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Case Report

A numerical treatment for the Gilson-Pickering equation using collocation method with error estimation

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ABSTRACT

This paper presents a finite element scheme for numerical solutions of the Gilson-Pickering (G-P) equation by using septic B-spline functions as approximate functions. Firstly we study optimal-order L^2-error estimates for standard Galerkin semi-discrete approximation using smooth splines on a uniform mesh for periodic initial value problem of the G-P equation. A Von-Neumann stability analysis of the algorithm has been performed as well. Moreover, reliableness and practicalness of the presented method is demonstrated by analyzing behavior of single soliton. The L_2 and L_∞ error norms and two lowest invariants I_1 and I_2 of the equation have been computed to control proficiency and conservation properties of the suggested algorithm. Obtained numerical results have been illustrated with tables and graphics for easy visualization of properties of the problem modelled. Also the results indicate that our method is favorable.

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INTRODUCTION

Almost all physical processes encountered in nature are defined by various types of non-linear partial differential equations (NLPDEs). Many mathematical models are used to represent physical flows in various disciplines such as wave propagation, shallow water waves, reaction - diffusion models, biomechanical waves etc. [1]. Understanding the structure of these NLPDEs and seeking their solutions is of prime importance for scientists, as their solutions illuminate the way to understand the behavior of systems and help predict the development of the process in nature. Thus, many mathematicians focus their attention on solving NLPDEs. However, usually it is difficult to find their solutions analytically and sometimes it is almost impossible. Therefore many researchers have been working on to find efficient and high accurate numerical algorithms to overcome such

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problems [2]. In this study, the class of partial differential equations (PDEs) of third order, which are not completely linear, is taken into account. This class also includes different examples that have appeared recently in the literature and have been given outstanding motion wave solutions. These solutions consist of the sum of the exponential sums; we observe that these bases are seen in the presence of a linear inequality that appears to be a factor in motion wave reduction. In recent studies, it is seen that much attention has been paid to the development of numerical schemes. In this context, Claire Gilson and Andrew Pickering named the Gilson - Pickering (G-P) equation in 1995 developed a model [3]. Since this model has wide applications, it has recently managed to attract the attention of the scientific world. The model we consider in the study is as follows [4]:

$$u_t - \varepsilon u_{xxt} + 2\kappa \mathrm{u}_x - u u_{xxx} - \alpha u u_x - \beta u_x u_{xx} = 0, \tag{1}$$

where ε, κ, α, β are nonzero real constants. This equation includes many other nonlinear models, such as:

a. Fuchssteiner-Fokas-Camassa-Holm equation [5,6], with $\varepsilon = 1$, $\alpha = -3$ and $\beta = 2$, which is a completely integrable nonlinear partial differential equation that arises at different levels of approximation in shallow water theory;

b. Fornberg-Whitham equation [7,8] with $\varepsilon = 1$, $\alpha = -1$, $\kappa = 0.5$ and $\beta = 3$, which was developed to analyze the qualitative characteristics of wave breakage and admits a wave of the highest height;

c. Rosenau-Hyman equation [9,10], with $\varepsilon = 0$, $\alpha = 1$, $\kappa = 0$ and $\beta = 3$, which arises in the study of the influence of nonlinear dispersion on the structure of patterns in liquid drops.

When we use a transformation $\xi = x - ct$, Eq. (1) which is a PDE, can be written as

$$(2k - c)u_\xi + (\varepsilon c - u(\xi))u_{\xi\xi\xi} - \alpha u u_\xi - \beta u_\xi u_{\xi\xi} = 0.$$

Thus we convert the equation to the above non-linear ordinary differential equation (ODE) and the ODE can be written as the following system of autonomous ODEs

$$u' = z, z' = v, v' = \frac{z}{u(\xi) - \varepsilon c}[(2k - c) - \alpha u - \beta v]. \tag{2}$$

This system has two equilibrium points $\{(0,0,0), (\frac{2k - c}{\alpha}, 0, 0)\}, u \neq c\varepsilon$ as (0,0,0) always satisfies

$$0 = z, 0 = v, 0 = \frac{z}{u(\xi) - \varepsilon c}[(2k - c) - \alpha u - \beta v].$$

The evolution of such u(ξ) is presented in Figure 1.

Because of the great importance of G-P equation in nonlinear equations, many analytical methods have been used such as, Bernoulli sub-equation function method [11], a knot meshless method [12], first integral method [7], G'/G method [13]. Exact travelling wave solutions of the G-P equation are investigated in [14]. Applicability of the first integral method to the regularized Gilson-Pickering equation under a parameter condition has been studied in [15]. H. M. Baskonus has applied an analytical method based on Bernoulli differential equation to extract new complex soliton solutions to the Gilson - Pickering model [16]. The invariance and multiplier approach has been applied to recover a few of the conserved quantities of the equation by G. Ebadi et al. [10,17]. But unfortunately, there are very few literature for G-P equation that focus on numerical studies. T. Ak et al. have discussed the quasiperiodic and chaotic behaviors of the perturbed Gilson-Pickering equation by analyzing phase portrait analysis, time series analysis and Poincare section [18]. The existence of smooth and non-smooth travelling wave solutions under different parametric conditions are well studied here as well.

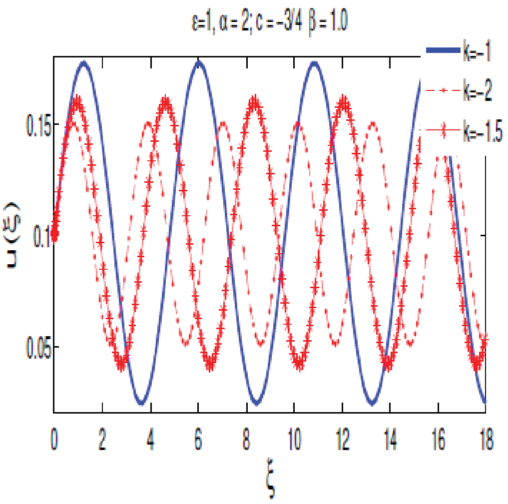
The finite element method is one of the most effective techniques used in the solution of PDEs. B-spline basis functions have drawn attention in the theory of approximation, solution of boundary- value problems and PDEs by considering numerical properties. These functions are very useful for numerical calculations. Collocation method based on the B-spline basis functions is very effective technique in point of it has a programmable computation approach and easy to apply. Thus require less computational efforts by comparison other existent methods [19]. As collocation points increase in the collocation method, the problem will be met at more points that force the approximate solution to approach the exact solution. In these days, among various numerical methods [20-25], finite element method is also commonly applied to the problems for numerical solutions [26-43].

In this work, we present collocation method with septic B-splines for the numerical solution of the G-P equation. The paper has been designated as follows: In Section 2, we analyze Galerkin semi-discrete approximation for the G-P equation. In Section 3, a short discussion of septic B-splines is given. The proposed higher order B-spline collocation scheme has been implemented to the G-P model in Section 4. Section 5 shows the stability analysis of the numerical scheme and it is followed by Section 6 which contains numerical examples of the behavior of single soliton. Finally, in Section 7, a brief conclusion about the presented method is given.

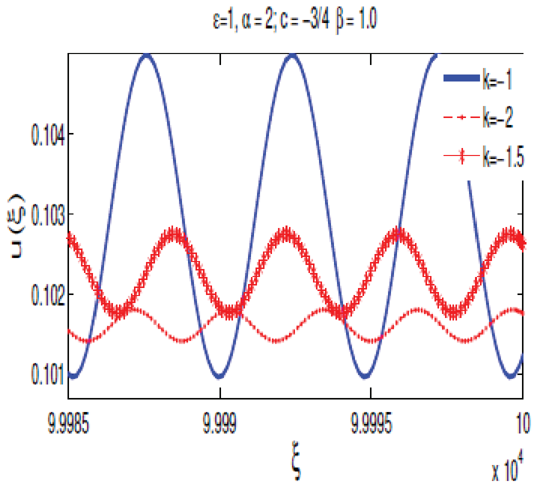
Galerkin Semi-Discrete Approximation for the G-P Equation

In this section, we analyze optimal-order L^2-error for a standard spatial Galerkin semi-discrete approximation considering a uniform mesh and smooth splines for the following G-P equation with the initial and boundary values

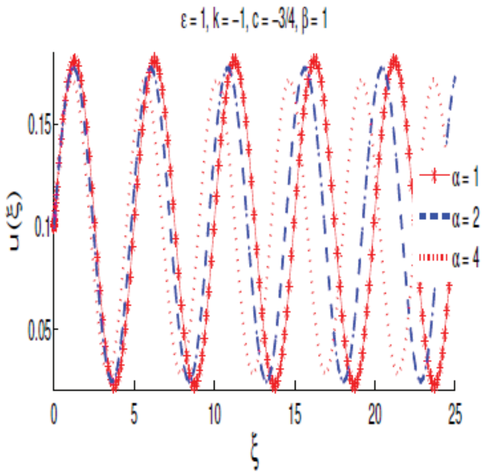
$$u_t - \varepsilon u_{xxt} + 2\kappa \mathrm{u}_x - u u_{xxx} - \alpha u u_x - \beta u_x u_{xx} = 0,\ x \quad \Omega = [a,b],\ t \in [0,T], \tag{3}$$



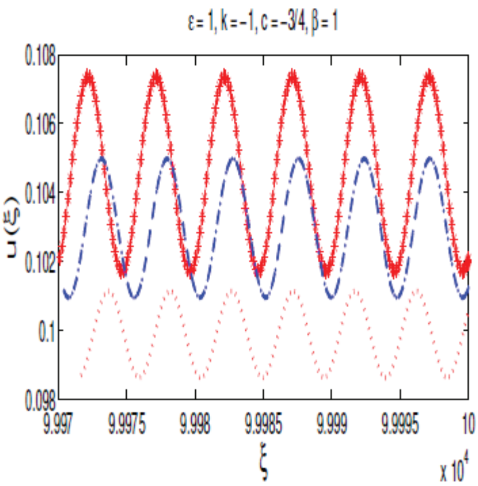
(a) Initial evolution of solutions



(b) ξ long evolutions



(c) Initial evolution of solutions



(d) ξ long evolutions

Figure 1. Solutions of (2): Here figures on the left (a and c) show initial evolutions for a set of parameters and the figures on the right (b and d) show long ξ behaviour of the solutions for another set of parameters. Here in figures (a and b) we vary k and in figures (c, and d) we vary α to demonstrate the solutions behaviour for long ξ.

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{4}$$

$$u(a,t) = u(b,t), \quad u_x(a,t) = u_x(b,t), \quad u_{xx}(a,t) = u_{xx}(b,t), \quad t \in [0,T], \tag{5}$$

where $u_0(x)$ is a given $L = (b - a)$ -periodic function smooth enough.

We next establish the notation to be used throughout the article. For integer $m \geq 0$, we denote H^m (Ω) the Sobolev space of order m on Ω, $H^m_{per}(\Omega)$ the subspace of H^m (Ω) consisting of L-periodic functions. We denote C^m (Ω), $m \geq 0$, m-times continuously differentiable functions on Ω and $C^m_{per}(\Omega)$ L-periodic such functions, $(\bullet,\bullet)$ denoted the inner product of L^2 (Ω) and the corresponding norm by $\|\bullet\|$. $W^m_\infty(\Omega)$ and L^∞ (Ω) norms are indicated by $\|\bullet\|_{m,\infty}$ and $\|\bullet\|_\infty$, respectively.

Here N is integer and $h = \dfrac{b-a}{N}, x_i = a + ih, i = 0,1,\ldots,N, J_i = [x_{i-1}, x_i]$. For integer $r \geq 2$, we denote the N-dimensional space of L-periodic smooth spline

$$S^r_h = \{\chi \in C^{r-2}_{per}(\Omega),\ \ \chi\big|_{J_i} \in P_{r-1}(J_i),\ i = 1,2,\ldots,N\},$$

where $P_{r-1}(J_i)$ denote set of polynomials of degree less or equal to $r-1$ on J_i. The following statement for S_h^r is well known [44,45]:

Theorem If v is L-periodic and $v \in H^s(\Omega) \cap W_\infty^m(\Omega)$, then there exists a $\chi \in S_h^r$ such that

$$\sum_{j=0}^{s-1} h^j \left\| v - \chi \right\|_j \le C h^s \left\| v \right\|_s, \quad 1 \le s \le r, \tag{6}$$

and

$$\sum_{j=0}^{m-1} h^j \left\| v - \chi \right\|_{j,\infty} \le C h^m \left\| v \right\|_{m,\infty}, \quad 1 \le m \le r. \tag{7}$$

Also, for all $\chi \in S_h^r$, the following inverse properties hold

$$\left\| \chi \right\|_\beta \le C h^{-(\beta-\alpha)} \left\| \chi \right\|_\alpha, \quad 0 \le \alpha \le \beta \le r-1. \tag{8}$$

$$\left\| \chi \right\|_{s,\infty} \le C h^{-(s+1/2)} \left\| \chi \right\|, \quad 0 \le s \le r-1, \tag{9}$$

where C is a generic positive constant independent of h, but may hold different real values at different occasions. In [46], Thomée et al. demonstrated that there occurs a basis $\left\{ \phi_j \right\}_{j=1}^N$ of S_h^r with $\sup(\phi_j) = O(h)$, in fact if v is a smooth enough L-periodic function, the associated quasi-interpolant $Q_h v = \sum_{j=1}^N v(x_j)\phi_j$ satisfies

$$\left\| Q_h v - v \right\| \le C h^r \left\| v^{(r)} \right\|. \tag{10}$$

Using the approximations (6)-(10), it can be seen that following result also possesses for quasi-interpolant. For $r \ge 3, j = 0,1,2, v \in H_{per}^r(\Omega) \cap W_\infty^r(\Omega)$ and $V = Q_h v$, we have [45]

$$\left\| V - v \right\|_j \le C h^{r-j} \left\| v \right\|_r, \tag{11}$$

$$\left\| V - v \right\|_{j,\infty} \le C h^{r-j-1/2} \left\| v \right\|_{r,\infty}, \tag{12}$$

$$\left\| V \right\|_j \le C, \quad \left\| V \right\|_{j,\infty} \le C. \tag{13}$$

Noting that $\beta u_x u_{xx} + u u_{xxx} = \frac{1}{2}(\beta - 1)\left(u_x^2\right)_x + (u u_{xx})_x$ and taking integration by parts, we describe standard Galerkin semi-discretization of G-P equation as follows. We seek $u_h : [0,T] \to S_h^r$, for $r \ge 3$ satisfying for $t \in [0,T]$ the equations:

$$\left(u_{ht}, \phi\right) + \varepsilon(u_{htx}, \phi') - 2\kappa(u_h, \phi') - \alpha(u_h u_{hx}, \phi) + \tfrac{1}{2}(\beta - 1)(u_{hx}^2, \phi') + (u_h u_{hxx}, \phi') = 0,$$
$$\forall \phi \in S_h^r, \; t \in [0,T], \tag{14}$$

$$u_h(0) = Q_h u_0. \tag{15}$$

In the same approach as [45], applying an energy procedure we demonstrate an optimal-order L^2 estimate for the error of semi-discrete approximation described by the initial-value problem (14)-(15).

Theorem Let $u(x,t)$ be sufficiently smooth solutions of (3)-(5) in $\Omega \times [0,T]$, h sufficiently small and $r \ge 3$. Then, $\exists!\, u_h$ of the semidiscrete nonlinear problem (14)-(15) on $[0, T]$ such that

$$\max_{0 \le t \le T} \left\| u(t) - u_h(t) \right\| \le C h^r, \tag{16}$$

holds where C is a constant independent of h.

Septic B-Spline Approximation

In the present work, G-P equation is assumed with the boundary conditions $u \to 0$ while $x \to \pm\infty$, x and t which generally denote time and space, respectively. To obtain the solution on the interval $[a,b]$ division $a = x_0 < x_1 < \cdots < x_{N-1} < x_N = b$ of the space domain is imagined scattered uniformly with $h = \frac{b-a}{N} = x_{m+1} - x_m$ for $m = 1, (1), N$. The set of seventh degree B-spline functions $\{\emptyset_{-3}(x), \emptyset_{-2}(x), \ldots, \emptyset_{N+2}(x), \emptyset_{N+3}(x)\}$, at the knots x_m can be written on problem domain $[a,b]$ as [47]:

$$\emptyset_m(x) = \begin{cases} a^7, & [x_{m-4}, x_{m-3}] \\ a^7 - 8b^7, & [x_{m-3}, x_{m-2}] \\ a^7 - 8b^7 + 28c^7, & [x_{m-2}, x_{m-1}] \\ a^7 - 8b^7 + 28c^7 - 56d^7, & [x_{m-1}, x_m] \\ e^7 - 8f^7 + 28g^7 - 56h^7, & [x_m, x_{m+1}] \\ e^7 - 8f^7 + 28g^7, & [x_{m+1}, x_{m+2}] \\ e^7 - 8f^7, & [x_{m+2}, x_{m+3}] \\ e^7, & [x_{m+3}, x_{m+4}] \\ 0, & di\breve{g}er. \end{cases} \tag{17}$$

where $a = (x - x_{m-4})^7$, $b = (x - x_{m-3})^7$, $c = (x - x_{m-2})^7$, $d = (x - x_{m-1})^7$, $e = (x_{m+4} - x)^7$, $f = (x_{m+3} - x)^7$, $g = (x_{m+2} - x)^7$, $h = (x_{m+1} - x)^7$. B-spline functions have some confidental features like smoothness, local support and ability of handling local phenomena, which make them appropriate to solve linear and nonlinear partial differential equations easily and sensitively. Using septic B-spline basis functions give rise to a global solution for which both the functions and up to their sixth order derivatives are continuous [48]. Collocation method has two excellent advantages: establishing method does not include integrations and resulting matrix system is banded with small band width. Therefore, B-splines when associate with the collocation ensures a simple solution procedure of partial differential equations [49].

Approximate solution $u_N(x,t)$ for analytical solution $u(x,t)$ is sought in the following equality,

$$u_N\ x,t \quad \overset{N\ 3}{\underset{m\ 3}{}} \quad {}_m\ x \quad {}_m\ t \tag{18}$$

where $\sigma_m(t)$ are time dependent unknown coefficients specified from the boundary conditions [50]. The septic B-spline functions are employed to overcome the higher order derivatives in the equation and when the bases are chosen at a high degree, generally better numerical results are obtained [51]. Since each septic B-spline covers eight consecutive elements, each finite element interval $\left[x_m, x_{m+1}\right]$ is covered with eight septic B-spline functions. In each element, applying the transformation $h\xi = x - x_m, 0 \le \xi \le 1$ to the spesific region $\left[x_m, x_{m+1}\right]$ is planned to more easily practicable region [0,1]. Thus, septic B-splines depending on variable ξ over the finite element [0,1] are defined as [52]:

$$\begin{aligned}
&{}_{m\ 3} \quad 1 \quad 7 \quad 21\ ^2 \quad 35\ ^3 \quad 35\ ^4 \quad 21\ ^5 \quad 7\ ^6 \quad ^7,\\
&{}_{m\ 2} \quad 120 \quad 392 \quad 504\ ^2 \quad 280\ ^3 \quad 84\ ^5 \quad 42\ ^6 \quad 7\ ^7,\\
&{}_{m\ 1} \quad 1191 \quad 1715 \quad 315\ ^2 \quad 665\ ^3 \quad 315\ ^4 \quad 105\ ^5 \quad 105\ ^6 \quad 21\ ^7,\\
&{}_{m} \quad 2416 \quad 1680\ ^2 \quad 560\ ^4 \quad 140\ ^6 \quad 35\ ^7,\\
&{}_{m\ 1} \quad 1191 \quad 1715 \quad 315\ ^2 \quad 665\ ^3 \quad 315\ ^4 \quad 105\ ^5 \quad 105\ ^6 \quad 35\ ^7,\\
&{}_{m\ 2} \quad 120 \quad 392 \quad 504\ ^2 \quad 280\ ^3 \quad 84\ ^5 \quad 42\ ^6 \quad 21\ ^7,\\
&{}_{m\ 3} \quad 1 \quad 7 \quad 21\ ^2 \quad 35\ ^3 \quad 35\ ^4 \quad 21\ ^5 \quad 7\ ^6 \quad 7\ ^7,\\
&{}_{m\ 4} \quad ^7.
\end{aligned} \tag{19}$$

Using Eq. (18) and septic B-splines (19), nodal values of u_m and its derivatives are calculated in terms of element parameters σ_m in the following form

$$\begin{aligned}
&u_N\ x_m,t \quad {}_{m\ 3} \quad 120\ {}_{m\ 2} \quad 1191\ {}_{m\ 1} \quad 2416\ {}_{m} \quad 1191\ {}_{m\ 1} \quad 120\ {}_{m\ 2} \quad {}_{m\ 3},\\
&u_m \quad \tfrac{7}{h} \quad {}_{m\ 3} \quad 56\ {}_{m\ 2} \quad 245\ {}_{m\ 1} \quad 245\ {}_{m\ 1} \quad 56\ {}_{m\ 2} \quad {}_{m\ 3}\ ,\\
&u_m \quad \tfrac{42}{h^2} \quad {}_{m\ 3} \quad 24\ {}_{m\ 2} \quad 15\ {}_{m\ 1} \quad 80\ {}_{m} \quad 15\ {}_{m\ 1} \quad 24\ {}_{m\ 2} \quad {}_{m\ 3}\ ,\\
&u_m \quad \tfrac{210}{h^3} \quad {}_{m\ 3} \quad 8\ {}_{m\ 2} \quad 19\ {}_{m\ 1} \quad 19\ {}_{m\ 1} \quad 8\ {}_{m\ 2} \quad {}_{m\ 3}\ .
\end{aligned} \tag{20}$$

Implementation of Collocation Method

First of all, using (18) and (20) in Eq. (3), the following general form equation is reached for the linearization technique:

$$\begin{aligned}
&{}_{m\ 3} \quad 120\ {}_{m\ 2} \quad 1191\ {}_{m\ 1} \quad 2416\ {}_{m} \quad 1191\ {}_{m\ 1} \quad 120\ {}_{m\ 2} \quad {}_{m\ 3}\\
&\tfrac{42}{h^2} \quad {}_{m\ 3} \quad 24\ {}_{m\ 2} \quad 15\ {}_{m\ 1} \quad 80\ {}_{m} \quad 15\ {}_{m\ 1} \quad 24\ {}_{m\ 2} \quad {}_{m\ 3}\\
&\tfrac{14}{h} \quad {}_{m\ 3} \quad 56\ {}_{m\ 2} \quad 245\ {}_{m\ 1} \quad 245\ {}_{m\ 1} \quad 56\ {}_{m\ 2} \quad {}_{m\ 3}\\
&\tfrac{210}{h^3} Z_{m1} \quad {}_{m\ 3} \quad 8\ {}_{m\ 2} \quad 19\ {}_{m\ 1} \quad 19\ {}_{m\ 1} \quad 8\ {}_{m\ 2} \quad {}_{m\ 3}\\
&\tfrac{7}{h} Z_{m1} \quad {}_{m\ 3} \quad 56\ {}_{m\ 2} \quad 245\ {}_{m\ 1} \quad 245\ {}_{m\ 1} \quad 56\ {}_{m\ 2} \quad {}_{m\ 3}\\
&\tfrac{7}{h} Z_{m2} \quad {}_{m\ 3} \quad 56\ {}_{m\ 2} \quad 245\ {}_{m\ 1} \quad 245\ {}_{m\ 1} \quad 56\ {}_{m\ 2} \quad {}_{m\ 3} \quad 0,
\end{aligned} \tag{21}$$

where

$$Z_{m1} \quad u_m \quad {}_{m\ 3} \quad 120\ {}_{m\ 2} \quad 1191\ {}_{m\ 1} \quad 2416\ {}_{m} \quad 1191\ {}_{m\ 1} \quad 120\ {}_{m\ 2} \quad {}_{m\ 3}$$

and

$$Z_{m2} \quad u_{m\ xx} \quad \tfrac{42}{h^2} \quad {}_{m\ 3} \quad 24\ {}_{m\ 2} \quad 15\ {}_{m\ 1} \quad 80\ {}_{m} \quad 15\ {}_{m\ 1} \quad 24\ {}_{m\ 2} \quad {}_{m\ 3}\ .$$

Let's assume that σ_i is linearly interpolated using the Crank-Nicolson finite difference approach and its time derivative $\dot{\sigma}_i$ is separated by the forward finite difference formula:

$$_i \quad \frac{{}^{n\ 1}_{i} \quad {}^{n}_{i}}{2}, \quad _i \quad \frac{{}^{n\ 1}_{i} \quad {}^{n}_{i}}{t}. \tag{22}$$

Hence, the above operations allows us to derive a recursion relationship between σ_i^{n+1} and σ_i^n for as [53]:

$$\begin{aligned}
&{}_1\ {}^{n\ 1}_{m\ 3} \quad {}_2\ {}^{n\ 1}_{m\ 2} \quad {}_3\ {}^{n\ 1}_{m\ 1} \quad {}_4\ {}^{n\ 1}_{m} \quad {}_5\ {}^{n\ 1}_{m\ 1} \quad {}_6\ {}^{n\ 1}_{m\ 2} \quad {}_7\ {}^{n\ 1}_{m\ 3}\\
&{}_7\ {}^{n}_{m\ 3} \quad {}_6\ {}^{n}_{m\ 2} \quad {}_5\ {}^{n}_{m\ 1} \quad {}_4\ {}^{n}_{m} \quad {}_3\ {}^{n}_{m\ 1} \quad {}_2\ {}^{n}_{m\ 2} \quad {}_1\ {}^{n}_{m\ 3},
\end{aligned} \tag{23}$$

where

$$\begin{aligned}
&{}_1 \quad 1 \quad A \quad B\ 2 \quad Z_{m1} \quad CZ_{m1} \quad DZ_{m2}\ ,\\
&{}_2 \quad 120 \quad 24A \quad 56B\ 2 \quad Z_{m1} \quad 8CZ_{m1} \quad 56DZ_{m2}\ ,\\
&{}_3 \quad 1191 \quad 15A \quad 245B\ 2 \quad Z_{m1} \quad 19CZ_{m1} \quad 245DZ_{m2}\ ,\\
&{}_4 \quad 2416 \quad 80A\ ,\\
&{}_5 \quad 1191 \quad 15A \quad 245B\ 2 \quad Z_{m1} \quad 19CZ_{m1} \quad 245DZ_{m2}\ ,\\
&{}_6 \quad 120 \quad 24A \quad 56B\ 2 \quad Z_{m1} \quad 8CZ_{m1} \quad 56DZ_{m2}\ ,\\
&{}_7 \quad 1 \quad A \quad B\ 2 \quad Z_{m1} \quad CZ_{m1} \quad DZ_{m2}\ , m \quad 0,1, \quad ,N,\\
&A \quad \tfrac{42}{h^2}\ , \quad B \quad \tfrac{7}{2h} \quad t, \quad C \quad \tfrac{105}{h^3} \quad t, \quad D \quad \tfrac{7}{2h} \quad t.
\end{aligned} \tag{24}$$

If we take a look at the algebraic system (23) we obtained above, the number of linear equations are less than the number of unknown coefficients, that is, the system contains of $(N + 1)$ equation $(N + 7)$ unknown time dependent parameters. The simplest way to find a unique solution is to remove six unknowns $\sigma_{-3}, \sigma_{-2}, \sigma_{-1}, \ldots, \sigma_{N+1}, \sigma_{N+2}, \sigma_{N+3}$ from the system. This procedure is applied using the boundary conditions with the values of u and after eliminating unknowns, following matrix-vector system is obtained

$$Ed^{n+1} = Fd^n \tag{25}$$

where $d^n = (\sigma_0, \sigma_1, \ldots, \sigma_N)^T$. Now, initial parameters σ^0 are established using initial condition and derivatives at the boundaries;

$$\begin{aligned}
&u_N(x,0) = u(x_m,0), m = 0,1,2,\ldots,N\\
&(u_N)_x(a,0) = 0, \qquad (u_N)_x(b,0) = 0,\\
&(u_N)_{xx}(a,0) = 0, \qquad (u_N)_{xx}(b,0) = 0,
\end{aligned}$$

Thus, initial vector d^0 can be determined in the following system of algebraic equations in matrix form:

$$Wd^0 = b,$$

where

$$W \quad \begin{bmatrix} 1536 & 2712 & 768 & 24 & & & & & & & \\ \frac{82731}{81} & \frac{210568.5}{81} & \frac{104796}{81} & \frac{10063.5}{81} & 1 & & & & & & \\ \frac{9600}{81} & \frac{96597}{81} & \frac{195768}{81} & \frac{96474}{81} & 120 & 1 & & & & & \\ & & & & & & & & & & \\ & & & 1 & 120 & 1191 & 2416 & 1191 & 120 & 1 & \\ & & & & 1 & 120 & \frac{96474}{81} & \frac{195768}{81} & \frac{96597}{81} & \frac{9600}{81} & \\ & & & & & 1 & \frac{10063.5}{81} & \frac{104796}{81} & \frac{210568.5}{81} & \frac{82731}{81} & \\ & & & & & & 24 & 768 & 2712 & 1536 & \end{bmatrix}$$

$$d^0 \quad 0, \ 1, \ 2, \ldots, \ N \ 2, \ N \ 1, \ N \ ^T$$

and

$$b \quad u \ x_0, 0 \ , u \ x_1, 0 \ , \ldots, u \ x_{N \ 1}, 0 \ , u \ x_N, 0 \quad ^T.$$

Stability Analysis

It is not easy to undertake the stability analysis of nonlinear partial differential equations. Most researchers solve with the problem by linearizing the partial differential equation [54]. We follow the Von-Neumann analysis for the stability of the scheme. In a typical amplitude mode, we can define the growth factor ξ of the error as follows:

$$\delta_m^n = \xi^n e^{imkh}, \tag{26}$$

where $i = \sqrt{-1}$, h is element size and k is mode number. Putting the Fourier mode (26) into the iterative system (23), which give the growth factor:

$$\xi = \frac{\rho_1 - i\rho_2}{\rho_1 + i\rho_2}, \tag{27}$$

where

$$\begin{aligned} &_1 \quad 2416 \quad 80A \quad 2382 \quad 30A \ \cos kh \quad 240 \quad 48A \ \cos 2kh \quad 2 \quad 2A \ \cos 3kh \ , \\ &_2 \quad 490B \ 2 \quad Z_{m1} \quad 38CZ_{m1} \quad 490DZ_{m2} \ \sin kh \\ &112B \ 2 \quad Z_{m1} \quad 16CZ_{m1} \quad 112DZ_{m2} \ \sin 2kh \\ &2B \ 2 \quad Z_{m1} \quad 2CZ_{m1} \quad 2DZ_{m2} \ \sin 3kh \ , \end{aligned} \tag{28}$$

and

$$A \quad \frac{42}{h^2}, \quad B \quad \frac{7}{2h} \ t, \quad C \quad \frac{105}{h^3} \ t, \quad D \quad \frac{7}{2h} \quad t, \quad m \quad 0, 1, \quad , N. \tag{29}$$

$|\xi| = 1$ is obtained when we take the modulus of Eq. (27). In this way, we demonstrate that scheme (23) is unconditionally stable under the present conditions.

NUMERICAL APPLICATIONS AND RESULTS

In this section, in order to demonstrate the performance of our algorithm, some numerical examples are considered. Two sets of parameters have been used and discussed for numerical simulations of the motion of a single solitary wave with precise solutions. The performance of the proposed method will be checked with the L_2 and L_∞ error norms given as [55]

$$L_2 = \|U_{tam} - U_N\|_2 \cong \sqrt{h \sum_{j=0}^{N} \left| U_j^{tam} - (U_N)_j \right|^2}, \tag{30}$$

$$L_\infty = \|U_{tam} - U_N\|_\infty \cong \max_j \left| U_j^{tam} - (U_N)_j \right| \tag{31}$$

G-P equation has only two invariants given by [56]

$$\begin{aligned} I_1 \quad &{}_a^b u dx \quad h \quad {}_{j \ 1}^{N} u_j^n, \\ I_2 \quad &{}_a^b \left[\tfrac{1}{6} \ 3u^2 \quad u_x^2 \quad 2 \ uu_{xx} \ \right] dx \quad h \quad {}_{j \ 1}^{N} \left[\tfrac{1}{6} \left(3 \left(u_j^n \right)^2 \quad u_x \ {}_j^2 \quad 2 \ q_j^n \ u_{xx} \ {}_j \right) \right]. \end{aligned} \tag{32}$$

Dispersion of a single solitary wave

The G-P equation has an exact solution of the form [57]

$$u(x,t) = A(-1 + \tanh^2[B(x - ct)]), \tag{33}$$

where $A = \frac{3(c-2\kappa)\varepsilon c}{\alpha\varepsilon c - c + 2\kappa}$ and $B = \frac{1}{2}\sqrt{-\frac{2\kappa - c}{\varepsilon c}}$. Note that, ε, κ, c and β are arbitrary real numbers. We will consider the G-P equation with the boundary-initial conditions which are

$$u(x,0) = A(-1 + \tanh^2[B(x)]), \tag{34}$$

where $u \to 0$ as $x \to \pm\infty$

Case 1

For the first numerical calculation, we choose the quantities $\varepsilon = 1$, $\kappa = -0.5$, $\alpha = -3$, $\beta = -1.5$, $c = 0.5$, $\Delta t = 0.01$ and $h = 0.1$ over the interval $x \in [-10,10]$. Numerical values of the invariants and error norms have been presented at some predefined times up to $t = 1$ in Table 1. It is observed from the table that the errors are noticeably small and invariants of solutions are almost unchanged as time grows. Numerical solution of single solitary wave is plotted at some fixed times from $t = 0$ to $t = 1$ in Figure 2. Curves are indistinguishable when numerical solution of a single wave with amplitude=-0.75 using presented algorithm is drawn in the same graphic at time $t = 1$. The continuous solution profile shown in the picture is found to be associated with the dispersing wave component of the solution in the figure. Also, distribution of error at time $t = 1$ has been depicted graphically in Figure 3.

Table 1. Here we present invariants which are conserved quantities and the relevant error norms for Case I.

t	I_1	I_2	L_2	L
0.0	1.7320506923	.6928203409	.0000000000	.0000000000
0.1	1.7320539139	.6922972521	.0102526817	.0063523550
0.2	1.7320559682	.6907326639	.0205660540	.0127017392
0.3	1.7320570436	.6881405274	.0309994467	.0190989473
0.4	1.7320573095	.6845438026	.0416095452	.0255739130
0.5	1.7320569166	.6799740306	.0524492501	.0320755635
0.6	1.7320559975	.6744707513	.0635667332	.0385876322
0.7	1.7320546674	.6680807859	.0750047221	.0454235307
0.8	1.7320530251	.6608574018	.0868000289	.0523507186
0.9	1.7320511534	.6528593872	.0989833149	.0593493612
1.0	1.7320491200	.6441500580	.1115790714	.0665106554

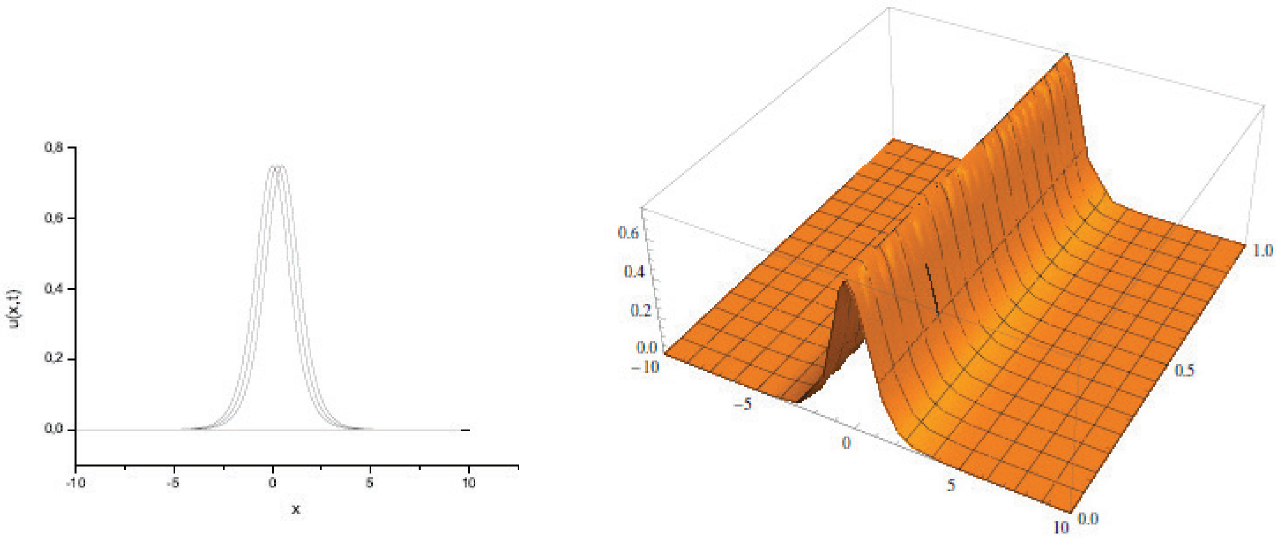


Figure 2. Behavior of single soliton for Case 1.

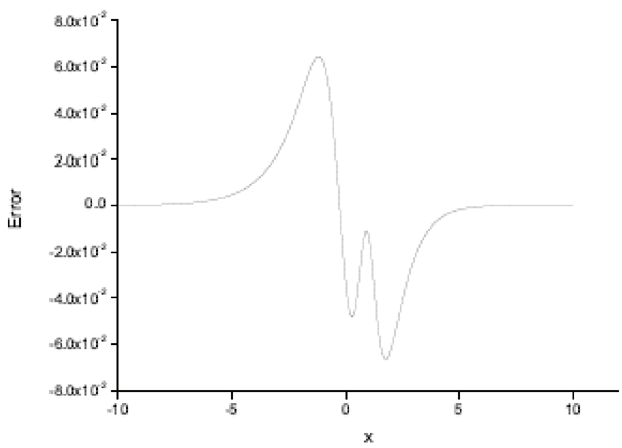


Figure 3. Error distributions for the parameters of Table 1 at t = 1.

Case 2

For the second numerical calculation, we consider the G-P equation with the parameters $\varepsilon = 1$, $h = 0.1$, $\kappa = 1$, $\alpha = -4$, $\beta = -1$, $c = 0.5$ and $\Delta t = 0.01$, to analyze the error norms and the quantities of the invariants for different space and time steps. In this case, single soliton has amplitude=0.83. Values of the invariants and error norms are presented in Table 2. Thus, we can see the effects of the amount of sorting points on the numerical method more easily. The values of invariants and errors hardly change as time progresses. It is observed from the Table 2 that our method is credible and efficient. In Figure 4, propagation of single solitary wave is displayed. Further, Figure 4 shows that the method we use in our article performs the propagation motion of a single solitary wave to the desired extent, moves at a constant speed, and maintains its shape and amplitude for a forward time as expected. Distribution of specific errors at time $t = 1$ are graphed in Figure 5.

Table 2. Invariant quantities and relevant error norms for Case 2.

t	I_1	I_2	L_2	L
0.0	-1.4907119030	.8281733851	.000000000	.000000000
0.1	-1.4907112084	.8261877045	.0192209295	.0128570987
0.2	-1.4907090396	.8202778182	.0387697276	.0260406734
0.3	-1.4907051404	.8105829002	.0589552705	.0396119351
0.4	-1.4906990801	.7973274456	.0800511183	.0537768219
0.5	-1.4906902479	.7808106929	.1022837858	.0688809698
0.6	-1.4906778456	.7613929741	.1258263864	.0848775401
0.7	-1.4906608774	.7394799982	.1507973202	.1018775502
0.8	-1.4906381394	.7155061610	.1772629349	.1199785118
0.9	-1.4906082057	.6899179608	.2052428023	.1392565064
1.0	-1.4905694141	.6631584850	.2347163210	.1603806847

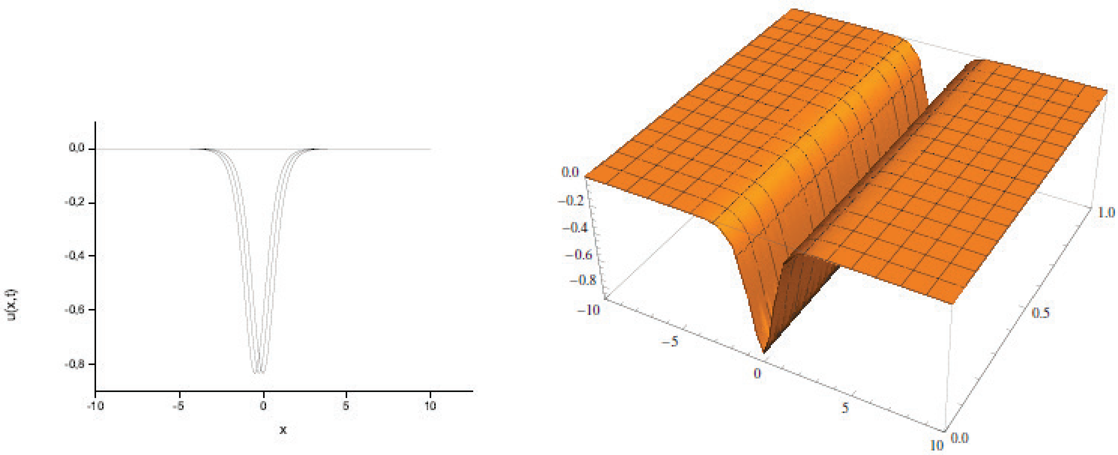


Figure 4. Behavior of single soliton for Case 2.

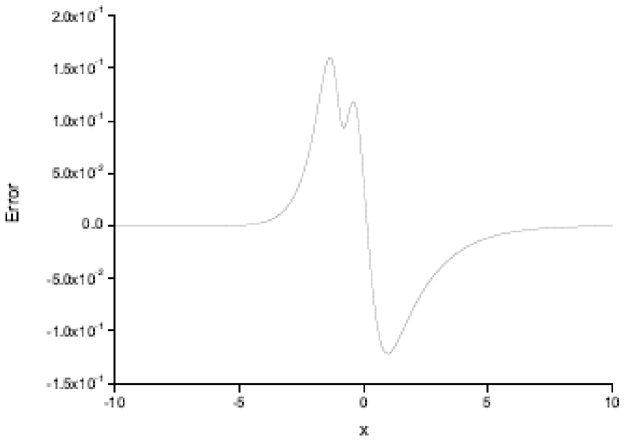


Figure 5. Error distributions for the parameters of Table 2 at t = 1.

CONCLUSION

In this paper, a finite element collocation scheme based on septic B-spline has been employed to investigate the propagation of non-linear dispersive solitary wave solutions of the G-P model. At first, we study Galerkin semi-discrete approximation for the equation. We have shown that our linearized scheme is unconditionally stable. In order to make numerical experiments, the algorithm has been studied together with the single solitary wave motion considering known analytical solution. The performance and validness of numerical algorithm has been measured by computing both L_2, L_∞ error norms and I_1, I_2 invariants. The results of the examples verify that our error norms are satisfactorily small. We conclude that the numerical scheme proposed here to approximate the solutions of G-P model is powerful, efficient and high accurate technique for solving a wide class of non-linear evolutional partial differential equations that arise in various scientific and engineering research. Moreover, the present approach of this study can be applied to other nonlinear evolution equations arising in different fields of nonlinear science.

AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

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