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Research Article

Completely weakly, quasi-regular semigroups characterized by soft union Quasi ideals, (generalized) bi-ideals and semiprime ideals

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ABSTRACT

In this paper, certain kinds of regularities of semigroups are studied by correlating soft set theory. Completely, weakly and quasi-regular semigroups are characterized by soft union quasi-ideals, soft union (generalized) bi-ideals and soft union semiprime ideals of a semigroup. It is proved that if every soft union quasi-ideal of a semigroup is soft union semiprime, then every quasi-ideal of a semigroup is semiprime and thus, if every quasi-ideal of a semigroup is semiprime, then the semigroup is completely regular. Also, it is obtained that the case when every soft union quasi-ideal (bi-ideal, generalized bi-ideal, respectively) of a semigroup is soft union semiprime is equivalent to the case when every quasi-ideal (bi-ideal, generalized bi-ideal, respectively) of a semigorup is semiprime, where the semigroup is completely semigroup. Similar characterizations are obtained for weakly and quasi-regular semigroups. By these characterizations, we intent to bring a new perspective to the regularities of semigroup theory via soft set theory. Further study can be focused on soft union tri quasi-ideals, soft union bi-quasi ideals, soft union lateral bi-quasi-ideals and soft union lateral tri-quasi ideals of a semigroup.

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INTRODUCTION

Molodtsov [1] introduced the principle concept of soft set to find solutions for uncertainty and vagueness problems in 1999. Since then, set theoretical aspects of soft sets especially operations of soft sets are studied in [2], [3], [4]. The theory of soft set has also many applications in different kinds of algebraic structures such as groups, semigroups, rings, semirings, near-rings, BCK/BCI algebras and BL-algebras [5–14].

In [15], Dar and Ali described the structure of generalized Jordan *-derivations of prime rings with involution. Further as a consequence, it was shown that any generalized Jordan *-derivation on a prime ring with involution is generalized X-inner, provided R is not a PI-ring. In cite [16],

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Dar and Khan studied the generalized derivations in rings with involution which behave like strong commutativity preserving mappings. In [17], certain identities involving multiplicative (generalized)-derivations on some appropriate subsets of the ring R was investigated. The results obtained in the paper [17] extends, unifies and complements several known results, since a great deal of calculation with commutators and anti-commutators are done.

In [18], Abu Arqub and Al-Smadi discussed a new definition of fuzzy fractional derivative, so-called fuzzy conformable and they proposed fuzzy conformable fractional integral softly. Also, uniqueness, existence, and other properties of solutions of certain fuzzy conformable fractional differential equations under strongly generalized differentiability were utilized. Furthermore, all needed requirements for characterizing solutions by equivalent systems of crisp conformable fractional differential equations were discussed. In [19], Abu Arqub et al. proposed a new method for solving fuzzy differential equations based on the reproducing kernel theory under strongly generalized differen- tiability. In [20], the analytic and approximate solutions of second-order, two-point fuzzy boundary value problems based on the reproducing kernel theory under the assumption of strongly generalized differentiability were discussed. In [21], Abu Arqub proposed the reproducing kernel Hilbert space method to obtain the exact and the numerical solutions of fuzzy Fredholm-Volterra integro-differential equations. The solution methodology is based on generating the orthogonal basis from the obtained kernel functions in which the constraint initial condition is satisfied. In the early 20th century, the researchers started to study the formal properties of semigroups since semigroups have a vital applications in language theory, coding theory, combinatorics, automata theory and mathematical analysis. A semigroup is an algebraic structure consisting of a nonempty set S together with an associative binary operation [24]. The theory of finite semigroups has been of particular importance in theoretical computer science since the 1950s because of the natural link between finite semigroups and finite automata via the syntactic monoid. In other areas of applied mathematics, semigroups are fundamental models for linear time-invariant systems. While in partial differential equations, a semigroup is associated to any equation whose spatial evolution is independent of time; in probability theory, semigroups are associated with well-konwn Markov processes. The researchers not only studied the general structure of the semigroup, but also their ideals and different kinds of semigroups as regards regularities such as regular semigroups, intra- regular semigroups, completely regular semigroups, quasi-regular semigroups and weakly regular semigroups [24–29].

In [30], Sezgin defined soft union semigroups, soft union left (right, two- sided) ideals, bi-ideals and soft semiprime ideals of a semigroup and obtained their basic properties. Also, regular and intra-regular semigroups were characterized by soft union semigroups and soft union left (right, two-sided) ideals and soft union bi-ideals. In [31], soft union interior ideals, quasi-ideals, generalized bi-ideals are defined, their basic properties with respect to soft set operations and soft union product are obtained and the interrelations of them are investigated. Also, regular and intra-regular semigroups are characterized by the properties of soft union interior ideals, soft union quasi-ideals and soft union generalized bi-ideals.

In this paper, soft right (left) ideals, soft quasi-ideals, soft (generalized) bi-ideals of a semigroup and soft union semiprime ideals are characterized by completely regular, weakly regular and quasi-regular semigroups. In Section 2, some basic definitions about semigroups, soft sets, different kinds of soft union ideals of a semigroup and their interrelations are recalled. In Section 3, completely regular semigroups are characterized as regards soft union quasi-ideals, soft union (generalized) bi-ideals and soft union semiprime ideals. It is shown that if every soft union quasi-ideal of S is soft union semiprime, then every quasi-ideal of S is semiprime and so if every quasi-ideal of S is semiprime, then S is completely regular. In Section 4, quasi-regular semigroups are studied in terms of soft union left (right) ideals, soft union quasi ideals and soft union (generalized) bi-ideals. In Section 5, weakly regular semigroups are characterized by soft union quasi- ideals and soft union bi-ideals of a semigroup. Our intent in this paper is to bring a new perspective to the regularities of semigroup theory via soft set theory.

PRELIMINARIES

In this part, some basic conceptions related to semigroups and certain kinds of semigroups, soft sets, soft union left (right) ideals, soft union quasi-ideals and soft union (generalized) bi-ideals of a semigroup are recalled. S denotes a semigroup throughout this paper. A semigroup is an algebraic structure consisting of a set together with an associative binary operation. A nonempty subset C of S is called a subsemigroup of S if $CC \subseteq C$; generalized bi-ideal of S if $CSC \subseteq C$; a quasi-ideal of S if $CS \cap SC \subseteq C$; semiprime if $\forall a \in S$, $a^2 \in C$ implies that $a \in C$. A subsemigroup D of S is called a bi-ideal of S if $DSD \subseteq D$. Quasi-ideal of S generated by $a \in S$ is defined as following: $Q[a] = \{a\} \cup (aS \cap Sa)$ and denoted by $Q[a]$.

An element b of S is completely regular, if there exists an element y in S such that $b = byb$ and $by = yb$. If all element of S is completely regular, then S is called a completely regular semigroup. S is left regular if for each element b of S, there exists an element y in S such that $b = yb^2$ if for each element b of S, there exists an element y in S satisfying $b = b^2y$, then S is called right regular semigroup. S is left (right) quasi-regular if all left (right) ideal of S is idempotent and is called quasi-regular if all both left ideal and right ideal of S is idempotent ([28]). For the other definitions of quasi-regular semigroups, we refer to [22, 23]. S is weakly-regular if for all y in S, y $(yS)^2$. S is called intra-regular if for all element b of S there exist elements y and z in S such that b =

yb^2z. For more about semigroups, ideals of semigroups and regularity of semigroups, we refer to [24–29]. From now on, U refers to an initial universe, E is a set of parameters, P (U) is the power set of U and A, B, C ⊆ E.

Definition 1 A soft set f_A over U is a set given by $f_A : E \to P(U)$ such that $f_A(x) = \emptyset$ if x is not in A.

We can represent a soft set over U by the set of ordered pairs as following:

$f_A = \{(x, f_A(x)) : x \in E, f_A(x) \in P(U)\}$ [1, 32].

It is clear to see that a soft set is a parametrized family of subsets of the set U. It is worth noting that the sets $f_A(x)$ may be arbitrary. Some of them may be empty, some may have nonempty intersection. If we define more then one soft set in a subset A of the set of parameters E, then the soft sets will be denoted by f_A, g_A, h_A etc. If we define more then one soft set in some subsets A, B, C etc. of parameters E, then the soft sets will be denoted by f_A, f_B, f_C etc., respectively. From now on, all the soft sets over U is denoted by S(U) and all the soft sets defined in this paper is the element of S(U).

Definition 2 A soft set by f_A is called a soft subset of f_B and denoted by $f_A \tilde{\subseteq} f_B$ if $f_A(x) \subseteq f_B(x)$ for all x ∈ E and is called a soft supset of f_B and denoted by $f_A \tilde{\supseteq} f_B$ if $f_A(x) \supseteq f_B(x)$ and is called soft equal to f_B and denoted by $f_A = f_B$ if $f_A(a) = f_B(x)$ for all x ∈ E [32] .

Definition 3 Let f_A and f_B be soft sets over U. Then, union of f_A and f_B, denoted by $f_A \tilde{\cup} f_B$, is defined as by $(f_A \tilde{\cup} f_B)(a)=f_A(a) \cup f_B(a)$ for all a ∈ E [32].

Definition 4 Let f_S and g_S be soft sets over U. Then, soft union product $f_S * g_S$ is defined by

$$(f_S * g_S)(a) = \begin{cases} \bigcap_{a=bc}\{f_S(b) \cup g_S(c)\}, & \text{if } \exists b, c \in S \text{ such that } a = bc, \\ U, & otherwise \end{cases}$$

for every a ∈ S [30]. In [30], it is proved that soft union product is associative.

Definition 5 A soft set f_S is called a soft union semigroup of S, if $f_S(ab) \subseteq f_S(a) \cup f_S(b)$ for all *x, y* ∈ S [30].

Definition 6 A soft set f_S is called a soft union left ideal of S over U if if $f_S(ab) \subseteq f_S(b)$ and is called a soft union right ideal of S over U if $f_S(ab) \subseteq f_S(a)$ for all a, b ∈ S. A soft set f_S is a soft union two-sided ideal (soft union ideal) of S if it is both soft union left and soft union right ideal of S [30].

If $f_S(x) = \emptyset$ for all x ∈ S, then f_S is a soft union left (right) ideal of S over U. We denote such a kind of soft union left (right) ideal by $\tilde{\theta}$. $f_S * \tilde{\theta} \tilde{\supseteq} \tilde{\theta}$ and $\tilde{\theta} * f_S \tilde{\supseteq} \tilde{\theta}$ is obvious. And f_S is an soft union left ideal of S if and only if $\tilde{\theta} * f_S \tilde{\supseteq} f_S$ and soft union right ideal of S over U if and only if $f_S * \tilde{\theta} \tilde{\supseteq} f_S$ [30].

Definition 7 A soft union semigroup f_S is called a *soft union bi-ideal* of S over U if $f_S(abc) \subseteq f_S(a) \cup f_S(c)$ for all a, b, c ∈ S [30].

Definition 8 A soft set f_S is called a *soft union quasi-ideal* of S over U if $f_S \tilde{\subseteq} (f_S * \tilde{\theta}) \tilde{\cup} (\tilde{\theta} * f_S)$ [31].

Definition 9 A soft set f_S is called a *soft union generalized bi-ideal* of S over U if $f_S(abc) \subseteq f_S(a) \cup f_S(c)$ for all a, b, c ∈ S [31].

Definition 10 A soft set f_S is called *soft union semiprime* if $f_S(a) \subseteq f_S(a^2)$ for all a ∈ S [30].

Definition 11 Let X be a subset of S. The soft characteristic function of the complement X, denoted by S_X^c, is defined as

$$\mathcal{S}_{X^c}(x) = \begin{cases} \emptyset, & \text{if } x \in X, \\ U, & \text{if } x \in S \setminus X \end{cases}$$

[30]. In [31], it is proved that when X is a nonempty subset of S, X is a quasi-ideal of S if and only if S_X^c is a soft union quasi-ideal of S over U. In [31], it is also proved that all soft union bi-ideals of S are also soft union generalized bi-ideals of S, all soft union quasi-ideals of S are also soft union bi-ideals of S, and all soft union left (right) ideals of S are also soft union quasi-ideals of S.

COMPLETELY REGULAR SEMIGROUPS

In this part, a completely regular semigroup is characterized as regards soft quasi-ideals, soft (generalized) bi-ideals of a semigroup and soft union semiprime ideals. In [33], for a semigroup S, it is proved that the condition where S is completely regular is equivalent to the condition where S is both left and right regular, namely, $x \in Sx^2$ and $x \in x^2 S$ for all x∈S. Hence, we have the following:

Theorem 1 If S is a completely regular semigroup, then for every soft union generalized bi-ideal k_S of S, $k_S(x) = k_S(x^2)$ for oll x ∈ S.

Proof Let S be a completely regular semigroup. Assume that k_S is a soft union generalized bi-ideal of S. Then, there is an element y ∈ S such that $x = x^2yx^2$ since S is completely regular by hypothesis. Hence,

$$\begin{aligned} k_S(x) &= k_S(x^2yx^2) \\ &\subseteq k_S(x^2) \cup k_S(x^2) \\ &= k_S(x^2) \end{aligned} \tag{1}$$

Hence, $k_S(x) \subseteq k_S(x^2)$. Now,

$$\begin{aligned} k_S(x^2) &= k_S(xx) \\ &= k_S(x(x^2yx^2)) \\ &= k_S(x(x^2yx)x) \\ &\subseteq k_S(x) \cup k_S(x) \\ &= k_S(x). \end{aligned} \tag{2}$$

Thus, $k_S(x^2) \subseteq k_S(x)$. By double-sided inclusion, $k_S(x) = k_S(x^2)$.

Theorem 2 Let S be a semigroup. If every soft union quasi-ideal of S is soft union semiprime, then every quasi-ideal of S is semiprime.

Proof Suppose that every soft union quasi-ideal of S is soft union semiprime and assume that Q is a quasi-ideal of S. Let $x^2 \in Q$ and $x \notin Q$. As the soft characteristic function S_Q^c is a soft union quasi-ideal of S, by hypothesis it is soft union semiprime. Thus, by Definition 4, $S_X^c(x) = U$ and

$S_Q{}^c\ (x^2) = \emptyset$. Now, by the definition of soft union semiprime, $S_Q{}^c\ (x) = U \subseteq S_Q{}^c\ (x^2) = \emptyset$. But this is a contradiction. Thus, $x \in Q$ and hence Q is semiprime, implying that every quasi-ideal of S is semiprime.

Theorem 3 Let S be a semigroup. If every quasi-ideal of S is semiprime, then S is completely regular.

Proof Suppose that every quasi-ideal of S is semiprime and let $x \in S$. Since the principal ideal $Q[x^2]$ generated by x^2 is a quasi-ideal and by the hypothesis a semiprime, and since $x^2 \in Q[x^2]$, then $x \in Q[x^2]$ by the definition of semiprime ideal. It is known that the soft characteristic function $S_{(Q_{[x^2]})}c$ is a soft union quasi-ideal of S when

$Q[x^2]$ is quasi-ideal [31]. Then, $S_{(Q_{[x^2]})}c(x^2) = S_{(Q_{[x^2]})}c(x) = \emptyset$ implying that $x \in Q[x^2] = \{x^2\} \cup (x^2S \cap S x^2)$. Since $x \neq x^2$ and so $x \notin \{x^2\}$, then $x \in (x^2S \cap S x^2)$ implying that S is completely regular according to [33].

Theorem 4 Let S be a semigroup S. Then, the followings are equivalent:

1) S is completely regular.
2) $k_S(x) = k_S(x^2)$ for every soft union generalized bi-ideal k_S of S and for all $x \in S$.
3) $k_S(x) = k_S(x^2)$ for every soft union bi-ideal k_S of S and for all $x \in S$.
4) $k_S(x) = k_S(x^2)$ for every soft union quasi-ideal k_S of S and for all $x \in S$.
5) Every soft union generalized bi-ideal of S is soft union semiprime.
6) Every soft union bi-ideal of S is soft union semiprime.
7) Every soft union quasi-ideal of S is soft union semiprime.
8) Every generalized bi-ideal of S is semiprime.
9) Every bi-ideal of S is semiprime.
10) Every quasi-ideal of S is semiprime.

Proof By Theorem 1, Theorem 2 and Theorem 3, (1) implies (2), (7) implies (10) and (10) implies (1), respectively. Since every soft union bi-ideal S of is a soft union generalized bi-ideal of S, (2) implies (3), (5) implies (6) and (8) implies (9). And since every soft union quasi-ideal of S is a soft union bi-ideal of S, (3) implies (4), (6) implies (7) and (9) implies (10). And by the definition of soft union semiprime, (4) implies (7), (3) implies (6), (2) implies (5). This completes the proof.

QUASI-REGULAR SEMIGROUPS

In this part, quasi-regular semigroups are studied in terms of soft union left (right) ideals, soft union quasi-ideals and soft union (generalized) bi-ideals of a semigroup S. In ([28]), it is proved that S is left (right) quasi-regular if and only if $x \in S\,x\,S\,x$ ($x \in x\,S\,x\,S$), namely, there exist elements $a, b \in S$ such that $x = axbx$ ($x = xaxb$).

Theorem 5 [30] A semigroup S is left (right) quasi-regular if and only if every soft union left (right) ideal is idempotent.

Proof In order to show that S is quasi-regular, we need to show that every soft union ideal of S is idempotent by Theorem 5. First, assume that k_S is a soft union right ideal of S. Since k_S is a soft union right ideal of S, $k_S * \tilde{\theta} \mathrel{\widetilde{\supseteq}} k_S$ holds [30]. Thus, $(\tilde{\theta} * k_S) \mathrel{\widetilde{\cup}} (k_S * \tilde{\theta}) \mathrel{\widetilde{\supseteq}} k_S * \tilde{\theta} \mathrel{\widetilde{\supseteq}} k_S$. Hence, k_S is a soft union quasi-ideal of S. Now, let $k_S = (k_S * \tilde{\theta})^2 \mathrel{\widetilde{\cup}} (\tilde{\theta} * k_S)^2$ holds for the soft union quasi ideal k_S of S. Since k_S is a soft union right ideal of S and $k_S * \tilde{\theta} \mathrel{\widetilde{\supseteq}} k_S$ holds, we have

$$
\begin{aligned}
k_S &= (k_S * \tilde{\theta})^2 \mathrel{\widetilde{\cup}} (\tilde{\theta} * k_S)^2 \\
&\mathrel{\widetilde{\supseteq}} (k_S * \tilde{\theta})^2 \\
&= (k_S * \tilde{\theta}) * (k_S * \tilde{\theta}) \\
&\mathrel{\widetilde{\supseteq}} k_S * k_S .
\end{aligned} \tag{3}
$$

Hence, $k_S \supseteq k_S * k_S$. Now, we need to show that $k_S * k_S \mathrel{\widetilde{\supseteq}} k_S$. Since $k_S \mathrel{\widetilde{\supseteq}} \theta$ and k_S is a soft right ideal of S, $k_S * k_S \mathrel{\widetilde{\supseteq}} k_S * \theta \mathrel{\widetilde{\supseteq}} k_S$ and so by double inclusion $k_S = (k_S)^2$. Thus, we obtain that every soft right ideal of S is idempotent, so S is right quasi-regular by Theorem 5. One can similarly show the left quasi-regularity. Thus, S is quasi-regular.

Theorem 7 If S is both left quasi-regular and intra-regular, then $l_S \mathrel{\widetilde{\cup}} m_S \mathrel{\widetilde{\cup}} k_S \mathrel{\widetilde{\supseteq}} l_S * m_S * k_S$ for every soft union generalized bi-ideal k_S, for every soft union left ideal l_S and every soft union right ideal m_S of S.

Proof Suppose that S is both left quasi-regular and intra-regular. Let k_S be any soft union generalized bi-ideal, l_S be any soft union left ideal and m_S be any soft union right ideal of S and x be any element of S. Since S is intra- regular, there exist elements $a, b \in S$ such that $x = ax^2b$. Also, as S is left quasi-regular, there exist $k, l \in S$ such that $x = kxlx$. Therefore,

$$
\begin{aligned}
x &= kxlx \\
&= k(axxb)lx \\
&= ((ka)x)((x(bl)x).
\end{aligned} \tag{4}
$$

Hence,

$$
\begin{aligned}
(l_S * m_S * k_S)(x) &= [l_S * (m_S * k_S)](x) \\
&= \bigcap_{x=((ka)x)((x(bl)x)} [l_S((ka)x)) \cup (m_S * k_S)(x(bl)x))] \\
&\subseteq l_S((ka)x)) \cup (m_S * k_S)(x(bl)x)) \\
&\subseteq l_S(x) \cup (\bigcap_{(x(bl))x=pr)} m_S(p) \cup k_S(r)) \\
&\subseteq l_S(x) \cup (m_S(x(bl)) \cup k_S(x)) \\
&\subseteq l_S(x) \cup m_S(x) \cup k_S(x) \\
&= (l_S \widetilde{\cup} m_S \widetilde{\cup} k_S)(x).
\end{aligned} \tag{5}
$$

Thus, $l_S \mathrel{\widetilde{\cup}} m_S \mathrel{\widetilde{\cup}} k_S \mathrel{\widetilde{\supseteq}} l_S * m_S * k_S$.

Theorem 8 If $l_S \mathrel{\widetilde{\cup}} m_S \mathrel{\widetilde{\cup}} k_S \mathrel{\widetilde{\supseteq}} l_S * m_S * k_S$ for every for every soft union quasi-ideal k_S, for every soft union left ideal l_S and every soft union right ideal m_S of S, then S is both intra-regular and left quasi-regular.

Proof Suppose that $l_S \mathrel{\widetilde{\cup}} m_S \mathrel{\widetilde{\cup}} k_S \mathrel{\widetilde{\supseteq}} l_S * m_S * k_S$ holds for every soft union-quasi- ideal k_S, for every soft union left ideal l_S and every soft union ideal m_S of S. Since l_S is a soft union left ideal of S, $\tilde{\theta} * l_S \mathrel{\widetilde{\supseteq}} l_S$ holds. Thus, $(\tilde{\theta} * l_S) \mathrel{\widetilde{\cup}} (l_S *$

$\tilde{\theta}) \supseteqq (\tilde{\theta} * l_S) \supseteqq l_S$. Hence, l_S is a soft union quasi-ideal of S. Moreover, since $\tilde{\theta}$ it self is a soft union right ideal of S,

$$\begin{aligned} l_S &= l_S \tilde{\cup} m_S \tilde{\cup} k_S \\ &= l_S * \tilde{\theta} * l \\ &= l_S * (\tilde{\theta} * l_S) \\ &\supseteqq l_S * l_S \end{aligned} \tag{6}$$

Hence, we have $l_S \supseteqq l_S * l_S$ for the soft union left ideal l_S of S. Now, $l_S * l_S \supseteqq \theta * l_S \supseteqq l_S$. Hence, we obtained that $l_S * l_S \supseteqq l_S$. By double-sided inclusion, $l_S = l_S * l_S$ for the soft union left ideal l_S of S. Hence S is left quasi-regular by Theorem 5.

Now, since $\tilde{\theta}$ itself is a soft union left ideal of S, hence soft union quasi-ideal of S,

$$\begin{aligned} l_S \tilde{\cup} m_S &= l_S \tilde{\cup} m_S \tilde{\cup} \tilde{\theta} \\ &= l_S * m_S * \tilde{\theta} \\ &= l_S * (m_S * \tilde{\theta}) \\ &\supseteqq l_S * m_S. \end{aligned} \tag{7}$$

Theorem 9 Let S be a semigroup S. Then, the followings are equivalent:

1) S is both left quasi-regular and intra-regular.
2) $l_S \tilde{\cup} m_S \tilde{\cup} k_S \supseteqq l_S * m_S * k_S$ for every union quasi-ideal k_S, for every soft union left ideal l_S and every soft union right ideal m_S of S.
3) $l_S \tilde{\cup} m_S \tilde{\cup} k_S \supseteqq l_S * m_S * k_S$ for every soft union bi-ideal k_S, for every soft union left ideal l_S and every soft union right ideal m_S of S.
4) $l_S \tilde{\cup} m_S \tilde{\cup} k_S \supseteqq l_S * m_S * k_S$ for every soft union generalized bi-ideal k_S, for every soft union left ideal l_S and every soft union right ideal m_S of S.

Proof (1) implies (4) is from Theorem 7, and (2) implies (1) is from Theorem 8. Since, every soft union bi-ideal S of is a soft union generalized bi-ideal of S, (4) implies (3), and since every soft union quasi-ideal of S is a soft union bi-ideal of S, (3) implies (2).

WEAKLY REGULAR SEMIGROUPS

In this part, weakly regular semigroup is characterized as regards soft union quasi-ideals and soft union (generalized) bi-ideals of a semigroup.

Theorem 10 [30] The following conditions are equivalent for a monoid S:

1) S is weakly regular.
2) $k_S \tilde{\cup} l_S \supseteqq k_S * l_S$ for every soft union right ideal k_S of S and for every soft union ideal l_S of S.

Theorem 11 Let S be a monoid. If S weakly regular, then $k_S \tilde{\cup} l_S \supseteqq k_S * l_S$ for every soft union generalized bi-ideal k_S of S and for every soft union ideal l_S of S.

Proof Assume that S is a weakly regular monoid and $k_S \tilde{\cup} l_S \supseteqq k_S * l_S$ holds for every soft union generalized bi-ideal k_S of S, for every soft union ideal l_S of S. Let x S. Then, $a \in (aS)^2$. Hence, a = axay for some x, y∈S. Also, since

$a = axay = ax(axay)y = (axa)(xay^2)$

$$\begin{aligned} (k_S * l_S)(a) &= \bigcap_{a=(axa)(xay^2)} (k_S(axa) \cup l_S(xay^2)) \\ &\subseteq [k_S(a) \cup k_S(a)] \cup l_S(a)) \\ &\subseteq k_S(a) \cup l_S(a) \\ &= (k_S \tilde{\cup} l_S)(a). \end{aligned} \tag{8}$$

Theorem 12 Let S be a monoid. If $k_S \tilde{\cup} l_S \supseteqq k_S * l_S$ for every soft union quasi- ideal k_S of S and for every soft union ideal l_S of S, then S is weakly regular.

Proof Let S be a monoid and $k_S \tilde{\cup} l_S \supseteqq k_S * l_S$ hold for every soft union quasi- ideal k_S of S and for every soft union ideal l_S of S. Since every soft union right ideal of S is a soft union quasi-ideal of S, then $k_S \tilde{\cup} l_S \supseteqq k_S * l_S$ holds for every soft union right k_S of S and for every soft union ideal l_S of S. Hence, S is a weakly regular semigroup by Theorem 10.

Theorem 13 The following conditions are equivalent for a monoid S:

1)S is weakly regular.

2)$k_S \tilde{\cup} l_S \supseteqq k_S * l_S$ for all soft union generalized bi-ideal k_S of S and for all soft union ideal l_S of S.

3) $k_S \tilde{\cup} l_S \supseteqq k_S * l_S$ for all soft union bi-ideal k_S of S and for all soft union ideal l_S of S.

4) $k_S \tilde{\cup} l_S \supseteqq k_S * l_S$ for all soft union quasi-ideal k_S of S and for all soft union ideal l_S of S.

Proof (1) implies (2) is by Theorem 11. Since every soft union bi-ideal of S is a soft union generalized bi-ideal of S, (2) implies (3) and since every soft union quasi ideal of S is a soft union bi-ideal of S, (3) implies (4) and by Theorem 12, (4) implies (1).

Theorem 14 Let S be a monoid. If S is weakly regular, then $k_S \tilde{\cup} l_S \tilde{\cup} m_S \supseteqq k_S * l_S * m_S$ for all soft union generalized bi-ideal k_S, for all soft union ideal l_S and for all soft union right ideal m_S of S.

Proof Assume that S is a weakly regular monoid and $k_S \tilde{\cup} l_S \tilde{\cup} m_S \supseteqq k_S * l_S * m_S$ holds for all soft union generalized bi-ideal k_S, for all soft union ideal l_S and for all soft union right ideal m_S of S. Let a ∈ S. Then, $a \in (aS)^2$. Hence, a = axay for some x, y ∈ S. Also, since xay = x(axay)y = (xax)(ay^2).

$$\begin{aligned} (k_S * l_S * m_S)(a) &= [k_S * (l_S * m_S)](a) \\ &= \bigcap_{a=axay} [k_S(a) \cup (l_S * m_S)(xay)] \\ &\subseteq k_S(a) \cup \{ \bigcap_{xay=pv} (l_S(p) \cup m_S(v))\} \\ &\subseteq k_S(a) \cup l_S(xax) \cup m_S(ay^2) \\ &\subseteq k_S(a) \cup l_S(a) \cup m_S(a) \\ &= (k_S \tilde{\cup} l_S \tilde{\cup} m_S)(a). \end{aligned} \tag{9}$$

Theorem 15 Let S be a monoid. If $k_S \tilde{\cup} l_S \tilde{\cup} m_S \supseteqq k_S * l_S * m_S$ for all soft union quasi-ideal k_S, for all soft union ideal l_S and for all soft union right ideal m_S of S, then S is weakly regular.

Proof Let $k_S \tilde{\cup} l_S \tilde{\cup} m_S \tilde{\supseteq} k_S * l_S * m_S$ hold for all soft union quasi ideal of k_S, soft union ideal l_S and soft union right ideal m_S of S. Since soft union right ideal m_S of S is also a soft union quasi-ideal of S, the soft union ideal l_S of S is also a soft union right ideal of S and θ itself is a soft union ideal of S,

$$\begin{aligned} m_S \tilde{\cup} l_S &= m_S \tilde{\cup} \theta \tilde{\cup} l_S \\ &= m_S * \tilde{\theta} * l_S \\ &= (m_S * \tilde{\theta}) * l_S \\ &\tilde{\supseteq} m_S * l_S. \end{aligned} \tag{10}$$

Thus, S is weakly regular by Theorem 10.

Theorem 16 The following conditions are equivalent for a monoid S:

1) S is weakly regular.
2) $k_S \tilde{\cup} l_S \tilde{\cup} m_S \tilde{\supseteq} k_S * l_S * m_S$ for all soft union generalized bi-ideal k_S of S, for all soft union ideal l_S of S and for all soft union right ideal m_S of S.
3) $k_S \tilde{\cup} l_S \tilde{\cup} m_S \tilde{\supseteq} k_S * l_S * m_S$ for all soft union bi-ideal k_S of S, for all soft union ideal l_S of S and for all soft union right ideal m_S of S.
4) $k_S \tilde{\cup} l_S \tilde{\cup} m_S \tilde{\supseteq} k_S * l_S * m_S$ for all soft union quasi-ideal k_S of S, for all soft union ideal l_S of S and for all soft union right ideal m_S of S.

Proof (1) implies (2) is by Theorem 14. Since every soft union bi-ideal of S is a soft union generalized bi-ideal of S, (2) implies (3) and since every soft union quasi-ideal of S is a soft union bi-ideal of S (3) implies (4). (4) implies (1) is by Theorem 15.

CONCLUSION

In this paper, we have characterized completely regular, weakly regular, quasi- regular semigroups by soft union quasi-ideals, soft union (generalized) bi-ideals of a semigroup and soft union semiprime ideals. A different point of view has been brought to semigroup theory as regards regularity via soft set theory. In the future works, completely regular, weakly regular, quasi-regular semigroups can be characterized by soft union tri quasi ideals, soft union m-quasi ideals, soft union m-bi-ideals, soft union left bi-quasi ideals, soft union lateral bi-quasi ideals, soft union right bi-quasi ideals, soft union left tri-quasi ideals, soft union lateral tri-quasi ideals, soft union right tri-quasi ideals of a semigroup.

AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

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