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Research Article

Solving high-order nonlinear differential equations using operational matrix based on exponential collocation method

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ABSTRACT

In this paper, the exponential approximation is applied to solve high-order nonlinear differential equations. The main idea of this method is based on the matrix representations of the exponential functions and their derivatives by using collocation points. To indicate the usefulness of this method we employ it for some well-known high-order nonlinear equations like Riccati, Lane-Emden and so on. The numerical approximate solutions are compared with available(existing) exact(analytical) solutions and the comparisons are made with other methods to show the accuracy of the proposed method. For convergence and error analysis of the method, criteria for a number of basis sentences presented. The method has been reviewed by several examples to show its validity and reliability. The reported examples illustrate that the method is appropriately efficient and accurate.

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INTRODUCTION

Differential equations have a remarkable role in several scientific and engineering phenomena that have always been considered in physical and technical applications and they are appeared in various areas such as mathematics, physics and engineering sciences ([1-4]). Among these equations, nonlinear differential equations are the most important. Many of these nonlinear ordinary differential equations have no exact(analytical) solution, which this is why so many researchers are interested in employing numerical methods, therefore, numerical approximation methods may be utilized to acquire approximate solutions. Numerical methods for solving nonlinear differential equations, especially high-order nonlinear differential equations, have always been considered. For example, some methods like the Homotopy Analysis Method (HAM) [5], the Adomian Decomposition Method (ADM)[6], the Homotopy Perturbation Method (HPM) ([7-12]), the Variational Iteration Method (VIM)[13], and so on, have been employed for solving some nonlinear differential equations. Most of the methods, which used to solve nonlinear differential equations, convert the equation to a system of nonlinear equations and then solve the system

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This paper was recommended for publication in revised form by Hijaz Ahmad theoretically or numerically. Also, the system of nonlinear equations can also be solved with software.

In recent years, Yüzbaşi et al. have applied the collocation method based on exponential approximation to solve some problems like pantograph equation, the linear neutral delay differential, Fredholm integro-differential difference equations and so on ([14-19]).

In this work, we will extend the exponential collocation method for approximating the solution of the high-order nonlinear differential equations in the general formulation as

$$\sum_{k=0}^{m}\sum_{s=0}^{n} Q_{k,s}(x)u^{s}(x)u^{(k)}(x) + \sum_{k=1}^{m}\sum_{s=1}^{m} P_{k,s}(x)u^{(s)}(x)u^{(k)}(x) = h(x), \quad (1)$$

subject to initial and/or boundary conditions

$$\sum_{k=0}^{m-1}\left(a_{ik}u^{(k)}(a) + b_{ik}u^{(k)}(b) + c_{ik}u^{(k)}(c)\right) = \alpha_i, \quad i = 0,1,\dots,m-1 \quad (2)$$

where $u(x)$ E $C^m[a, b]$ is an unknown function and $u^{(O)}(x)$ denotes function $u(x)$ itself, namely $u^{(O)}(x) = u(x)$. Here, $Q_{k,s}$, $P_{k,s}$ and h are known given functions that are defined on the interval $[a, b]$. Also a_{ik}, b_{ik}, c_{ik} and a_i are real or complex constants and c E $[a, b]$. We try to find the approximate solution of Equation (1) with condition (2) as series of exponential functions. Exponential functions or exponential polynomials are based on the linearly independent exponential basis set

$$\mathcal{B} = \{1, e^{-x}, e^{-2x}, \dots\}. \quad (3)$$

To begin with, we assume that the unique solution $u(x)$ of equation (1) can be expressed as a exponential series of the form

$$u(x) = \sum_{n=0}^{\infty} a_n e^{-nx}, \quad (4)$$

then by truncation this power series after the $(N + 1)$ st term, their linear combination defined by the expansion

$$u(x) \simeq u_N(x) = \sum_{n=0}^{N} a_n e^{-nx}, \quad (5)$$

which, coefficients a_n are unknown and N is an arbitrary positive integer such that $N \geq m$. Note that $u(x)$ is the exact solution and $u_N(x)$ is the approximate solution of the problem.

The organization of this article is structured as follows: In Section 2, we express briefly required mathematical elementary and matrix relations for exponential functions of the method. In Section 3, we present the numerical implementation of matrix operation of method. Section 4 involves the error analysis and method convergence. For this purpose, assuming the exact solution of equation (1) is analytic, an upper bound for the absolute error of the approximate solution is given in terms of the Taylor truncation error of the exact solution. Section 5 contains numerical examples, where approximate solutions corresponding to various N values are obtained using the proposed method that numerical experiments are examined to illustrate efficiency and accuracy of the method, and results are reported. Finally, last section consists of a brief conclusion.

PRELIMINARIES AND MATRIX RELATIONS

In this section, we outline operational matrices of the exponential method we will use in order to solve equation. The method, was employed to obtain approximate solutions of high-order nonlinear differential equations. In the first step, we create the differentiation matrices that are the basic tools of the current approach. Differentiation matrices make this method more suitable for managing high-order differential equations. By constructing an operational matrix, it is easy to derive high-order derivatives of the unknown in terms of values at collocation points. Firstly, we inscribe the approximated solution $u_N(x)$ defined by linear combination (5) of equation (1) in the matrix form as,

$$u(x) = \mathbf{E}(x)\mathbf{A}, \quad (6)$$

where

$$\mathbf{E}(x) = [1 \quad e^{-x} \quad e^{-2x} \dots \quad e^{-Nx}],$$

and

$$\mathbf{A} = [a_1 \quad a_2 \quad \dots \quad a_N]^T.$$

Taking advantage of the linearity of expansion , we can compute the derivative of u by differentiating the basic functions. The derivatives of u are obtained as follows

$$u'(x) = \sum_{n=1}^{N}(-na_n)e^{-nx}, \quad (7)$$

and for higher order derivatives of u we present a matrix form. Next, we explain how to create a differentiation matrix through the method, and we extract and create a matrix $\mathbf{D}$ so that the equations are in the collocation points. The derivative of the approximate solution can also be expressed as a product of matrices. Namely, $\mathbf{E}(x)$ has a relation with its first derivative $E'(x)$ that is demonstrated by

$$\mathbf{E}'(x) = \mathbf{E}(x)\mathbf{D},$$

where the operational (differentiation) matrix $\mathbf{D}$ corresponding to above relation is represented

$$\mathbf{D}=\begin{bmatrix}0&0&0&\cdots&0\\0&-1&0&\cdots&0\\0&0&-2&\cdots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\dots&-N\end{bmatrix},$$

and that, after repeating the procedure k

$$\mathbf{E}^k(x)=\mathbf{E}(x)\mathbf{D}^k,\quad k=0,1,2,\dots, \tag{8}$$

holds for any nonnegative integer k, that $\mathbf{D}^{\circ}$ is the identity matrix which its dimension is: (N + 1) x (N + 1). Note that

$$\mathbf{D}^k=\begin{bmatrix}0&0&0&\cdots&0\\0&(-1)^k&0&\cdots&0\\0&0&(-2)^k&\cdots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\dots&(-N)^k\end{bmatrix}.$$

By using of the matrix relations (6) and (8), we can write matrix representation as

$$u^{(k)}(x)=\mathbf{E}(x)\mathbf{D}^k\mathbf{A},\qquad k=0,1,\dots,m. \tag{9}$$

After replacing the collocation points $\{x_i\}_{i=0}^{N}$ in , we turn into the following system of matrix equations as

$$u^{(k)}(x_i)=\mathbf{E}(x_i)\mathbf{D}^k\mathbf{A},\qquad i=0,1,\dots,N,$$

that, in the matrix form, we have

$$\mathbf{U}^{(k)}=\mathbf{E}.\mathbf{D}^k.\mathbf{A},$$

where

$$\mathbf{E}=\begin{bmatrix}\mathbf{E}(x_0)\\\mathbf{E}(x_1)\\\vdots\\\mathbf{E}(x_N)\end{bmatrix}=\begin{bmatrix}1&e^{-x_0}&e^{-2x_0}&\dots&e^{-Nx_0}\\1&e^{-x_1}&e^{-2x_1}&\dots&e^{-Nx_1}\\\vdots&\vdots&\vdots&\ddots&\vdots\\1&e^{-x_N}&e^{-2x_N}&\dots&e^{-Nx_N}\end{bmatrix},\qquad \mathbf{U}^{(k)}=\begin{bmatrix}u^{(k)}(x_0)\\u^{(k)}(x_1)\\\vdots\\u^{(k)}(x_N)\end{bmatrix}.$$

As the same way, by putting the collocation points into the $u^r(x)u^{(k)}(x)$, and using the above relations, the following matrix representation is obtained as

$$\begin{bmatrix}u^r(x_0)u^{(k)}(x_0)\\u^r(x_1)u^{(k)}(x_1)\\\vdots\\u^r(x_N)u^{(k)}(x_N)\end{bmatrix}=\begin{bmatrix}u^r(x_0)&0&\dots&0\\0&u^r(x_1)&\dots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\dots&u^r(x_N)\end{bmatrix}\begin{bmatrix}u^{(k)}(x_0)\\u^{(k)}(x_1)\\\vdots\\u^{(k)}(x_N)\end{bmatrix}=(\overline{\mathbf{U}})^r\mathbf{U}^{(k)},$$

so that

$$\mathbf{U}=\mathbf{E}.\mathbf{A},$$

where

$$\overline{\mathbf{E}}=\begin{bmatrix}\mathbf{E}(x_0)&0&\dots&0\\0&\mathbf{E}(x_1)&\dots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\dots&\mathbf{E}(x_N)\end{bmatrix},\qquad \overline{\mathbf{A}}=\begin{bmatrix}\mathbf{A}&0&\dots&0\\0&\mathbf{A}&\dots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\dots&\mathbf{A}\end{bmatrix}.$$

Implementation of Matrix Operation

In this section, we explain how to employ the exponential collocation method for the problem . For this purpose, we use the following procedure. To acquire an exponential series solution of equation under the conditions, the operational matrix method is applied as follows. The foundation of this method is based on calculating the unknown coefficients using the collocation points. In the first step, the collocation points are placed in (1) as

$$\sum_{k=0}^{m}\sum_{s=0}^{n}Q_{k,s}(x_i)u^s(x_i)u^{(k)}(x_i)+\sum_{k=1}^{m}\sum_{s=1}^{m}P_{k,s}(x_i)u^{(s)}(x_i)u^{(k)}(x_i)=h(x_i),\quad i=0,1,\dots,N,$$

and afterwards the above system can be expressed in the following matrix form

$$\sum_{k=0}^{m}\sum_{s=0}^{n}\mathbf{Q}_{k,s}(\overline{\mathbf{U}})^s\mathbf{U}^{(k)}+\sum_{k=1}^{m}\sum_{s=1}^{m}\mathbf{P}_{k,s}\mathbf{U}^{(s)}\mathbf{U}^{(k)}=\mathbf{H},$$

where

$$\mathbf{Q}_{k,s}=\begin{bmatrix}Q_{k,s}(x_0)&0&\dots&0\\0&Q_{k,s}(x_1)&\dots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\dots&Q_{k,s}(x_N)\end{bmatrix},\qquad \mathbf{P}_{k,s}=\begin{bmatrix}P_{k,s}(x_0)&0&\dots&0\\0&P_{k,s}(x_1)&\dots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\dots&P_{k,s}(x_N)\end{bmatrix},$$

and

$$\mathbf{H}=[h(x_0)\quad h(x_1)\quad\dots\quad h(x_N)]^T.$$

After the substitution of the above relations, we obtain to the following matrix equation

$$\sum_{k=0}^{m}\sum_{s=0}^{n}\mathbf{Q}_{k,s}(\overline{\mathbf{E}}.\overline{\mathbf{A}})^s\mathbf{E}\mathbf{D}^k\mathbf{A}+\sum_{k=1}^{m}\sum_{s=1}^{m}\mathbf{P}_{k,s}\mathbf{E}\mathbf{D}^s\mathbf{A}\mathbf{E}\mathbf{D}^k\mathbf{A}=\mathbf{H},$$

such that

$$\left(\sum_{k=0}^{m}\sum_{s=0}^{n}\mathbf{Q}_{k,s}(\overline{\mathbf{E}}.\overline{\mathbf{A}})^s\mathbf{E}\mathbf{D}^k+\sum_{k=1}^{m}\sum_{s=1}^{m}\mathbf{P}_{k,s}\mathbf{E}\mathbf{D}^s\mathbf{A}\mathbf{E}\mathbf{D}^k\right)\mathbf{A}=\mathbf{H}. \tag{10}$$

Briefly, (10) can also be presented as follows

$$\mathbf{W}\mathbf{A}=\mathbf{H},$$

where

$$\mathbf{W}=\sum_{k=0}^{m}\sum_{s=0}^{n}\mathbf{Q}_{k,s}(\overline{\mathbf{E}}.\overline{\mathbf{A}})^s\mathbf{E}\mathbf{D}^k+\sum_{k=1}^{m}\sum_{s=1}^{m}\mathbf{P}_{k,s}\mathbf{E}\mathbf{D}^s\mathbf{A}\mathbf{E}\mathbf{D}^k. \tag{11}$$

Here, (10) conforms to a nonlinear system of the $(N+1)$ algebraic equations with the unknown coefficients a_n. Finally, in order to impose initial and/or boundary conditions, we try to acquire a matrix presentation of the conditions (2). By employing the relation(9) at points a, b and c, the matrix presentation of conditions that related to the coefficients matrix A becomes

$$\left(\sum_{k=0}^{m-1}\{a_{ik}\mathbf{E}(a)+b_{ik}\mathbf{E}(b)+c_{ik}\mathbf{E}(c)\}\mathbf{D}^k\right)\mathbf{A}=\alpha_i,\quad i=0,1,\dots,m-1.$$

Error Analysis and Convergence

In this section, we investigate error analysis and convergence of the method. We suppose that $u(x)$ be an infinitely differentiable function on interval $[a,b]$ and $u_N(x)$ be the approximated solution of $u(x)$ at collocation points x_i Here, we give an upper bound for the absolute error in terms of the Taylor truncation error of the exact solution.

A criteria for convergence:

If we consider an $(N + 1)$-term truncation approximation of exponential series, that is

$$u_N(x)=\sum_{n=0}^{N}a_n\,e^{-nx},$$

we can investigate the error of the method from the residual function by employing (1) as follows:

$$R_N(x)=\sum_{k=0}^{m}\sum_{s=0}^{n}Q_{k,s}(x)u_N^s(x)u_N^{(k)}(x)+\sum_{k=1}^{m}\sum_{s=1}^{m}P_{k,s}(x)u_N^{(s)}(x)u_N^{(k)}(x)-h(x).\quad(12)$$

In order to discuss the convergency and error analysis of the main problem with the initial and the boundary condition let us introduce the following norm.

Definition. The least-square norm is defined by

$$\|u(x)\|_2=\left(\int_a^b\omega(x)|u(x)|^2dx\right)^{\frac{1}{2}},\quad(13)$$

where $\omega(x)$ as weight function is non-negative[20] .

Now compute the error

$$\|R_N(x)\|_2=\left(\int_a^b\omega(x)|R_N(x)|^2dx\right)^{\frac{1}{2}},\quad(14)$$

for the desired approximation. Since the series (5) converges, we can specify the reliable N for an proper error, $\varepsilon > 0$, by employing the Cauchy condition

$$\|u_N(x)-u_M(x)\|_2^2\le\varepsilon,\quad(15)$$

that $N, M \geq N_0 - 1$, for some fixed integer number N_0. In special case, let us consider $h(x) = 0$ and $M = N - 1$ for $u(x)\ E\ C^m[0,1]$, then

$$2(a_N)^2\int_0^1 e^{-(2N+1)x}\,dx\le\varepsilon,$$

that

$$\frac{2(a_N)^2}{2N+1}\left(1-e^{-(2N+1)}\right)\le\varepsilon,$$

and finally

$$N\ge\frac{(a_N)^2}{\varepsilon}\left(1-e^{-(2N+1)}\right)-1,$$

which by ignoring the small term, we have the following criteria

$$N\ge\frac{(a_N)^2}{\varepsilon}-1.$$

It is a necessary criteria to stop the procedure of calculations. Obviously, If it increases the number of series sentences, then the accuracy of the approximate solution will increase. In fact, this inequality offers a necessary criteria to stop calculations process when ε be an arbitrary Cauchy error and a_N be the last coefficient.

Error bound for the solution

In this part, we relate the error bound for the approximate solution $u_N(x)$ to the truncation error of the Taylor polynomial corresponding to the exact solution.

Theorem. Suppose $u(x)$ and $u_N(x)$ denote the exact and the approximate solutions of problem , respectively. If $u(x) \in C^{N+1}[a,b]$, then

$$|u(x)-u_N(x)|\le|R_N^T(x)|+|u_N^T(x)-u_N(x)|,$$

where $u_N^T(x)$ denotes the N-th degree Taylor polynomial of u around the point $x = q \in [a,b]$ and R_N^T expresses its remainder term.

Proof. See [21].

As a result, this theorem is very helpful to find an upper bound of the absolute error in terms of the Taylor truncation error of the exact solution. Notice that this is not an a priori error bound; it only serves as a means to compare the actual error to this Taylor truncation error.

Illustrative Examples

In this section, we apply the method explained in Section 2 to several high-order nonlinear equations and compare the resulting approximate solutions with some other methods present in the literature. All the calculations have been performed using MAPLE. We solve four examples by using the method, and report the numerical results along with comparison with other methods. The absolute error

$$e(x)=|u_{exact}-u_{approx}|,$$

has been used to show that this method is efficient and reasonably accurate.

Example 1. As a first example, consider the Riccati differential equation ([23]-[24]) as

$$u'(x)-u(x)+2u^2(x)=0,\qquad x\in[0,1],$$

with initial condition $u(0) = 1$, which the exact solution of this problem is

$$u(x) = \frac{1}{2 - e^{-x}}.$$

For implementation of the method, from for $m = 1$, $n = 1$ the coefficients are

$$Q_{0,0} = -1, \qquad Q_{0,1} = 2, \qquad Q_{1,0} = 1, \qquad Q_{1,1} = 0,$$

and

$$P_{k,s} = 0, \quad \forall k, s = 0,1.$$

Table.1 compares the absolute errors of the solutions obtained by the present method, the Bessel Polynomials Method [22] , the Taylor Method [23] and the Decomposition [24] Method for $N = 3$. It can be concluded that for each choice of the parameter , the present method outperforms the aforementioned ones for most, if not all, of the sample points taken from [0,1]. The values in the table also imply that the absolute error functions resulting from the present method is more evenly distributed over the interval [0,1] compared to the other methods. This procedure has also been carried out for the values $N = 3$ and $N = 10$. The obtained bounds are shown in Table 2 together with the maximum actual errors corresponding to these N values. Furthermore, the last column indicates that increasing N decreases the absolute error by a significant amount. The data in the table can be seen in a visual setting in Figure.1. This figure shows the graph of absolute error function with $N = 3$ (top left), graph of numerical and exact solution with $N = 3$ (top right), graph of absolute error function with $N = 10$ (bottom left) and graph of numerical and the exact solution with $N = 10$ (bottom right).

Example 2. As a high-order complex problem, examine following fifth-order nonlinear ODE([25]-[26])

$$u^{(5)}(x) - e^{-x}u^2(x) = 0,$$

Table 2. The absolute error of the solution for Example 1 with $N = 3$ and = 10.

x	N=3	N=10
0	0	0
0.10	9.3727×10^{-4}	9.5461×10^{-8}
0.20	2.2564×10^{-3}	7.0035×10^{-8}
0.30	3.0230×10^{-3}	5.4983×10^{-8}
0.40	3.1385×10^{-3}	4.5255×10^{-8}
0.50	2.7799×10^{-3}	3.6935×10^{-8}
0.60	2.1748×10^{-3}	3.0999×10^{-8}
0.70	1.5184×10^{-3}	2.6045×10^{-8}
0.80	9.5098×10^{-4}	2.2166×10^{-8}
0.90	5.5932×10^{-4}	1.9100×10^{-8}
1.00	3.8725×10^{-4}	1.5947×10^{-8}

with the boundary conditions

$$u(0) = u'(0) = u''(0) = 1, \quad u(1) = u'(1) = e,$$

so that, the exact solution of this nonlinear differential equation is exponential function as $u(x) = e^x$.

For implementation of the method, from for $m = 5$, $n = 1$ the coefficients are

$$Q_{5,0} = 1, \qquad Q_{0,1} = -e^{-x},$$

and other coefficients are zero. The behavior of absolute error is reported in Table.3 and compared by Shifted Chebyshev Polynomial Method (SCPM) and Chebyshev collocation matrix method (CCMM) at same conditions with $N = 6$. Figure 2 shows graphs of absolute error function(left) and graph of numerical and the exact solution (right) with $N = 6$.

Example 3. For third example, consider the Lane-Emden equation([29]-[30]) as follows

$$u''(x) + \frac{\alpha}{x}u'(x) + g(x,u) = h(x),$$

Table 1. Comparison of the absolute errors of present method and other methods for Example 1.

x	Present Method N=3	Bessel P.M. [22] N=3	Taylor M. [23] N=5	Decomposition M. [24] N=4
0	0	0	0	0
0.20	2.2564×10^{-3}	4.9431×10^{-3}	1.5238×10^{-4}	5.8137×10^{-3}
0.40	3.1385×10^{-3}	4.3154×10^{-3}	1.1431×10^{-2}	8.1660×10^{-2}
0.60	2.1748×10^{-3}	1.3279×10^{-3}	1.2118×10^{-1}	3.7369×10^{-1}
0.80	9.5098×10^{-4}	3.2640×10^{-3}	6.1674×10^{-1}	1.0891×10^{00}
1	3.8725×10^{-4}	3.9361×10^{-2}	2.1293×10^{00}	2.4876×10^{00}

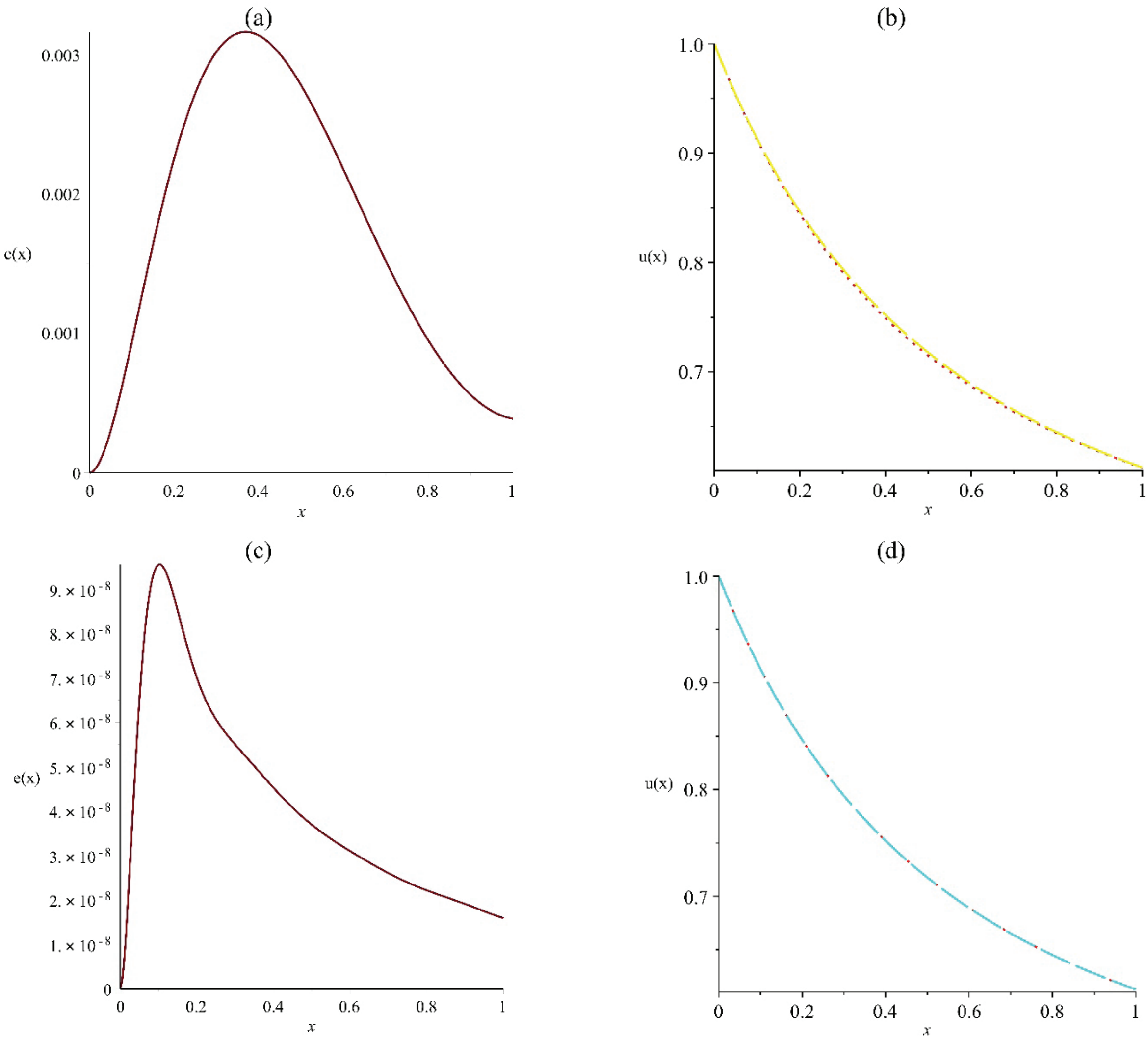

Figure 1. Graph of absolute error function with N = 3(a), graph of numerical and exact solution with N = 3(b), graph of absolute error function with N = 10(c) and graph of numerical and exact solution with N = 10(d) for Example 1.

Table 3. The comparison the absolute error of the solution with other methods for Example 2

x_i	**Present Method** N=6	**Shifted Chebyshev(SCPM)[25]** N=6	**Chebyshev M.(CCMM)[25]** N=6
0.00	0	0	0
0.10	1.5980×10^{-8}	4.5×10^{-8}	8.0×10^{-8}
0.20	8.8263×10^{-8}	2.3×10^{-7}	4.0×10^{-7}
0.30	1.9951×10^{-7}	4.1×10^{-7}	1.0×10^{-6}
0.40	3.0443×10^{-7}	2.4×10^{-7}	2.0×10^{-6}
0.50	3.6247×10^{-7}	4.9×10^{-7}	3.0×10^{-6}
0.60	3.5266×10^{-7}	1.6×10^{-6}	4.0×10^{-6}
0.70	2.7780×10^{-7}	2.7×10^{-6}	4.0×10^{-6}
0.80	1.6295×10^{-7}	2.7×10^{-6}	3.0×10^{-6}
0.90	5.1427×10^{-8}	1.3×10^{-6}	1.0×10^{-6}
1.00	0	0	2.0×10^{-2}

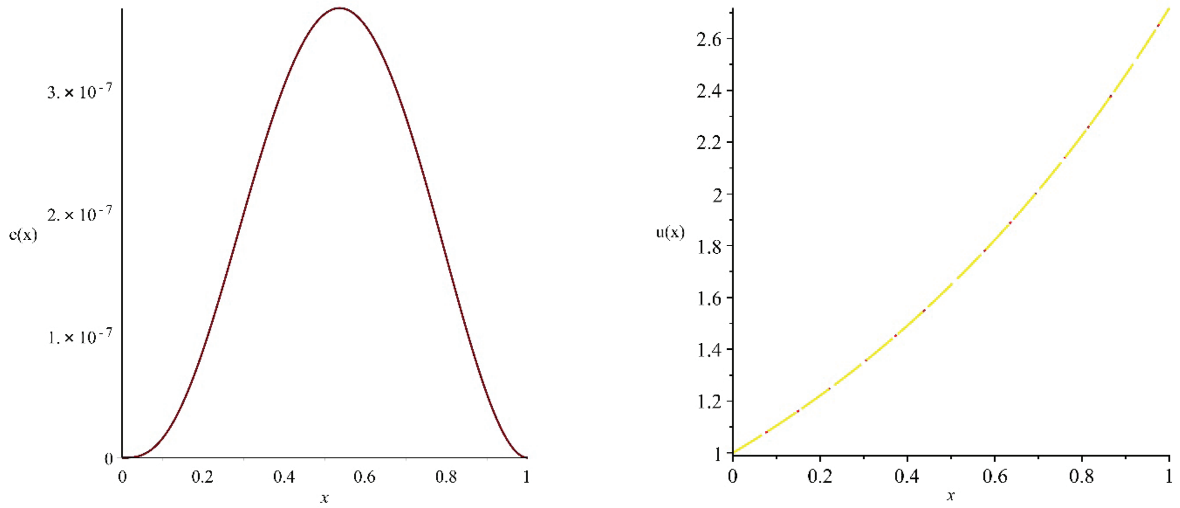

Figure 2. Graph of absolute error function(left) and graph of numerical and exact solution(right) with N = 6 for Example 2.

with initial conditions

$$u(0) = \alpha_0, \qquad u'(0) = \alpha_1.$$

In special case, for $g(x,u) = u^n(x)$ and $h(x) = 0$, this equation is the standard Lane-Emden equation. In this example, let $n = 5$, $\alpha = 2$ and $h(x) = 0$ then

$$u''(x) + \frac{2}{x}u'(x) + u^5(x) = 0,$$

with initial conditions

$$u(0) = 1, \qquad u'(0) = 0.$$

The exact solution of this problem is well-known as $u(x) = \left(1 + \frac{x^2}{3}\right)^{-1/2}$.

For implementation of the method, from for $m = 2$, $n = 4$ the coefficients are

$$Q_{2,0} = 1, \qquad Q_{0,4} = 1, \qquad Q_{1,0} = \frac{2}{x},$$

and other coefficients are zero. The mentioned procedure has also been carried out for the values $N = 5$ and $N = 10$. The obtained bounds are shown in Table 4 together with the maximum actual errors corresponding to these N values. Figure 3 shows the absolute error function with $N = 5$ (top left), graph of numerical and the exact solution with $N = 5$ (top right), graph of absolute error function with $N = 10$ (bottom left) and comparison of numerical and the exact solution with $N = 10$ (bottom right).

Example 4. As another example, consider a nonlinear third-order differential equation([29]-[30]) as follows

Table 4. The absolute error obtained by the Method for Example 3

x_i	N=5	N=10
0.00	0	0
0.10	6.0264×10^{-4}	1.7758×10^{-5}
0.20	1.5729×10^{-3}	2.4552×10^{-5}
0.30	2.3039×10^{-3}	2.5351×10^{-5}
0.40	2.6684×10^{-3}	2.4950×10^{-5}
0.50	2.7316×10^{-3}	2.3806×10^{-5}
0.60	2.6093×10^{-3}	2.2199×10^{-5}
0.70	2.4046×10^{-3}	2.0302×10^{-5}
0.80	2.1854×10^{-3}	1.8219×10^{-5}
0.90	1.9816×10^{-3}	1.6059×10^{-5}
1.00	1.7908×10^{-3}	1.3868×10^{-5}

$$u^{(3)}(x) + u''(x) + u'(x)u(x) = -e^{-2x}$$

with conditions

$$u(0) = 1, \quad u'(0) = -1, \quad u''(0) = 1,$$

which its exact(analytical) solution is $u(x) = e^{-x}$. For implementation of the method, from for $m = 1$, $n = 1$ the coefficients are

$$Q_{0,0} = -1 \quad Q_{0,1} = 2, \qquad Q_{1,0} = -1, \qquad Q_{1,1} = 0,$$

and the rest of the coefficients are zero. For this example, when we assume the following approximation

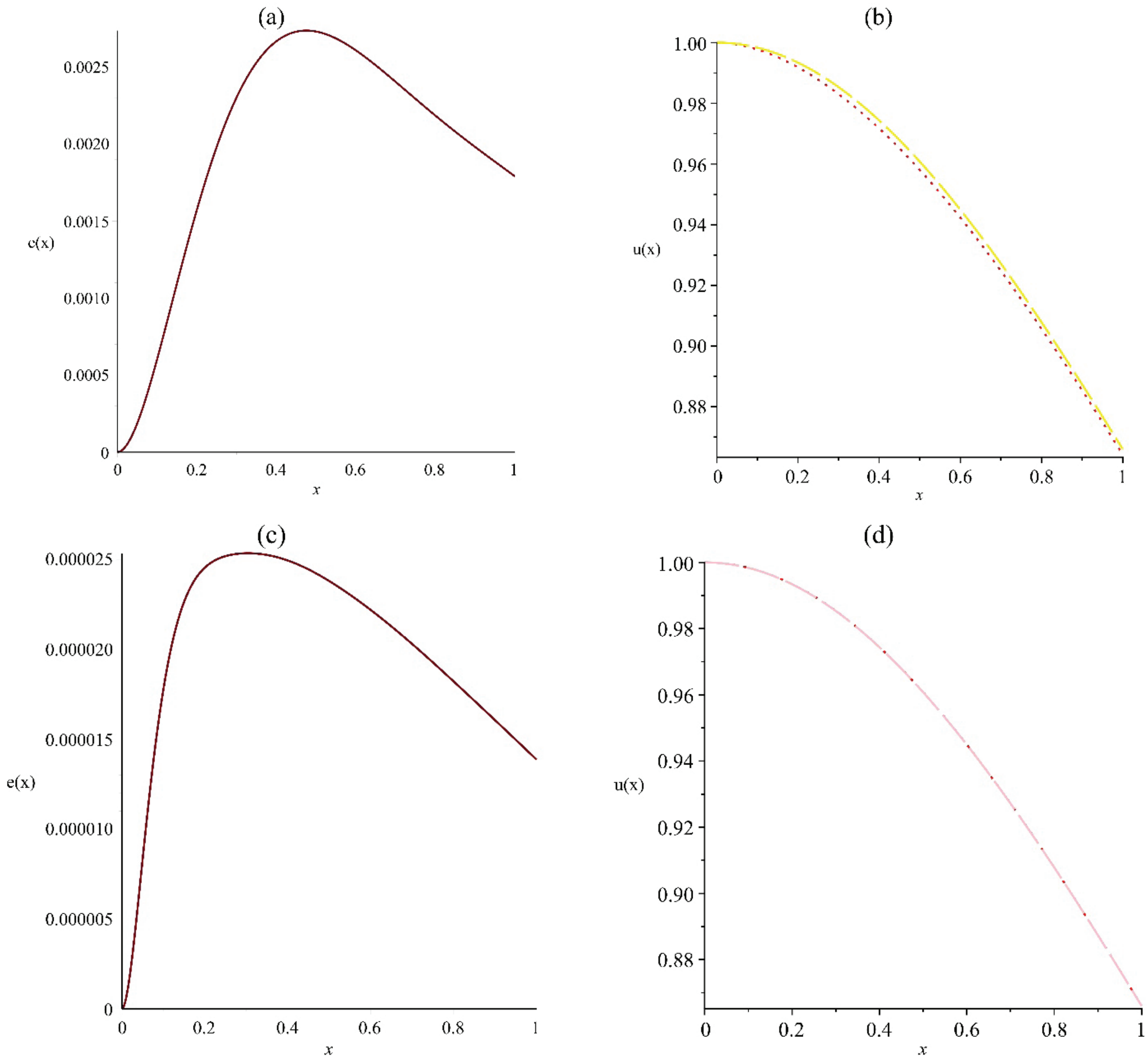

Figure 3. Graph of absolute error function with N = *5*(a), graph of numerical and exact solution with N = *5*(b), graph of absolute error function with N = *10*(c) and graph of numerical and exact solution with N = *10*(d) for Example 3.

$$u(x) \simeq u_N(x) = \sum_{n=0}^{N} a_n e^{-nx},$$

implementing the present method for any N yields the solution $u(x) = e^{-x}$, that is the exact solution of the problem. In fact, for any choice of N we get as a result of the algorithm, which means any N yields the exact solution. This is not a surprise since the scheme described in Section 2 makes it clear that the unknown coefficients of the approximate solution $u_N(x)$ obtained as a result are equal to the actual coefficients of the exact solution and we reach to exact solution means

$$a_0 = 0,\ a_1 = 1 \ \ and \ \ a_n = 0,\ \ for \ \ n \geq 2,$$

because its exact solution $u(x) = e^{-x}$ belongs to the basis set $B = \{1, e^{-x}, e^{-2x}, \ldots\}$

CONCLUSION

In this paper exponential approximation has been employed to solve high-order nonlinear differential equations. The method based on exponential functions and collocation method as operational matrix. As observed, there is no concern about approximating higher-order derivatives of the unknowns. Also, to demonstrate the accuracy and efficiency of the method, four examples with different order and complexity have been examined. Through the examples provided, we realize that the obtained numerical results are in good contract with the exact analytical solutions. As a result of comparisons with other methods, it has been observed that the method presented gives good results. In addition, it is realized that errors decrease when N values increase. As shown in the results obtained from computations, we conclude that implementation of this method will be very easy with less computational costs for similar problems.

AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

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