ABSTRACT

This manuscript aims to introduce the concept of orthogonal fuzzy b-metric-like spaces and discuss some fixed point results. A non-trivial example is imparted to illustrate the feasibility of the proposed methods. Finally, to validate the superiority of the obtained results, we provide an application to fractional differential equations.

INTRODUCTION

The concept of fuzzy sets was initiated by Zadeh [13], which gave a new aspect to research activity leading to the improvement of fuzzy systems. Afterward, several researchers contributed towards some basic significant results in fuzzy sets.

Kramosil and Michalek [11] introduced the concept of fuzzy metric spaces by generalizing the concepts of probabilistic metric spaces to fuzzy metric spaces. George and Veeramani [15] derived a Hausdorff topology initiated by a fuzzy metric to modify the concept of fuzzy metric spaces. Later on, fixed point theory via a fuzzy metric has been enriched with several different generalizations. Garbiec [26] displayed the fuzzy version of the Banach contraction principle in fuzzy metric spaces. For some necessary definitions, examples, and basic results, we refer to [13, 14] and the references therein.

As we know, fixed point theory plays a crucial role in proving the existence of solutions of different mathematical models and has a wide range of applications in different fields related to mathematics. This theory has intrigued many researchers. Recently, Harandi [17] initiated the concept of metric-like spaces, which generalizes the notion of metric spaces in a nice way. Alghamdi et al. [2] used the concept metric-like spaces to introduce the notion of b-metric-like spaces. Since then, several authors have...
worked on metric-like spaces and b-metric-like spaces. For more details, refer to [5-8]. In this sequel, Shukla and Abbas [2] generalized the concept of metric-like spaces and introduced fuzzy metric-like spaces. Recently, Javed et al. [27] introduced the concept of fuzzy b-metric-like spaces and they proved several fixed point results. For more details on this topic, see [1, 3, 4, 10, 11, 17, 20, 28-34]. Eshaghi Gordji et al. [21] introduced the concept of orthogonal sets. More details can be found in [22-25].

In this article, we aim to generalize the concept of fuzzy b-metric-like spaces by introducing orthogonal fuzzy b-metric-like spaces. We also prove some related fixed point results in the setting of orthogonal fuzzy b-metric-like spaces. Moreover, we give examples and an application of fractional differential equations to support our obtained results that shows the superiority of present notions in the existing literature.

First, we write some shortcut notations used throughout this paper: CTM for continuous triangular norm, BML for b-metric-like, FBM for fuzzy b-metric, FBML for fuzzy b-metric-like, and s.t. for such that.

**Definition 1.1.** [1] A binary operation $*: [0, 1] \times [0, 1] \rightarrow [0, 1]$ is called a CTM if it satisfies the following assertions:
1. $a * b = b * a$, $(\forall) a, b \in [0, 1]$;
2. $a * 1 = a$, $(\forall) a \in [0, 1]$;
3. $(a * b) * c = a * (b * c)$, $(\forall) a, b, c \in [0, 1]$;
4. If $a \leq b$ and $b \leq d$, with $a, b, c, d \in [0, 1]$, then $a * b \leq c * d$.

Some fundamental examples of a t-norm are $a * b = a \cdot b$, $a * b = \min \{a, b\}$ and $a * b = \max\{a + b - 1, 0\}$.

**Definition 1.2.** [2] A BML on a set $K \neq \emptyset$ is a function $\mu: K \times K \rightarrow [0, +\infty)$ so that for all $e, k, z \in K$ and $u \geq 1$, it satisfies the following conditions:
1. If $\mu(e, k) = 0 \Rightarrow e = k$;
2. $\mu(e, k) = \mu(k, e)$;
3. $\mu(e, k) \leq u[\mu(e, z) + \mu(z, k)]$.

The pair $(K, \mu)$ is called a BML space.

**Example 1.1.** [2] Let $K = [0, \infty)$. Define $\mu: K \times K \rightarrow [0, +\infty)$ by

$$\mu(e, k) = e + k.$$

Then $(K, \mu)$ is called a BML space with $u = 2$.

**Example 1.2.** [2] Let $K = [0, \infty)$. Define $\mu: K \times K \rightarrow [0, +\infty)$ by

$$\mu(e, k) = (\max\{e, k\})^2.$$

Then $(K, \mu)$ is a BML space with $u = 2$.

**Definition 1.3.** [3] A 3-tuple $(K, \Delta, \ast)$ is said to be an FML space if $K \neq \emptyset$ is a random set, $\ast$ is a CTM and $\Delta$ is a fuzzy set on $K \times K \times (0, \infty)$ meeting the points below for all $e, k, z \in K$, $r, s > 0$:

- FL1) $\Delta(e, k, r) > 0$;
- FL2) If $\Delta(e, k, r) = 1$, then $e = k$;
- FL3) $\Delta(e, k, r) = \Delta(k, e, r)$;
- FL4) $\Delta(e, z, r + s) \geq \Delta(e, k, r) \ast \Delta(k, z, s)$;
- FL5) $\Delta(e, k, r^+) : (0, \infty) \rightarrow [0, 1]$ is continuous.

**Example 1.3.** [3] Let $K = \mathbb{R}^+$, $p \in \mathbb{R}^+$ and $m > 0$. Define a t-norm by $a \ast b = ab$ and the fuzzy set $\Delta$ on $K \times K \times (0, \infty)$ by

$$\Delta(e, k, r) = \frac{pr}{pr + m(\max\{e, k\})^2}, \forall e, k \in K, r > 0.$$

Then $(K, \Delta, \ast)$ is an FML space.

**Definition 1.4.** [4] A 3-tuple $(K, \Delta, \ast)$ is said an FBM if $K$ is a random (non-empty) set, $\ast$ is a CTM and $\Delta$ is a fuzzy set on $K \times K \times (0, \infty)$ meeting the points below for all $e, k, z \in K$, $r, s > 0$ and a provided real number $u \geq 1$:

- B1) $\Delta(e, k, r) > 0$;
- B2) If $\Delta(e, k, r) = 1$, then $e = k$;
- B3) $\Delta(e, k, r) = \Delta(k, e, r)$;
- B4) $\Delta(e, z, r + s) \geq \Delta(e, k, r) \ast \Delta(k, z, s)$;
- B5) $\Omega(e, k, r^+) : (0, \infty) \rightarrow [0, 1]$ is continuous.

**Example 1.4.** [5] Let $\Delta(e, k, r) = e^{-\frac{mr}{r}}$, where $p > 1$ is a real number. It is then simple to show that $\Delta$ is an FBM with $b = 2^{-p-1}$.

**Definition 1.5.** [27] A 4-tuple $(K, \Delta, \ast, u)$ is named an FBML space if $K \neq \emptyset$ is a random set, $\ast$ is a CTM and $\Delta$ is a fuzzy set on $K \times K \times (0, \infty)$ meeting the following points below for all $e, k, z \in K$, $r, s > 0$:

- B1) $\Delta(e, k, r) > 0$;
- B2) If $\Delta(e, k, r) = 1$, then $e = k$;
- B3) $\Delta(e, k, r) = \Delta(k, e, r)$;
- B4) $\Delta(e, z, u(r + s)) \geq \Delta(e, k, r) \ast \Delta(k, z, s)$, for $u \in \mathbb{N}$; 
- B5) $\Omega(e, k, r^+) : (0, \infty) \rightarrow [0, 1]$ is continuous.

**Example 1.5** [27] Take $K = (0, \infty)$ . Given a t-norm as $a \ast b = ab$, then

$$\Delta(e, k, r) = \left[ e^{-\frac{(e+k)^2}{r}} \right]^{-1}, \forall e, k \in K, r > 0$$

is a FBML. But, it is not a FBM.

**Lemma 1.1.** [9] If for some $v \in (0, 1)$ and $e, k \in K$,

$$\Delta(e, k, r) \geq \Delta\left(e, k, \frac{r}{v}\right), \quad r > 0,$$

then $e = k$.

**Definition 1.6.** [21] Assume that $K \neq \emptyset$ and $\perp \in K \times K$ is a binary relation. Suppose there exists $e_0 \in K$ such that $e_0 \perp e$ or $e \perp e_0$ for all $e \in K$. Thus, we say that $K$ is an orthogonal set (O-set). Further, we denote an orthogonal set by $(K, \perp)$.

**Example 1.6.**
- Let $K = [0, \infty)$ and define $e \perp k$ if $ek = \min\{e, k\}$, then by putting $e_0 = 1$, $(K, \perp)$ is an O-set.
ii. Suppose $\Delta$ is the set of scalar matrices of order $2 \times 2$ with entries from natural numbers (i.e. $Q = [a\ b; c\ d]$ for all $a \in N$). Define the relation $\perp$ by $A \perp B$ if $\det(A) \leq \det(B)$.

Then by taking $A = l_1, (Q, \perp)$ is an O-set.

**Definition 1.7.** [21] Suppose that $(K, \perp)$ is an O-set. A sequence $\{e_n\}$ for all $n \in N$ is called an O-sequence if $(\forall n, e_{n+1} \perp e_n)$ or $(\forall n, e_{n+1} \perp e_n)$.

**Definition 1.8.** [21] A metric space $(K, d)$ is an orthogonal metric space if $(K, \perp)$ is an O-set. Further, $\zeta: K \to K$ is $\perp$-continuous at $e \in K$ if for each O-sequence $\{e_n\}$ for all $n \in N$ in $K$ so that $\lim_{n \to \infty} d(e_n, e) = 0$ then $\lim_{n \to \infty} d(\zeta e_n, \zeta e) = 0$.

Furthermore, $\zeta$ is $\perp$-continuous at each $e \in K$ if $\zeta$ is $\perp$-continuous at each $e \in K$. Also, $\zeta$ is $\perp$-preserving if $\zeta e \perp \zeta k$ such that $e \perp k$. Finally, $K$ is orthogonally complete (O-complete) if every Cauchy O-sequence is convergent.

**RESULTS AND DISCUSSIONS**

In this section, we introduce orthogonal BML spaces and orthogonal FBML spaces. We will prove some fixed point results in the class of orthogonal FBML spaces.

**Definition 2.1.** Let $K \neq \emptyset$ be an orthogonal set and $u \geq 1$. A function $d: K \times K \to \mathbb{R}^+$ is called orthogonal BML if it meets the below points: $\forall e, k, z \in K$,

- **OB1)** If $d(e, k) = 0 \Rightarrow e = k$ such that $e \perp k$ and $k \perp e$;
- **OB2)** $d(e, k) = d(k, e)$ such that $e \perp k$ and $k \perp e$;
- **OB3)** $d(e, k) \leq u[d(e, z) + d(z, k)]$ such that $e \perp z$, $z \perp k$ and $e \perp k$.

Then the set $K$ is named an orthogonal BML space and is denoted by $(K, d, u, \perp)$.

**Example 2.1.** Let $K = \mathbb{R}$. The set $K$ is orthogonal if $e \perp k$ iff $e \in \{[e], [k]\}$, where $e, k \in K$, and define $d(e, k) = (e + k)^p$ for all $e, k \in \mathbb{R}$. Clearly, $d$ is an orthogonal BML space. But, it is not a metric-like space. It suffices to take $e, k \in \mathbb{R}$ and an odd $p$, then clearly $d(e, k)$ is not in $\mathbb{R}^+$.

**Remark 2.1.** Every BML space is an orthogonal BML space, but the converse is not true.

**Definition 2.2.** A 5-tuple $(K, \Delta, * , u, \perp)$ is called an orthogonal FBML space if $K \neq \emptyset$ is a random orthogonal set $(K, \perp)$, * is a CTM and $\Delta$ is a fuzzy set on $K \times K \times \mathbb{R}^+$.

- **L1) $\Delta(e, k, r) > 0$, $\forall e, k \in K, r > 0$ such that $e \perp k$ and $k \perp e$;**
- **L2) $\Delta(e, k, r) = \Delta(k, e, r)$, $\forall e, k \in K, r > 0$ such that $e \perp k$ and $k \perp e$;**
- **L3) $\Delta(e, k, r) = \Delta(k, e, r)$, $\forall e, k \in K, r > 0$ such that $e \perp k$ and $k \perp e$;**
- **L4) $\Delta(e, z, u(r + s)) = \Delta(e, k, r) * \Delta(k, z, s)$, $\forall e, k, z \in K, r, s > 0$ such that $e \perp z$, $z \perp k$ and $e \perp k$;**
- **L5) $\Delta(e, k, r) = \Delta(k, e, r)$, $\forall e, k \in K, r > 0$ such that $e \perp k$ and $k \perp e$.**

Now, we shall show that $(K, \Delta, \ast, u)$ is not an FBML space. For $e, k \in K$, from (B4),

$$\Delta(e, z, u(r + s)) = \Delta(e, k, r) * \Delta(k, z, s), \forall e, k, z \in K, r, s > 0$$

We have

$$\frac{u(r+s)}{u(r+s)+(e+z)^p} \geq \frac{r}{r+(e+k)^p} \cdot \frac{s}{u(r+s)+(e+z)^p}$$

In particular, assume that $p = 3 = u$, $r = s = \frac{1}{2}$ and $e = k = z = -1$, then

$$\frac{3}{2+1} \geq \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{2}$$

which is a contradiction.

**Example 2.2.** Let $K = \mathbb{R}$ and define a t-norm as $a \ast b = ab$. Given a binary relation $\perp$ as $e \perp k$ iff $e, k \in \{[e], [k]\}$, then for all $e, k \in K, r > 0$ and $p$ belongs to odd positive integer,

$$\Delta(e, k, r) = \frac{r}{r+(e+k)^p}$$

is an orthogonal FBML. But, clearly it is not an FBML.

**Proof.** $(B_1, L1), (B_1, L2), (B_1, L3)$ and $(B_1, L5)$ are obvious. Here, we prove $(B_1, L4)$. For an arbitrary integer $u$, we know that

$$\frac{(e+z)^p}{r+r+(e+k)^p} \geq \frac{r}{r+(e+k)^p} \cdot \frac{s}{u(r+s)+(e+z)^p}$$

and define a t-norm as $a \ast b = ab$. Given a binary relation $\perp$ as $e \perp k$ iff $e, k \in \{[e], [k]\}$, then for all $e, k \in K, r > 0$ and $p$ belongs to odd positive integer,

$$\Delta(e, k, r) = \frac{r}{r+(e+k)^p}$$

is an orthogonal FBML. But, clearly it is not an FBML.

**Proof.** $(B_1, L1), (B_1, L2), (B_1, L3)$ and $(B_1, L5)$ are obvious. Here, we prove $(B_1, L4)$. For an arbitrary integer $u$, we know that

$$\Delta(e, k, r) = \frac{r}{r+(e+k)^p}$$

is an orthogonal FBML. But, clearly it is not an FBML.
Then \( \Delta \) is an orthogonal BML. This implies that \( e = k \).

Now, assume that \( \mu(e, k) = (e + k)^n \). Then

\[
\frac{\mu(e, z)}{r + s} \leq \frac{u[\mu(e, k) + \mu(k, z)]}{r + s}
\]

\[
\frac{\mu(e, z)}{u(r + s)} \leq \frac{\mu(e, k)}{r} + \frac{\mu(k, z)}{s}
\]

\[
\Rightarrow \Delta(e, zu(r + s)) \geq \Delta(e, k, r) + \Delta(k, z, s).
\]

Hence, (B.12) and (B.14) are satisfied and \((K, \Delta_{\ast}, u, \perp)\) is an orthogonal FBML space. Now, we prove that \((K, \Delta_{\ast}, u)\) is not an FBML space. From (B.4),

\[
\Delta(e, zu(r + s)) \geq \Delta(e, k, r) + \Delta(k, z, s), \forall e, k, z \in K, r, s > 0.
\]

We have

\[
es^{-p} \leq \frac{(e+k)^p}{r} \cdot \frac{(k+z)^p}{s}.
\]

In particular, assume that \( p = 3 \). Then

\[
s^{3/2} \geq \frac{(1-1)^3}{2} \cdot \frac{(1-1)^3}{2} = e^{32}.
\]

This is wrong.

**Remark 2.2.** Every FBML space is an orthogonal FBML space, but the converse is not true.

**Definition 2.3.** A sequence \( \{e_n\} \) in an orthogonal FBML space \((K, \Delta_{\ast}, u, \perp)\) is said to be convergent if \( e \in K \), if

\[
\lim_{n \to \infty} \Delta(e_n, e, r) = \Delta(e, e, r), \forall r > 0.
\]

**Definition 2.4.** A sequence \( \{e_n\} \) in an orthogonal FBML space \((K, \Delta_{\ast}, u, \perp)\) is said to be Cauchy if

\[
\lim_{n \to \infty} \Delta(e_n, e_{n+p}, r), \forall r > 0, p \geq 1
\]

exists and is finite.

**Definition 2.5.** \( \zeta : K \to K \) is \( \perp \)-continuous at \( e \in K \) in an orthogonal FBML space \((K, \Delta_{\ast}, u, \perp)\) if for each O-sequence \( \{e_n\} \) for all \( n \in N \) in \( K \), \( \lim_{n \to \infty} \Delta(e_n, e, r) \) exists and is finite for all \( r > 0 \), then \( \lim_{r \to \infty} \Delta(\zeta e_n, e, r) \) exists and is finite for all \( r > 0 \). Furthermore, \( \zeta \) is \( \perp \)-continuous if \( \zeta \) is \( \perp \)-continuous at each \( e \in K \). Also, \( \zeta \) is \( \perp \)-preserving if \( \zeta e \perp \zeta k \), whence \( e \perp k \). Finally, \( K \) is orthogonally complete (O-complete) if every Cauchy O-sequence is convergent.

**Definition 2.6.** An orthogonal FBML space \((K, \Delta_{\ast}, u, \perp)\) is said to be complete if every Cauchy sequence \( \{e_n\} \) in \( K \), converges to some \( e \in K \) such that

\[
\lim_{n \to \infty} \Delta(e_n, e, r) = \Delta(e, e, r), \forall r > 0, p \geq 1.
\]

**Definition 2.7.** Let \((K, \Delta_{\ast}, u, \perp)\) be an orthogonal FBML space. A map \( \zeta : K \to K \) is an orthogonal contraction if \( \exists q \in (0,1) \) such that for every \( r > 0 \) and \( e, k \in K \) with \( e \perp k \), we have

\[
\Delta(\zeta e, \zeta k, qr) \geq \Delta(e, k, r).
\]

**Theorem 2.1.** Assume \((K, \Delta_{\ast}, u, \perp)\) is an orthogonal complete FBML space such that

\[
\lim_{r \to \infty} \Delta(e, e, r) = 1, \forall e, k \in K.
\]

Let \( \zeta : K \to K \) be \( \perp \)-continuous, \( \perp \)-contraction and \( \perp \)-preserving. Then, \( \zeta \) has a unique fixed point \( e \in K \). Furthermore,

\[
\lim_{n \to \infty} \Delta(\zeta^n e, e, r) = \Delta(e, e, r), \forall e \in K \text{ and } r > 0.
\]

**Proof:** Since \((K, \Delta_{\ast}, u, \perp)\) is an Orthogonal complete FBML space, there exists \( e_0 \in K \) such that

\[
e_0 \perp k, \forall k \in K.
\]

That is, \( e_0 \perp \zeta e_0 \). Assume that

\[
e_0 = \zeta e_0 , e_0 = \zeta^2 e_0 = \zeta e_1 , \ldots, e_n = \zeta^n e_0, \forall n \in N.
\]

Since \( \zeta \) is \( \perp \)-preserving, \( \{e_n\} \) is an O-sequence. Now, since \( \zeta \) is an \( \perp \)-contraction, we can get

\[
\Delta(e_{n+1}, e_n, qr) \geq \Delta(\zeta e_n, \zeta e_{n-1}, qr) \geq \Delta(e_n, e_{n-1}, r)
\]

for all \( n \in N \) and \( r > 0 \). Note that \( \Delta \) is \( b \)-nondecreasing on \((0, \infty)\). Therefore, by applying the above expression, we can deduce

\[
\Delta(e_{n+1}, e_n, r) \geq \Delta(\zeta e_n, \zeta e_{n-1}, q) \geq \Delta(e_n, e_{n-1}, r)
\]

for all \( n \in N \) and \( r > 0 \). Thus, from (3), we have

\[
\Delta(e_{n+1}, e_n, r) \geq \Delta(e_n, e_{n-1}, q) \geq \Delta(e_{n-1}, e_{n-2}, q^2) \geq \Delta(e_{n-2}, e_{n-3}, q^3) \geq \cdots \geq \Delta(e_0, e_1, q^n)
\]

for all \( r > 0 \). Thus, from (3), we have

\[
\Delta(e_{n+1}, e_n, r) \geq \Delta(e_n, e_{n-1}, q) \geq \Delta(e_{n-1}, e_{n-2}, q^2) \geq \Delta(e_{n-2}, e_{n-3}, q^3) \geq \cdots \geq \Delta(e_0, e_1, q^n).
\]
Here, \( u \) is an arbitrary positive integer. We know that
\[
\lim_{n \to \infty} \Delta(e, k, r) = 1, \forall e, k \in K \text{ and } r > 0.
\]
Thus, from (4), we get
\[
\lim_{n \to \infty} \Delta(e_n, e_{n+p}, r) \geq 1 \cdot 1 \cdot ... \cdot 1 = 1.
\]
So, \( \{e_n\} \) is a Cauchy O-sequence. The hypothesis of O-completeness of the FBML space \((K, \Delta, u, \bot)\) ensures that there exists \( e \in K \) such that \( \Delta(e_n, e, r) \to 1 \) as \( n \to +\infty, \forall r > 0 \). Now, since \( \zeta \) is an \( \bot \)-continuous mapping, one writes \( \Delta(e_{n+1}, \zeta e, r) = \Delta(e_n, \zeta e, r) \to 1 \) as \( n \to +\infty \). Now, we have
\[
\Delta(e_n, \zeta e, r) \geq \Delta(e_n, e_{n+1}, r) \cdot \Delta(e_{n+1}, \zeta e, r).
\]
Taking limit as \( n \to +\infty \), we get \( \Delta(e_n, e_{n+1}, r) \to 1 \) and hence \( \zeta e = e \). Therefore, \( e \) is a fixed point of \( \zeta \) and \( \Delta(e_n, e, r) \to 1, \forall r > 0 \).

Now, we show the uniqueness of the fixed point of the mapping \( \zeta \). Assume that \( e \) and \( k \) are two fixed points of \( \zeta \) such that \( e \neq k \). One writes \( e_0 \perp e \) and \( e_0 \perp k \).

Since \( \zeta \) is \( \perp \)-preserving, we get
\[
\zeta^n e_0 \perp \zeta^n e \text{ and } \zeta^n e_0 \perp \zeta^n k.
\]
for all \( n \in \mathbb{N} \). So from (1), we can derive
\[
\Delta(\zeta^n e_0, \zeta^n e, r) \geq \Delta(\zeta^n e_0, \zeta^n e, qr) \geq \Delta(e_0, e, r/q^n)
\]
and
\[
\Delta(\zeta^n e_0, \zeta^n k, r) \geq \Delta(\zeta^n e_0, \zeta^n k, qr) \geq \Delta(e_0, k, r/q^n).
\]
Consequently,
\[
\Delta(e_n, k, r) = \Delta(\zeta^n e, \zeta^n k, r)
\]
\[
\geq \Delta(\zeta^n e_0, \zeta^n e, r/2u) \cdot \Delta(\zeta^n e_0, \zeta^n k, r/2u)
\]
\[
\geq \Delta(e_0, e, r/2uq^n) \cdot \Delta(e_0, k, r/2uq^n) \to 1 \quad \text{as } n \to +\infty.
\]
So, \( e = k \), hence \( e \) is the unique fixed point.

**Corollary 2.1.** Let \((K, \Delta, u, \bot)\) be an O-complete fuzzy \( \bot \)-metric space. Let \( \zeta : K \to K \) be an \( \bot \)-contraction and \( \bot \)-preserving. Also, assume that if \( \{e_n\} \) is an O-sequence with \( e_n \to e \in K \), then \( e \perp e_n \) for all \( n \in \mathbb{N} \). Then, \( \zeta \) has a unique fixed point \( e \in K \). Furthermore, \( \lim_{n \to \infty} \Delta(\zeta^n e, e, r) = \Delta(e, e, r) \) for all \( e \in K \) and \( r > 0 \).

**Proof:** We can similarly derive as in the proof of Theorem 2.1 that \( \{e_n\} \) is a Cauchy sequence and converges to \( e \in K \). Hence, \( e \perp e_n \) for all \( n \in \mathbb{N} \). From (1), we can get
\[
\Delta(\zeta e, e_{n+1}, r) = \Delta(\zeta e, \zeta e_n, r) \geq \Delta(\zeta e, \zeta e_n, r) \geq \Delta(e, e_n, r)
\]
and
\[
\lim_{n \to \infty} \Delta(\zeta e, e_{n+1}, r) = 1.
\]
Thus,
\[
\Delta(e, e, r) \geq \Delta(e_n, e_{n+1}, r) \cdot \Delta(e_{n+1}, e_{n+2}, r).
\]
Taking limit as \( n \to +\infty \), we get \( \Delta(e, e, r) = 1 \) and hence \( e = e \). The rest of proof is similarly as in Theorem 2.1.

**Example 2.4.** Let \( K = [-2, 2] \) and define a binary relation \( \bot \) by
\[
e \perp k \iff e, k \in [|e|, |k|].
\]
Define \( \Delta \) by
\[
\Delta(e, k, r) = e^{-\frac{(e+k)^3}{r}}, \text{ for all } e, k \in K \text{ and } r > 0.
\]
Take the \( t \)-norm: \( a \ast b = a \cdot b \). Then \( \Delta \) is an orthogonal complete FBML space, but it is not a FBML space. Also, observe that \( \lim_{r \to \infty} \Delta(e, k, r) = 1, \forall e, k \in K \).

Define \( \zeta : K \to K \) by
\[
\zeta(e) = \begin{cases} 
\frac{e}{4}, & e \in [-2, 0] \\
0, & e \in (0, 2].
\end{cases}
\]
Then, it satisfies the following:
1. If \( e \in [-2, 0] \text{ and } k \in (0, 2] \), then \( \zeta(e) = \frac{e}{4} \) and \( \zeta(k) = 0 \).
2. If \( e, k \in [-2, 0] \), then \( \zeta(e) = \frac{e}{4} \) and \( \zeta(k) = \frac{k}{4} \).
3. If \( e, k \in (0, 2] \), then \( \zeta(e) = \frac{e}{4} \) and \( \zeta(k) = 0 \).
4. If \( e \in (0, 2] \text{ and } k \in [-2, 0] \), then \( \zeta(e) = 0 \) and \( \zeta(k) = \frac{k}{4} \).

We have \( e \perp k \iff e, k \in [|e|, |k|] \). This implies that \( \zeta(e), \zeta(k) \in \{\zeta(e), |\zeta(k)|\} \). Hence \( \zeta \) is \( \perp \)-preserving. Let \( \{e_n\} \) be an arbitrary o-sequence in \( K \) that \( \{e_n\} \) converges to \( e \in K \). We have
\[
\lim_{n \to \infty} \Delta(e_n, e, r) = \lim_{n \to \infty} e^{-\frac{(e_n+e)^3}{r}} = \Delta(e, e, r)
\]
as \( \{e_n\} \) converges to \( e \). We can easily see that if \( \lim_{n \to \infty} \Delta(e_n, e, r) \) exists and is finite, then \( \lim_{n \to \infty} \Delta(\zeta e_n, \zeta e, r) \) exists and is finite for all \( e \in K \text{ and } r > 0 \). Hence, \( \zeta \) is
orthogonally continuous. But, ζ is not continuous. For this, take $e_n, e \in [-2, 0]$, so

$$\lim_{n \to \infty} \Delta(\zeta e_n, \zeta e, r) = \lim_{n \to \infty} \Delta\left(\frac{e_n}{\sqrt{4}} \frac{r}{\sqrt{4}}\right) = \lim_{n \to \infty} e^{-\frac{(e_n + e)^2}{4\sqrt{4}}}.$$ 

As $e_n \to e$ as $n \to \infty$ and taking $e = -2$, we have $\lim_{n \to \infty} \Delta(\zeta e_n, \zeta e, r) = e^{-r^2} > 1$, which is wrong. We have 4 cases for $q \in \left[\frac{1}{2}, 1\right)$.

Case 1) If $e \in [-2, 0]$ and $k \in (0, 2]$. Then $\zeta e = \frac{e}{4}$ and $\zeta k = \frac{k}{4}$. We have

$$\Delta(\zeta e, \zeta k, qr) = \Delta\left(\frac{e}{4}, \frac{k}{4}, qr\right) = e^{-\frac{(e - k)^2}{16qr}} \geq e^{-\frac{(e - k)^2}{r^2}} = \Delta(e, k, r).$$

Case 2) If $e, k \in [-2, 0]$, then $\zeta e = 0$ and $\zeta k = 0$. We have

$$\Delta(\zeta e, \zeta k, qr) = \Delta(0, 0, qr) = e^0 \geq e^{-\frac{(e - k)^2}{r^2}} = \Delta(e, k, r).$$

Case 3) If $e \in (0, 2]$ and $k \in [-2, 0]$, then $\zeta e = 0$ and $\zeta k = \frac{k}{4}$. Here,

$$\Delta(\zeta e, \zeta k, qr) = \Delta\left(0, \frac{k}{4}, qr\right) = e^{-\frac{k^2}{16qr}} \geq e^{-\frac{k^2}{r^2}} = \Delta(e, k, r).$$

Case 4) If $e \in (0, 2]$ and $k \in [-2, 0]$, then $\zeta e = 0$ and $\zeta k = \frac{k}{4}$. We have

$$\Delta(\zeta e, \zeta k, qr) = \Delta\left(0, \frac{k}{4}, qr\right) = e^{-\frac{\frac{e}{4} \frac{r}{\sqrt{4}q}}{qr}} \geq e^{-\frac{(e - k)^2}{r^2}} = \Delta(e, k, r).$$

From all 4 cases, we obtain that

$$\Delta(\zeta e, \zeta k, qr) \geq \Delta(e, k, r).$$

Hence, ζ is an orthogonal contraction. But, ζ is not a contraction. Indeed, taking $e = -2$ and $k = 1$, one gets

$$\Delta(\zeta e, \zeta k, qr) = e^{-\frac{(\frac{e}{4})^2}{\sqrt{4qr}}} = e^{-\frac{r}{4qr}} \geq 1.$$ 

This is wrong.

All the conditions of Theorem 2.1 are satisfied and ζ has a unique fixed point, which is 0.

**Theorem 2.2.** Assume that $(K, \Delta, *, u, \perp)$ is an orthogonal complete FBML space such that

$$\lim_{r \to \infty} \Delta(e, k, r) = 1, \forall e, k \in K and r > 0.$$ 

Let $\zeta: K \to K$ be an $\perp$-continuous, $\perp$-contraction and $\perp$-preserving. Suppose that there exist $q \in \left(0, \frac{1}{u}\right]$ and $r > 0$ such that

$$\Delta(\zeta e, \zeta k, qr) \geq \min\{\Delta(e, e, r), \Delta(\zeta k, k, r)\}$$ 

for all $e, k \in K, r > 0$. Then ζ has a unique fixed point $e_\perp \in K$.

**Proof.** Since $(K, \Delta, *, u, \perp)$ is an orthogonal complete FBML space, there exists $e_0 \in K$ such that

$$e_0 \perp k, \forall k \in K.$$ 

Thus, $e_0 \perp \zeta e_0$. Consider,

$$e_1 = \zeta e_0, e_2 = \zeta^2 e_0 = \zeta e_1, \ldots, e_n = \zeta^n e_0 = \zeta e_{n-1}, \forall n \in N.$$ 

Since $\zeta$ is $\perp$-preserving, $\{e_n\}$ is an O-sequence. Note that $\Delta$ is b-non-decreasing on $(0, \infty)$, so

$$\Delta(e_{n+1}, e_n, r) \geq \Delta(e_n, e_{n-1}, qr) = \Delta(\zeta e_n, \zeta e_{n-1}, qr) \geq \min\{\Delta(e_n, e_n, r), \Delta(\zeta e_{n-1}, e_{n-1}, r)\}.$$ 

Two cases occur:

Case 1. If $\Delta(e_{n+1}, e_n, r) \geq \Delta(e_n, e_{n-1}, qr)$, then

$$\Delta(e_{n+1}, e_n, r) = \Delta(e_n, e_{n-1}, qr) = \Delta(e_{n+1}, e_n, qr).$$ 

Then by Lemma 1.1, we get $e_n = e_{n+1}$ for all $n \in \mathbb{N}$ and $r > 0$.

Case 2. If $\Delta(e_{n+1}, e_n, r) \geq \Delta(e_n, e_{n-1}, qr)$, then

$$\Delta(e_{n+1}, e_n, r) \geq \Delta(e_n, e_{n-1}, qr) \geq \Delta(e_{n+1}, e_n, r) \geq \Delta(e_n, e_{n-1}, r)$$ 

for all $n \in \mathbb{N}$ and $r > 0$. Then by Theorem 2.1, $\{e_n\}$ is a Cauchy orthogonal sequence. By completeness of $(K, \Delta, *, u, \perp)$, there exists $e_\perp \in K$ such that

$$\lim_{n \to \infty} \Delta(e_n, e_\perp, r) = 1, \forall r > 0.$$ 

We know that $\zeta$ is an $\perp$-continuous mapping, then

$$\lim_{n \to \infty} \Delta(e_{n+1}, \zeta e_\perp, r) = \lim_{n \to \infty} \Delta(e_n, \zeta e_\perp, r) \to 1.$$ 

Now, we prove that $e_\perp$ is a fixed point for $\zeta$. Let $r_1 \in (qu, 1)$ and $r_2 = 1 - r_1$. Then

$$\Delta(e_\perp, e_\perp, r) \geq \Delta\left(e_\perp, e_{n+1}, \frac{r_1}{u}\right) \geq \Delta\left(e_\perp, e_n, \frac{r_2}{u}\right) \geq \Delta\left(e_\perp, e_n, \frac{r_1}{u}\right) \geq \Delta\left(e_\perp, e_n, \frac{r_2}{u}\right) \geq \min\left\{\Delta\left(e_\perp, e_\perp, \frac{r_1}{u}\right), \Delta\left(e_n, e_{n+1}, \frac{r_1}{u}\right)\right\} \geq \min\left\{\Delta\left(e_\perp, e_\perp, \frac{r_2}{u}\right), \Delta\left(e_n, e_{n+1}, \frac{r_2}{u}\right)\right\}.$$ 

Taking $n \to \infty$, we get
Here, and from Lemma 1.1, we have $\varepsilon e^* = e^*$. Let $e^*$ and $k^*$ be two different fixed points of $\varepsilon$. We have $e_0 \perp e^*$ and $e_0 \perp k^*$. Since, $\varepsilon$ is $\perp$-preserving, we can write $\varepsilon^n e_0 \perp \varepsilon^n e^*$ and $\varepsilon^n e_0 \perp \varepsilon^n k^*$ for all $n \in \mathbb{N}$.

We have

Then, it satisfies the following:

1. If $e, k \in [-2, \frac{2}{3}]$, then $\varepsilon(e) = \frac{e}{4}$ and $\varepsilon(k) = \frac{k}{4}$.
2. If $e, k \in \left[\frac{2}{3}, 1\right]$, then $\varepsilon(e) = 1 - e$ and $\varepsilon(k) = 1 - k$.
3. If $e, k \in (1, 2)$, then $\varepsilon(e) = e - \frac{1}{2}$ and $\varepsilon(k) = k - \frac{1}{2}$.
4. If $e \in [-2, \frac{2}{3}]$ and $k \in \left[\frac{2}{3}, 1\right]$, then $\varepsilon(e) = \frac{e}{4}$ and $\varepsilon(k) = 1 - k$.
5. If $e \in [-2, \frac{2}{3}]$ and $k \in (1, 2)$, then $\varepsilon(e) = e - \frac{1}{2}$ and $\varepsilon(k) = \frac{k}{4}$.
6. If $e \in \left[\frac{2}{3}, 1\right]$ and $k \in (1, 2)$, then $\varepsilon(e) = 1 - e$ and $\varepsilon(k) = k - \frac{1}{2}$.
7. If $e \in (1, 2)$ and $k \in \left[\frac{2}{3}, 1\right]$, then $\varepsilon(e) = e - \frac{1}{2}$ and $\varepsilon(k) = 1 - k$.
8. If $e \in (1, 2)$ and $k \in [-2, \frac{2}{3}]$, then $\varepsilon(e) = e - \frac{1}{2}$ and $\varepsilon(k) = \frac{k}{4}$.
9. If $e \in \left[\frac{2}{3}, 1\right]$ and $k \in [-2, \frac{2}{3}]$, then $\varepsilon(e) = 1 - e$ and $\varepsilon(k) = k - \frac{1}{2}$.

We have $e \perp k \iff e, k \in \{|e|, |k|\}$. This implies that $\varepsilon(e), \varepsilon(k) \in \{|\varepsilon(e)|, |\varepsilon(k)|\}$. Hence, $\varepsilon$ is $\perp$-preserving. Let $(e_n)$ be an arbitrary o-sequence in $K$ so that $|e_n|$ converges to $e \in K$. We have

We can easily see that if $\lim_{n \to \infty} \Delta(e_n, e, r)$ exists and is finite, $\lim_{n \to \infty} \Delta(e_n, e, r)$ also exists and is finite for all $e \in K$ and $r > 0$. Hence, $\varepsilon$ is orthogonal continuous. But, $\varepsilon$ is not continuous. For this, take $e_n \to e$ as $n \to \infty$ and taking $e = -2$ and $r = \frac{1}{8}$, we have

As $e_n \to e$ as $n \to \infty$ and taking $e = -2$ and $r = \frac{1}{8}$, we have

This is wrong. Hence, all the conditions of Theorem 2.2 are satisfied and 0 is the unique fixed point of $\varepsilon$.

**APPLICATION**

Within this part, we apply Theorem 2.1 to investigate the existence and uniqueness of a solution of a nonlinear fractional differential equation (see [18]) given by

$$D^\alpha e(t) = f(t, e(t)) \quad (t \in (0, 1), \alpha \in (1, 2))$$

with boundary conditions

$$e(0) = 0, e'(0) = Ie(t) \quad (t \in (0, 1)),$$
where $D^\alpha_f$ is the Caputo fractional derivative of order $\alpha$ defined by
\[
D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^n(s) \, ds
\]

and $f: [0,1] \times \mathbb{R} \to \mathbb{R}^+$ is a continuous function. Let $K = C([0,1], \mathbb{R})$ be endowed with the supremum $|e| = \sup_{t \in [0,1]} |e(t)|$.

The Riemann-Liouville fractional integral of order $\alpha$ (see [19]) is given by
\[
I^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f(s) \, ds \quad (\alpha > 0).
\]

**Theorem 3.1.** Assume that
i. $f: [0,1] \times \mathbb{R} \to \mathbb{R}^+$ is a continuous function,
ii. $e(t): [0,1] \to \mathbb{R}$ is continuous,
So that
\[
|f(t, e) + f(t, k)| \leq L|e + k|
\]
for all $t \in [0,1]$ and for all $e, k \in K$ such that $e(t) + k(t) \geq 0$. $L$ is a constant with $L/L < 1$ where
\[
L = \frac{1}{\Gamma(\alpha + 1) + \frac{2k^{\alpha+1} \Gamma(\alpha)}{(2-k^2) \Gamma(\alpha + 1)}}
\]
Then the differential equation (3) has a unique solution.

**Proof.** We take the following orthogonal relation on $K$:
\[
eq k \iff e(t) + k(t) \geq 0 \quad \forall \ t \in [0,1]
\]
Also, we take
\[
\Delta(e, k, r) = e^{\frac{(e(t) + k(t))}{r}}.
\]
For all $e, k \in K$, we consider $|e + k| = \sup_{t \in [0,1]} |e(t) + k(t)|$.
(K, $\Delta$, $\ast$, $u$, $\perp$) is a complete orthogonal fuzzy BML space. Observe that it is not a fuzzy BML space. We define a mapping $\zeta: K \to K$ by
\[
\zeta(e)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, e(s)) \, ds
+ \frac{2t}{(2-k^2) \Gamma(\alpha)} \int_0^k \int_0^s (s-m)^{\alpha-1} f(m, e(m)) \, dm \, ds
\]
for all $t \in [0,1]$. Note that the equation (3.1) has a solution $e \in K$ iff $e(t) = \zeta(e)(t)$ for all $t \in [0,1]$. To check the existence of a fixed point of $\zeta$, we are going to show that $\zeta$ is $\perp$-preserving, $\perp$-contraction and $\perp$-continuous.

For all $t \in [0,1]$, $e(t) \perp k(t)$ means that $e(t) + k(t) \geq 0$. Clearly, from (4), we have $\zeta(e(t) + k(t)) \geq 0$. It implies that $\zeta(e(t) \perp k(t))$. Hence, $\zeta$ is $\perp$-preserving. For all $t \in [0,1]$ and $e(t) \perp k(t)$, we get
\[
\Delta(\zeta e, \zeta k, r) = e^{\frac{(\zeta e(t) + \zeta k(t))^3}{r}}.
\]
Also,
\[
\zeta e(t) + \zeta k(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, e(s)) \, ds
+ \frac{2t}{(2-k^2) \Gamma(\alpha)} \int_0^k \int_0^s (s-m)^{\alpha-1} f(m, e(m)) \, dm \, ds
+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, k(s)) \, ds
+ \frac{2t}{(2-k^2) \Gamma(\alpha)} \int_0^k \int_0^s (s-m)^{\alpha-1} f(m, k(m)) \, dm \, ds.
\]
From the fact that $e(t) + k(t) \geq 0$, we can take $e(t) + k(t) = |e(t) + k(t)|$, since $\zeta$ is $\perp$-preserving, which means that $\zeta(e(t) + k(t)) = |\zeta(e(t) + k(t))|$. We have
\[
|\zeta(e(t) + k(t))| = \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, e(s)) \, ds
+ \frac{2t}{(2-k^2) \Gamma(\alpha)} \int_0^k \int_0^s (s-m)^{\alpha-1} f(m, e(m)) \, dm \, ds
+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, k(s)) \, ds
+ \frac{2t}{(2-k^2) \Gamma(\alpha)} \int_0^k \int_0^s (s-m)^{\alpha-1} f(m, k(m)) \, dm \, ds \right|
\]
\[
\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, e(s)) + f(s, k(s))| \, ds
+ \frac{2t}{(2-k^2) \Gamma(\alpha)} \int_0^k \int_0^s (s-m)^{\alpha-1} |f(m, e(m)) + f(m, k(m))| \, dm \, ds
\]
\[
\leq \frac{4t \Gamma(\alpha) + k}{(2-k^2) \Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, e(s)) + f(s, k(s))| \, ds
+ \frac{4t \Gamma(\alpha) + k}{(2-k^2) \Gamma(\alpha)} \int_0^k \int_0^s (s-m)^{\alpha-1} |f(m, e(m)) + f(m, k(m))| \, dm \, ds
\]
\[
\leq \frac{4t \Gamma(\alpha) + k}{(2-k^2) \Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, e(s)) + f(s, k(s))| \, ds
+ \frac{4t \Gamma(\alpha) + k}{(2-k^2) \Gamma(\alpha)} \int_0^k \int_0^s (s-m)^{\alpha-1} |f(m, e(m)) + f(m, k(m))| \, dm \, ds
\]
From the fact $L/L < 1$ and (5), we get
\[
\Delta(\zeta e, \zeta k, r) = e^{\frac{(\zeta e(t) + \zeta k(t))^3}{r}} \geq e^{\frac{(e(t) + k(t))^3}{r}} = \Delta(e, k, r).
\]
It implies that $\zeta$ is an $\perp$-contraction.

Suppose that $\{e_n\}$ is an O-sequence in $K$ such that $\{e_n\}$ converges to $e \in K$. Because $\zeta$ is $\perp$-preserving, $\{\zeta e_n\}$ is an O-sequence for each $n \in \mathbb{N}$. Also, because $\zeta$ is an $\perp$-contraction, we have

$$\Delta(\zeta e_n(t), \zeta e(t), qr) \geq \Delta(e_n(t), e(t), r).$$

As $\lim_{n \to \infty} \Delta(e_n(t), e(t), r)$ exists and is finite for all $r > 0$, it is clear that $\lim_{n \to \infty} \Delta(\zeta e_n(t), \zeta e(t), qr)$ exists and is finite.

Hence, $\zeta$ is $\perp$-continuous. Thus, all the conditions of Theorem 2.1 are satisfied, and so $e(t)$ is the unique fixed point of $\zeta$.

**CONCLUSION**

In this manuscript, we introduced the notion of orthogonal fuzzy b-metric like spaces as a combination of orthogonal sets and fuzzy b-metric like spaces. This new setting has many applications and opens the door to extend and generalize some known related fixed point results in (fuzzy) b-metric like spaces. At the end, we solve a fractional differential equation and we gave some concrete examples illustrating the new concepts. This work can be extend in the structure of orthogonal control fuzzy metric-like spaces, Intutionistic fuzzy b-metric-like spaces, neutrosophic metric-like spaces etc.

**AUTHORSHIP CONTRIBUTIONS**

Authors equally contributed to this work.

**DATA AVAILABILITY STATEMENT**

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

**CONFLICT OF INTEREST**

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

**ETHICS**

There are no ethical issues with the publication of this manuscript.

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