Q) \tilde{\cap} (\tilde{H}, P)_0 \tilde{\lhd} (\tilde{H}, P)_0$. Suppose $(\tilde{J}, Q) \tilde{\lhd} (\tilde{H}, P)_0$ such that $\tilde{J}(\rho) \neq \{0\}$ and $\tilde{J}(\rho) \subseteq \tilde{K}(\rho)$ for all $\rho \in Supp(\tilde{K}, Q)$. Then, $\tilde{J}(\rho)\tilde{K}(\rho) \cap \tilde{K}(\rho)\tilde{J}(\rho) \cap \tilde{K}(\rho) * \tilde{J}(\rho) \subseteq \tilde{J}(\rho)\tilde{H}(\rho)_0 \cap \tilde{H}(\rho)_0 \tilde{J}(\rho) \cap \tilde{H}(\rho)_0 * \tilde{J}(\rho) \subseteq \tilde{J}(\rho)$. This implies that $\tilde{J}(\rho) \lhd_q \tilde{K}(\rho)$ for all $\rho \in Supp(\tilde{K}, Q)$. Since $(\tilde{K}, Q)$ is a SNF, we have $(\tilde{K}, Q)$ is soft Q-simple and so $\tilde{K}(\rho)$ is Q-simple for all $\rho \in Supp(\tilde{K}, Q)$. Consequently, $\tilde{K}(\rho) = \tilde{J}(\rho)$ for all $\rho \in Supp(\tilde{K}, Q)$. This shows that $\tilde{K}(\rho)$ is a MQI of $\tilde{H}(\rho)_0$ for all $\rho \in Supp(\tilde{K}, Q)$. Hence, $(\tilde{K}, Q)$ is a SMQI of $(\tilde{H}, P)_0$.

Theorem 5.9. Let $(\tilde{K}, Q)$ be a SQI of a SN $(\tilde{H}, P)$ over N such that $\tilde{K}(\rho) \neq \{0\}$ for all $\rho \in Supp(\tilde{K}, Q)$. If $(\tilde{K}, Q)$ is a SMQI of $(\tilde{H}, P)_0$, then $(\tilde{K}, Q)$ is a SMQI of $(\tilde{H}, P)$.

Proof. Let $(\tilde{K}, Q)$ be a SMQI of $(\tilde{H}, P)_0$ such that $\tilde{K}(\rho) \neq \{0\}$ for all $\rho \in Supp(\tilde{K}, Q)$. Suppose $(\tilde{J}, Q) \tilde{\lhd} (\tilde{H}, P)$ such that $\tilde{J}(\rho) \neq \{0\}$ and $\tilde{J}(\rho) \subseteq \tilde{K}(\rho)$ for all $\rho \in Supp(\tilde{K}, Q)$. Now, $\tilde{J}(\rho)\tilde{H}(\rho)_0 \cap \tilde{H}(\rho)_0\tilde{J}(\rho) \cap \tilde{H}(\rho)_0 * \tilde{J}(\rho) \tilde{\subseteq} J(\rho)\tilde{H}(\rho) \cap \tilde{H}(\rho)\tilde{J}(\rho) \cap \tilde{H}(\rho) * \tilde{J}(\rho) \subseteq \tilde{J}(\rho)$. This implies that $\tilde{J}(\rho) \lhd_q \tilde{H}(\rho)_0$ for all $\rho \in Supp(\tilde{K}, Q)$. Since $\tilde{K}(\rho)$ is a MQI of $\tilde{H}(\rho)_0$, we have $\tilde{K}(\rho) = \tilde{J}(\rho)$ for all $\rho \in Supp(\tilde{K}, Q)$. Since $\tilde{K}(\rho) \lhd_q \tilde{H}(\rho)$ and $\tilde{K}(\rho) = \tilde{J}(\rho)$, $\tilde{K}(\rho)$ is a MQI of $\tilde{H}(\rho)$ for all $\rho \in Supp(\tilde{K}, Q)$. Hence, $(\tilde{K}, Q)$ is a SMQI of $(\tilde{H}, P)$.

Theorem 5.10. Let $(\tilde{K}, Q)$ be a SQI of a SN $(\tilde{H}, P)$ over N such that $\tilde{K}(\rho) \neq \{0\}$ for all $\rho \in Supp(\tilde{K}, Q)$. Then, the following conditions are equivalent:

(i) $(\tilde{K}, Q)$ is a soft subnear-field of $(\tilde{H}, P)$,

(ii) $(\tilde{K}, Q)$ is a SMQI of $(\tilde{H}, P)0$ and each $\tilde{K}(\rho)$ contains a left cancellable element of $\tilde{H}(\rho)_0$ for all $\rho \in Supp(\tilde{K}, Q)$,

(iii) $(\tilde{K}, Q)$ is a SMQI of $(\tilde{H}, P)0$ and each $\tilde{K}(\rho)$ contains an idempotent element which is a left identity element of $\tilde{H}(\rho)_0$ for all $\rho \in Supp(\tilde{K}, Q)$,

(iv) $(\tilde{K}, Q)$ is a SMQI of $(\tilde{H}, P)0$ and each $\tilde{K}(\rho)$ contains a non-zero distributive element of $\tilde{H}(\rho)_0$ for all $\rho \in Supp(\tilde{K}, Q)$.

Proof. $(i) \Rightarrow (ii)$ It is clear from Theorems 5.5 and 5.8.

$(ii) \Rightarrow (iii)$ Let $e \in \tilde{K}(\rho)$ be an idempotent and left cancellable element of $\tilde{H}(\rho)_0$. Since $ee = e$ and $eey = ey$ gives $ey = y$ for all $y \in \tilde{H}(\rho)_0$, we have e is the left identity of $\tilde{H}(\rho)_0$ for all $\rho \in Supp(\tilde{K}, Q)$.

(iii) ⇒ (iv) Suppose $\tilde{H}(\rho)_0$ has a left identity e. Then, by Theorem 5.5, e is a non-zero distributive element of $\tilde{H}(\rho)_0$ for all $\rho \in Supp(\tilde{K}, Q)$.

$(iv) \Rightarrow (i)$ Let $e \in \tilde{K}(\rho)$ be a non-zero distributive and idempotent element of $\tilde{H}(\rho)_0$. Then $e \in \tilde{H}(\rho)_d$ and $e \in \tilde{H}(\rho)_E$, where $\tilde{H}(\rho)_E$ is the set of all idempotent element of $\tilde{H}(\rho)$. Thus, $e \in \tilde{K}(\rho) \cap \tilde{H}(\rho)_d \cap \tilde{H}(\rho)_E \neq \{0\}$. By Theorem 2 of [40], $\tilde{K}(\rho)$ is a subnear-field of $\tilde{H}(\rho)$ for all $\rho \in Supp(\tilde{K}, Q)$. Therefore, $(\tilde{K}, Q)$ is a SNF of $(\tilde{H}, P)$.

CONCLUSION

In this paper, we have introduced the notions of soft quasi-ideal, soft minimal quasi-ideal, soft left (resp. right) N-subgroup and soft invariant subnear-ring of a soft near-ring. We have also introduced the concepts of soft zerosymmetric near-ring, soft constant near-ring, soft near-field and soft Q-simple near-ring over a near-ring. We have obtained the properties of these notions with illustrated examples. We have provided the characterizations of soft quasi- ideals and soft near-fields over a near-ring. In future, one can extend the concepts of soft quasi-ideals to other algebraic substructures of AG-groupoid, BCK/BCI-algebra, semiring, hemiring, ring, Γ-near-ring, LA-semihypergroup, Γ-hypersemigroup, etc. Further, we aim to apply the concepts of this paper to soft linear algebraic codes in coding theory through the application of soft sets which is an approximated collection of codes and get some interesting results.

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The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

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