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Research Article

Observation of experts, attitudes through multi-criteria decision-making

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ABSTRACT

The aim of this paper is to obtain optimum fuzzy soft constants through multi-criteria decision-making approaches. TOPSIS and VIKOR are utilized for this purpose and results are compared with those obtained through Bonferroni mean. The hesitant fuzzy soft set is taken as initial data in decision-making methods. Hesitant fuzzy Bonferroni means and distance measures for TOPSIS and VIKOR are calculated in the structure of hesitant fuzzy set and hesitant fuzzy soft set. OFSCs are chosen from the constants which rank the alternatives in the decision-making process. By using the system of linear differential equations based on OFSCs, the future approach of people with respect to their decisions is analyzed and is observed through phase portrait and a line graph of that system of differential equations. To explain the proposed idea, explanative examples for two techniques are also given. These examples illustrate that if two persons select the two different alternatives, they do not favor each other after that decision.

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INTRODUCTION

Fuzzy set theory was instigated by Zadeh [29] which provides a suitable framework to model several problems with uncertain and ambiguous data. It is identified by a function defined on a set and takes values in the interval [0,1]. The value of a function determines the degree of membership of an element in the universe of discourse. There are certain situations where fuzzy set theory cannot be effectively employed. To overcome difficulties arising to model the problems that could not be modeled with the environment of fuzzy set theory, Molodtsov [13] introduced soft set which is a set associated with a set of parameters. Maji et al. [15] combined the concepts of soft sets and fuzzy sets and defined fuzzy soft sets (FSS). FSS theory is more applicable in intelligent systems, identification problems, pattern recognition, optimization, and control theory. Roy and Maji [19] successfully applied the fuzzy soft set theory to decision-making problems.

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Hesitancy is one of the most important factors which stops us from taking decisions at the right time. So it is very important to incorporate a hesitancy attitude to model decision-making processes. Hesitant fuzzy set (HFS) is a generalization of a fuzzy set which was introduced by Torra [22]. HFS can be more accurately reflect the people' hesitancy in stating their preferences about objects as compared to the fuzzy set. Xia and Xu [26] presented a series of aggregation operators for the elements of HFS. In recent times, some applications of HFS were represented by Faruk and Serif [12,16]. Hesitant fuzzy multi-criteria decision-making (MCDM) method was presented by Hu et al. [9]. Beg and Rashid [2] used HFS to model 2-tuple linguistic information. Some applications of the hesitant fuzzy soft set (HFSS) in MCDM were studied by Wang et al. [24].

MCDM facilitates the decision-making process when the situation of selecting the best alternative is complicated. Some of the methods are surveyed by Aruldoss et al. [1]. A multi-criteria decision analysis method called TOPSIS, an abbreviation of "The technique for order of preference by similarity to ideal solution" is developed by Hwang and Yoon [8]. Simplicity, rationality, intelligibility and good computational efficiency are some advantages of TOPSIS. TOPSIS is based on distances between two sets or elements and the obtained solution is close to the ideal situation and far from the worst situation. Xu and Xia [25] presented some basic distance and similarity measures for hesitant fuzzy sets. Chen [5] extended the TOPSIS for solving multi criteria decision making problems in the framework of fuzzy sets (see also, [3]). Moreover, this multi-criteria method is utilized by many researchers [7,6,18,10] to solve the decision-making problems based on the fuzzy soft set, intuitionistic fuzzy soft set and hesitant fuzzy soft set theories. Another MCDM method is VIKOR which is also based on "closeness to the ideal", and is the compromised ranking method. A comparative analysis of VIKOR and TOPSIS is presented by Opricovic and Tzeng [17]. The multi-criteria aggregation function is introduced by Yager [28] which is initially defined as Bonferroni mean (BM) which is a mean type aggregation operator. BM facilitates to cater inter-related arguments of individuals in the group decision-making process. Hesitant fuzzy Bonferroni mean (HFBM) is a BM that is calculated for hesitant fuzzy elements as defined by Zhu et al. [30]. By using BM and weighted Bonferroni mean (WBM) based on FSS, Beg et al. [4] found the optimum fuzzy soft constants (OFSCs). By using OFSCs, they developed a system of fuzzy soft differential equations and discussed different cases observed after the decision to analyze the human attitude. These different cases arise according to satisfaction or dissatisfaction, encouragement or discouragement of other people for the decisions. Solutions of system of linear differential equations, their stability, phase portraits have been discussed by Strogatz [21]. He also presented a dynamic model of love which was also discussed by Sprott [20]. A variety of research articles is published on decision-making approaches that rank a finite set of alternatives. However, no work except [4,11,23,14] has been done on what happens after a decision has been taken. There is a gap between the development of a system of linear equations and MCDM which is carried through in this research by utilizing TOPSIS and VIKOR. For this purpose, hesitant fuzzy soft matrices are considered as initial data to rank the alternatives. Then a system of differential equations is developed by using the obtained OFSCs to analyze the human attitude.

PRELIMINAries

Let us recall some basic definitions and known results.

Definition 1

[9] Let U be a universe of discourse. The hesitant fuzzy set A on U is identified by a function h_A on U that returns a subset of [0,1].

In the sequel, by hesitant fuzzy set, we mean a discrete hesitant fuzzy set where each $h_A(x)$ is represented as a set of finite values, $\{x_1, x_2, \dots, x_n\}$.

The following is an example of discrete hesitant fuzzy set.

Example 1

A bank manager intimates five different options to a customer who wants to open a bank account. Let U = $\{u_1,u_2,u_3,u_4,u_5\}$ be a universe of discourse, where u_1, u_2, u_3, u_4 and u_5 represent saving account, basic checking account, interest bearing checking account, money-market deposit account and certificate of deposit, respectively. A hesitant fuzzy set h over U can be represented as:

$$h = \left\{ {}^{\{0.2,0.3,0.5\}}/_{u_1}, {}^{\{0.4,0.8\}}/_{u_2}, {}^{\{0.2,0.3,0.6\}}/_{u_3}, {}^{\{0.5,0.6\}}/_{u_4}, {}^{\{0.3,0.4,0.6,0.7\}}/_{u_5} \right\}$$

Here one of the hesitant fuzzy element

$$\left\{ {}^{\{0.2,0.3,0.5\}}/_{u_1} \right\}$$

represents that there are three possible degrees of membership 0.2, 0.3 and 0.5 for saving account which can be interpreted as some hesitancy.

Definition 2

[24] Let H(U) be the set of all hesitant fuzzy sets in U and A⊆E (a set of parameters). A pair (F,A) is called a hesitant fuzzy soft set over U, where F is a mapping given by F:A⟶H(U).

Example 2

Let U=$\{u_1, u_2, u_3, u_4, u_5\}$ be the same set as given in above example and a set E=$\{e_1, e_2, e_3\}$ of parameters representing emergency cash, interest rate and credit card facility, respectively. Then a hesitant fuzzy soft set can be described as

$$F(e_1) = \left\{ {}^{\{0.2,0.5\}}/_{u_1}, {}^{\{0.1,0.7\}}/_{u_2}, {}^{\{0.1,0.3,0.6\}}/_{u_3}, {}^{\{0.4,0.5,0.9\}}/_{u_4}, {}^{\{0.1,0.5\}}/_{u_5} \right\}$$
$$F(e_2) = \left\{ {}^{\{0.2,0.4\}}/_{u_1}, {}^{\{0.3,0.4,0.7\}}/_{u_2}, {}^{\{0.1,0.3,0.6\}}/_{u_3}, {}^{\{0.4,0.5\}}/_{u_4}, {}^{\{0.3,0.5,0.7,0.9\}}/_{u_5} \right\}$$
$$F(e_3) = \left\{ {}^{\{0.2\}}/_{u_1}, {}^{\{0.3,0.4,0.7\}}/_{u_2}, {}^{\{0.1,0.3,0.6,0.8\}}/_{u_3}, {}^{\{0.4,0.5\}}/_{u_4}, {}^{\{0.3,0.5,0.8\}}/_{u_5} \right\}$$

Here, one of the particular element

$$F(e_1) = {}^{\{0.2,0.5\}}/_{u_1}$$

represents that a value {0.2,0.5} is assigned to saving account based on the criteria of emergency cash. The hesitant fuzzy soft set given above can be written in matrix form or tabular form as shown in Table 1

Table 1. Representation of Hesitant Fuzzy Soft Set

	e_1	e_2	e_3
u_1	{0.2,0.5}	{0.2,0.4}	{0.2}
u_2	{0.1,0.7}	{0.3,0.4,0.7}	{0.3,0.4,0.7}
u_3	{0.1,0.3,0.6}	{0.1,0.3,0.6}	{0.1,0.3,0.6,0.8}
u_4	{0.4,0.5,0.9}	{0.4,0.5}	{0.4,0.5}
u_5	{0.1,0.5}	{0.3,0.5,0.7,0.9}	{0.3,0.5,0.8}

To apply the decision-making methods while taking HFSS as initial data, it is essential to define distance and similarity measures. Some distance measures between two hesitant fuzzy sets (HFSs) have been defined by Xu and Xia [25]. Hesitant weighted hamming distance between two hesitant fuzzy sets is defined as:

Definition 3

[25] Let M and N be two HFSs on the universe X={x_1, x_2, ... , x_n} and the weight of each element x_i∈X is w_i, where i∈{1, 2, ... , n}, w_i∈[0,1] and $\sum_{i=1}^{n} w_i = 1$ then the hesitant weighted hamming distance between two hesitant fuzzy sets is given by

$$d_{hwh}(M,N) = \sum_{i=1}^{n} w_i \left[\frac{1}{l_{x_i}} \sum_{j=1}^{l_{x_i}} \left| h_M^{\sigma(j)}(x_i) - h_N^{\sigma(j)}(x_i) \right| \right]$$

where

$$l_{x_i} = max(|h_M(x_i)|, |h_N(x_i)|),$$

$h_M^{\sigma(j)}(x_i)$ and $h_N^{\sigma(j)}(x_i)$ represent the jth largest value in $h_M^{\sigma(j)}(x_i)$ and $h_N^{\sigma(j)}(x_i)$ respectively.

$d_{hwh}(M,N)$ cannot be calculated when $l(h_M(x_i)) \neq l(h_M(x_i))$. In this case, a shorter one is extended and a value is added several times in it. This value depends on the decision-makers' risk preferences. Optimists add the maximum value and pessimists add the minimum value.

Definition 4

[4] Let l,m be two natural numbers and $x_i \geq 0$ where $i \in \{1,2,\ldots,n\}$ then Bonferroni mean $B^{l,m}$ is defined as

$$B^{l,m} = \left(\frac{1}{n(n-1)} \sum_{i,j=1, i\neq j}^{n} x_i^l x_j^m \right)^{\frac{1}{l+m}}$$

Definition 5

[4] Let l, m be two natural numbers and $x_i \geq 0$ ($i = 1,2,\ldots,n$) and $w_i(\geq 0)$ be the weights for x_i with the condition $\sum_{i=1}^{n} w_i = 1$, then weighted Bonferroni mean (WBM) is defined as

$$WB^{l,m} = \left(\frac{1}{n(n-1)} \sum_{i,j=1, i\neq j}^{n} (w_i x_i)^l (w_j x_j)^m \right)^{\frac{1}{l+m}}$$

Definition 6

[30] For a hesitant fuzzy element (HFE) h,

$$s(h) = \frac{1}{\#h} \sum_{\gamma \in h} \gamma$$

is called a score function of h, where #h is the number of elements in h. Thus a score function of a hesitant fuzzy element gives an average value of all elements in h.

Consistent with [27], some operational laws for any three hesitant fuzzy elements h,h_{1}, h_{2} and a scalar λ are defined as:

1. $h^\lambda = \cup_{\gamma \in h} \{\gamma^\lambda\}$
2. $\lambda h = \cup_{\gamma \in h} \{1 - (1-\gamma)^\lambda\}$
3. $h_1 \oplus h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\gamma_1 + \gamma_2 - \gamma_1\gamma_2\}$
4. $h_1 \otimes h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\gamma_1\gamma_2\}$

Definition 7

[30] Let h_1, h_2, ... , h_n be hesitant fuzzy elements and l,m be two natural numbers, then hesitant fuzzy Bonferroni mean HFBM is defined as

$$HFB^{l,m}(h_1, h_2, \ldots, h_n) = \left(\frac{1}{n(n-1)} \left(\bigoplus_{i,j=1, i\neq j}^{n} h_i^l \otimes h_j^m \right) \right)^{\frac{1}{l+m}}$$

Definition 8

[30] Let h_1, h_2, ... , h_n be hesitant fuzzy elements with weight vector W=[w_1, w_2, ... , w_n] for which $\sum_{i=1}^{n} w_i = 1$

and l,m two natural numbers, then weighted hesitant fuzzy Bonferroni mean WHFBM is defined as

$$WHFB^{l,m}(h_1,h_2,\dots,h_n)=\left(\frac{1}{n(n-1)}\left(\bigoplus_{i,j=1,i\neq j}^{n}(w_ih_i)^l\otimes(w_jh_j)^m\right)\right)^{\frac{1}{l+m}}$$

Definition 9

[3] Hesitant fuzzy soft sets $R^l=\left[\left(x_{ij}^l\right)\right],(l=1,2,\dots,K)$ can be aggregated by a matrix X, where $X=\left[\left(x_{ij}\right)\right]$, $x_{ij}=\left\{x\left|x\in x_{ij}^{(l)}\,and\; r_{p_{ij}}\leq x\leq r_{q_{ij}}for\; all\; l\right.\right\}$ with

$$r_{p_{ij}}=\min\left\{\min_{l=1}^{K}(\max x_{ij}^{(l)}),\max_{l=1}^{K}(\max x_{ij}^{(l)})\right\}$$

$$\text{and } r_{q_{ij}}=\max\left\{\min_{l=1}^{K}(\max x_{ij}^{(l)}),\max_{l=1}^{K}(\max x_{ij}^{(l)})\right\}$$

Beg et al. [4] considered the following system of linear fuzzy soft differential equations

$$\frac{dP_1}{dt}=-(0.8198P_1+0.8263P_2)$$
$$\frac{dP_2}{dt}=-(0.8198P_1+0.8263P_2) \quad (1)$$

where P_1 and P_2 are two variables which represent the attitude of two persons after taking a decision at the time t. While dP_1/dt and dP_1/dt represent the change in persons' attitudes after some time due to that decision and $a_{P_j}^i$ (i,j=1,2) are optimum fuzzy soft constants (OFSCs) taken as signed fuzzy numbers which denote the influence on *ith* person of his internal feelings and *jth* person's feelings. Positive and negative sign is assigned to $a_{P_j}^i$ according to encouragement or discouragement of a person about a decision. Stability of system (1) depends upon eigen values of the matrix

$$\begin{bmatrix} a_{P_1}^1 & a_{P_2}^1 \\ a_{P_1}^2 & a_{P_2}^2 \end{bmatrix}$$

Analysis of the Future Attitudes of Experts Based on Fuzzy Soft Differential Equations

In this section, we develop the algorithms to analyze the human attitude of two persons after taking a decision, which are based on the following MCDM methods

- Bonferroni mean
- TOPSIS
- VIKOR

where hesitant fuzzy soft sets are considered as initial data.

An attitude analyzer method through hesitant fuzzy Bonferroni mean

This method consists of the following steps:

Step 1: Let $A=\{A_1, A_2,\dots, A_n\}$ be a set of objects/alternatives and $B=\{B_1, B_2,\dots, B_n\}$ be a set of attributes/criteria. Decision advisors present their opinions in the form of $m\times n$ hesitant fuzzy soft sets $D_1, D_2,\dots, D_p$.

Step 2: Calculate Bonferroni hesitant fuzzy soft set (BHFSS) $B_{m\times n}$ by applying definition 7 on $D_1, D_2,\dots, D_p$ and then calculate $B'_{m\times n}$ by using definition 6 whose entries are the score functions of respective entries of $B_{m\times n}$.

Step 3: Consider a weight vector $W=\{w_1, w_2, \dots, w_n\}$ such $w_n>0$ for j=1,2,...,n and $\sum_{j=1}^{n}w_j=1$. Then weighted Bonferroni fuzzy soft set (WBFSS) $C_{m\times 1}=[c_{i,1}]$, is computed where $c_{i,1}$, i=1,2,...,m are calculated by using definition 8 for *ith* entries of the set $B'_{m\times n}$.

Step 4: Optimal fuzzy soft constants a_j^i (i,j=1,2) are chosen from $c_{i,1}$, i=1,2,...,m. If two persons select same alternative, then

$$a_{P_j}^i=\begin{cases} max(c_{j,1}) & if \quad max\left(max(c_{j,1}),max(1-c_{j,1})\right)=max(c_{j,1}) \\ -max(1-c_{j,1}) & if \quad max\left(max(c_{j,1}),max(1-c_{j,1})\right)=max(1-c_{j,1}) \end{cases}$$

If two persons select different alternatives, then

$$a_{P_j}^i=\begin{cases} max(c_{j,1}) & if \quad i=j \\ -max(1-c_{j,1}) & if \quad i\neq j \end{cases}$$

Step 5: Develop a system of fuzzy soft differential equations as defined in [4] and check its stability to determine the next viewpoint of two persons.

In the following example, we discuss one of the cases discussed in [21] and draw phase portrait and analyze the human attitude of two persons after taking a decision.

Example 3

There are five different tiles and sanitary packages A_1, A_2, A_3, A_4, and A_5 in a store. Mr. Ali and Mr. Amir have to choose one of them for the construction of their houses. For this purpose they hire three contractors who give their feedback in connection with three attributes: B_1: beauty, B_2: long lasting, B_3: strength. We have to analyze their future attitude due to their decisions.

Step 1: The fuzzy soft sets representing the experts' opinions are given in Table 2.

Step 2: Using definition 7 with l=m=1, BHFSS $B_{5\times 3}$ is calculated as shown in Table 3.

and score functions of respective entries of $B_{5\times 3}$ are shown in Table 4. Calculation of the element of first row and first column of B is as follows:

Table 2. Hesitant Fuzzy Soft Set

	D_1	D_2	D_3
	B_1	B_1	B_1
	B_2	B_2	B_2
	B_3	B_3	B_3
A_1	{0.1,0.4}	{0.5}	{0.6}
	{0.2}	{0.4,0.7}	{0.7}
	{0.8}	{0.7}	{0.8,0.9}
A_2	{0.7}	{0.4}	{0.2,0.5}
	{0.3,0.5}	{0.6}	{0.8}
	{0.4}	{0.6}	{0.8}
A_3	{0.7}	{0.5,0.6}	{0.4}
	{0.4}	{0.5}	{0.2,0.6}
	{0.7,0.9}	{0.7}	{0.9}
A_4	{0.6}	{0.7}	{0.4,0.8}
	{0.5,0.6}	{0.7}	{0.4}
	{0.5}	{0.8}	{0.6}
A_5	{0.4}	{0.2,0.5}	{0.6}
	{0.1,0.3}	{0.4}	{0.3}
	{0.1,0.2}	{0.4}	{0.7}

$$HFB^{1,1}(\{0.1,0.4\},\{0.5\},\{0.6\}) = \left(\frac{1}{6}\begin{pmatrix}(\{0.1,0.4\}\otimes\{0.5\})\oplus(\{0.5\}\otimes\{0.1,0.4\})\\ \oplus(\{0.1,0.4\}\otimes\{0.6\})\oplus(\{0.6\}\otimes\{0.1,0.4\})\\ \oplus(\{0.5\}\otimes\{0.6\})\oplus(\{0.6\}\otimes\{0.5\})\end{pmatrix}\right)^{\frac{1}{2}}$$

$$= \left(\frac{1}{6}(\{0.05,0.20\}\oplus\{0.05,0.20\}\oplus\{0.06,0.24\}\oplus\{0.06,0.24\}\oplus\{0.30\}\oplus\{0.30\})\right)^{\frac{1}{2}}$$

$$= \left(\frac{1}{6}(\{0.0975,0.249,0.36\}\oplus\{0.1164,0.2856,0.4224\}\oplus\{0.51\})\right)^{\frac{1}{2}}$$

$$= \left(\frac{1}{6}(\{0.2025,0.3552,0.4787,0.3364,0.4634,0.5662,0.4344,0.5427,0.6303\}\oplus\{0.51\})\right)^{\frac{1}{2}}$$

$$= \left(\frac{1}{6}(\{0.6092,0.6840,0.7445,0.6748,0.737,0.7874,0.7228,0.7759,0.8188\})\right)^{\frac{1}{2}}$$

$$= (\{0.1449,0.1746,0.2034,0.1707,0.1995,0.2274,0.1925,0.2206,0.2477\})^{\frac{1}{2}}$$

$$= \{0.3806,0.4178,0.4509,0.4131,0.4466,0.4768,0.4387,0.4696,0.4976\}$$

Table 4. Fuzzy Soft Set B'

	B_1	B_2	B_3
A_1	0.443	0.4727	0.7802
A_2	0.4752	0.5966	0.5956
A_3	0.5464	0.4308	0.8421
A_4	0.6375	0.5463	0.6307
A_5	0.4455	0.2946	0.3923

Step 3: Let $W_1 = [(0.4\ \ 0.3\ \ 0.3)]$ and $W_2=[(0.7\ \ 0.2\ \ 0.1)]$ be two weight vectors for Mr. Ali and Mr. Amir respectively conforming to their personal choices. Then using WBM with $l = m = 1$ for B_1, B_2, B_3 in $B'_{5\times3}$, WBFSM is calculated for Mr. Ali as:

$$C_{5\times1} = \begin{bmatrix}0.2424\\0.2340\\0.2736\\0.2651\\0.1456\end{bmatrix}$$

Note that $WHFBM(A_5) < WHFBM(A_2\,) = WHFBM(A_1) < WHFBM(A_4) < WHFBM(A_3)$, that is, $0.1456 < 0.2340 < 0.2424 < 0.1456 < 0.2736$.

Now, WBFSM for Mr. Amir is calculated as:

$$C_{5\times1} = \begin{bmatrix}0.1865\\0.1881\\0.2121\\0.2137\\0.1183\end{bmatrix}$$

Note that, $WHFBM(A_5) < WHFBM(A_1) < WHFBM(A_2) = WHFBM(A_3) < WHFBM(A_4)$ that is, $0.1183 < 0.1865 < 0.1881 < 0.2121 < 0.2137$. Therefore A_3 is the best option for Mr. Ali and A_4 is the best option for Mr. Amir.

Table 3. Bonferroni Hesitant Fuzzy Soft Set B

	B_1	B_2	B_3
A_1	{0.3806,0.4131,0.4178, 0.4387,0.4466,0.4509, 0.4696,0.4768,0.4976}	{0.4136,0.424,0.4352, 0.466,0.475,0.4844, 0.5109,0.5189,0.5267}	{0.7536,0.7667,0.7745, 0.7778,0.7812,0.7849, 0.7875,0.7947,0.8013}
A_2	{0.4135,0.4360,0.4568, 0.4591,0.4785,0.4949, 0.4967,0.5126,0.5284}	{0.5596,0.5753,0.5828, 0.5903,0.5973,0.6042, 0.6112,0.6178,0.6307}	{0.5956}
A_3	{0.5284,0.5342,0.5399, 0.5411,0.5467,0.5521, 0.5533,0.5586,0.5638}	{0.3581,0.3944,0.4267, 0.3823,0.4362,0.4453, 0.4643,0.4727,0.4977}	{0.7667,0.7812,0.7932, 0.7949,0.807,0.8131,1}
A_4	{0.5638,0.5997,0.6096, 0.6315,0.6405,0.649, 0.668,0.6756,0.7}	{0.5284,0.5342,0.5399, 0.5411,0.5467,0.5521, 0.5533,0.5585,0.5638}	{0.6307}
A_5	{0.3865,0.4111,0.4257, 0.4338,0.4473,0.4603, 0.4675,0.4797,0.4977}	{0.2534,0.2722,0.2787, 0.2897,0.2957,0.3016, 0.3116,0.3171,0.3318}	{0.37,0.3782,0.3849, 0.3861,0.3926,0.3989, 0.4002,0.4064,0.4136}

Step 4: Since both persons select different stores so they may not be satisfied with each other's decision. Note that $a_{p_1}^1 = 0.2736, a_{P_2}^1 = -0.8817 = -(1-0.1183)$, $a_{P_1}^2 = -0.8544 = -(1-0.1456), a_{P_2}^2 = 0.2137$

Step 5: Finally consider the following system of fuzzy soft differential equations;

$$\frac{dP_1}{dt} = 0.2736P_1 - 0.881P_2$$
$$\frac{dP_2}{dt} = -0.8544P_1 + 0.2137P_2 \tag{2}$$

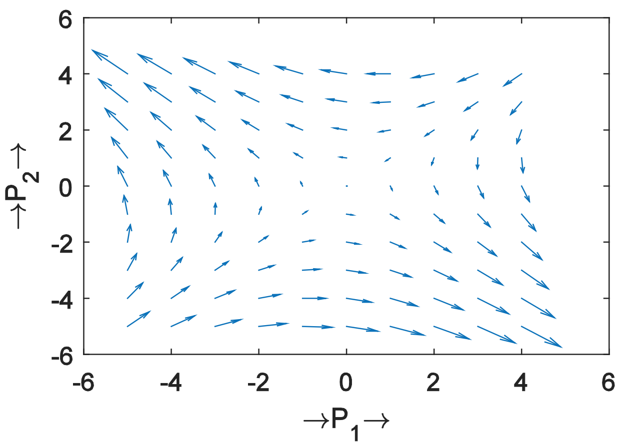

Figure 1. Phase Portrait for P_1 P_2-plane and phase portrait for differential equations (2).

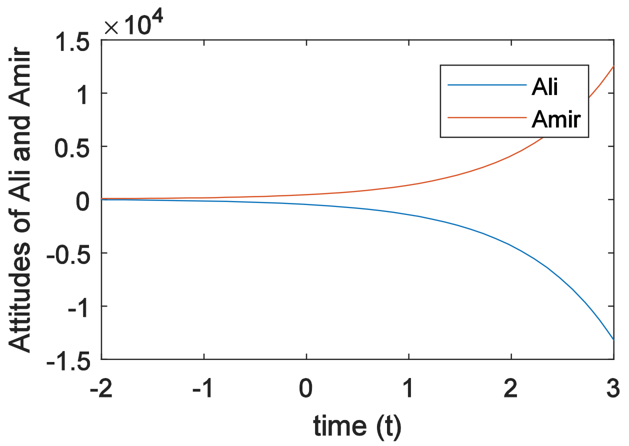

Figure 2. Line graph for differential equations (2).

Figure 1 and Figure 2 show that Mr. Ali and Mr. Amir disagree with each others' feelings over the time t.

Remark

When BM is applied to h_1, h_2, h_3 then $BM(h_1, h_2, h_3)$ consists of only one element if h_1, h_2, h_3 all have one element. If one of h_1, h_2, h_3 has two elements, then $BM(h_1, h_2, h_3)$ consists of nine elements. It is difficult to handle nine elements for further computation. And if two of h_1, h_2, h_3 have two elements, then $BM(h_1, h_2, h_3)$ consists of ninety elements. Then it is more difficult to handle ninety elements so we take the score function for those hesitant fuzzy elements for further computations.

An attitude analyzer method through TOPSIS

This method is composed of the following steps:

Step 1: Decision advisors present their opinions in the form of hesitant fuzzy soft sets.

Step 2: Find the collective decision set X by the aggregation formula defined in definition 9. Performance of alternative A_i with respect to attribute B_j is denoted as x_{ij} in an aggregated set X.

Step 3: To find the positive ideal solution and negative ideal solution, we have to find respective induced fuzzy soft sets. Induced fuzzy soft set is found by computing score functions of all hesitant fuzzy elements using definition 6. The values in the matrices presented in step 1 having maximum or minimum values in respective induced fuzzy soft sets are positive ideal solutions (PIS) $\tilde{A}^+ = (\tilde{V}_1^+ \quad \tilde{V}_2^+ \quad \ldots \quad \tilde{V}_n^+)$ and negative ideal solutions (NIS) $\tilde{A}^+ = (\tilde{V}_1^- \quad \tilde{V}_2^- \quad \ldots \quad \tilde{V}_n^-)$ respectively.

Step 4: Consider the weight vectors W_1 and W_2 for two persons according to their personal choices. Construct positive ideal separation matrix D^+ and negative ideal separation matrix D^- by the formulae

$$D^+ = \begin{bmatrix} d(x_{11}, \tilde{V}_1^+) + & \cdots & +d(x_{1n}, \tilde{V}_n^+) \\ d(x_{21}, \tilde{V}_1^+) + & \cdots & +d(x_{2n}, \tilde{V}_n^+) \\ \vdots & & \vdots \\ d(x_{m1}, \tilde{V}_1^+) + & \cdots & +d(x_{mn}, \tilde{V}_n^+) \end{bmatrix} = \begin{bmatrix} d(A_1, \tilde{A}^+) \\ d(A_2, \tilde{A}^+) \\ \vdots \\ d(A_m, \tilde{A}^+) \end{bmatrix}$$

and

$$D^- = \begin{bmatrix} d(x_{11}, \tilde{V}_1^-) + & \cdots & +d(x_{1n}, \tilde{V}_n^-) \\ d(x_{21}, \tilde{V}_1^-) + & \cdots & +d(x_{2n}, \tilde{V}_n^-) \\ \vdots & & \vdots \\ d(x_{m1}, \tilde{V}_1^-) + & \cdots & +d(x_{mn}, \tilde{V}_n^-) \end{bmatrix} = \begin{bmatrix} d(A_1, \tilde{A}^-) \\ d(A_2, \tilde{A}^-) \\ \vdots \\ d(A_m, \tilde{A}^-) \end{bmatrix}$$

where distance formula is used as defined in definition 3.

Step 5: Calculate the relative closeness (RC) of each alternative to the ideal solution as follows:

$$RC(A_i) = \frac{d(A_i, \tilde{A}^-)}{d(A_i, \tilde{A}^-) + d(A_i, \tilde{A}^+)}, i = 1,2, \ldots, m$$

Step 6: Rank all the alternatives according to the closeness coefficients, the alternative having the greater value $RC(A_i)$ is better.

Step 7: Find optimal fuzzy soft constants from $RC(A_i)$ and derive the system of linear fuzzy soft differential equations as defined (1) to study the feelings of people with respect to time.

An attitude analyzer method through VIKOR

This method comprises the following steps:

Step 1 - Step 4: Same as specified in the previous method (section 3.2).

Step 5: Calculate S_i, R_i and Q_i i=1,2,...,m by the following formulae:

$$S_i = \sum_{j=1}^{n} w_j \frac{d(\tilde{V}_j^+, x_{ij})}{d(\tilde{V}_j^+, \tilde{V}_j^-)}$$

$$R_i = \max_j \left(w_j \frac{d(\tilde{V}_j^+, x_{ij})}{d(\tilde{V}_j^+, \tilde{V}_j^-)} \right)$$

$$Q_i = \nu \frac{S_i - S^+}{S^- - S^+} + (1-\nu) \frac{R_i - R^+}{R^- - R^+}$$

where

$$S^+ = minS_i$$
$$S^- = maxS_i$$
$$R^+ = minR_i$$
$$R^- = maxR_i$$

and ν is the weight of the strategy of "the majority of criteria", here $\nu = 0.5$

Step 6: An alternative x' is selected as a compromised solution if Q' is minimum and x' is graded the best if the following two conditions are satisfied:

1. $Q(x'') - Q(x') \geq \frac{1}{m-1}$ where x'' is the alternative second position in ranking list by Q.
2. Alternative x' must also be the best ranked by S or/and R.

Step 7: Find OFSCs from Q_i and develop the system of linear fuzzy soft differential equations.

Example 4

Assume that there are five industries to invest the money namely; A_1: textile industry, A_2: food industries, A_3: cement industry, A_4: hotel and tourism, A_5: Sugar industry. Mr. Ali and Mr. Amir have to choose the best one. They hire three experts D_1, D_2 and D_3 for their valuable suggestions on the basis of four parameters: B_1- future growth, B_2- tax problems, B_3- quality, B_4- risk issues. Our objective is to analyze the attitude of two persons after some time due to their decisions which is carried through in three ways.

Through TOPSIS

Step 1: Three experts represent the alternatives with respect to the attributes in hesitant fuzzy soft set as shown in Tables 5-7.

Step 2: Aggregate all the decision matrices into the collective decision matrix as shown in Table 8. Description of one of the elements of Table 8 is as follows:

Table 5. Decision Set D1

	B_1	B_2	B_3	B_4
A_1	{0.2,0.4,0.5}	{0.2,0.5,0.6}	{0.1,0.2,0.4}	{0.3,0.7,0.9}
A_2	{0.1,0.4,0.8}	{0.2,0.4,0.6}	{0.2,0.5,0.6}	{0.5,0.6,0.7}
A_3	{0.1,0.4}	{0.5,0.9}	{0.3,0.4,0.6}	{0.4,0.5,0.7}
A_4	{0.2,0.4,0.5}	{0.1,0.2,0.3}	{0.7,0.8,0.9}	{0.1,0.5,0.9}
A_5	{0.1,0.4,0.8}	{0.4,0.8,0.9}	{0.5,0.7,0.9}	{0.3,0.4,0.9}

Table 6. Decision Set D2

	B_1	B_2	B_3	B_4
A_1	{0.1,0.4,0.7}	{0.2,0.4,0.7}	{0.3,0.5,0.8}	{0.4,0.5,0.6}
A_2	{0.2,0.4,0.6}	{0.2,0.5,0.7}	{0.5,0.6,0.8}	{0.3,0.7,0.9}
A_3	{0.1,0.3,0.5}	{0.1,0.4,0.5}	{0.4,0.7,0.8}	{0.3,0.7,0.8}
A_4	{0.2,0.4,0.8}	{0.2,0.4,0.9}	{0.3,0.7,0.8}	{0.3,0.4,0.7}
A_5	{0.2,0.7,0.8}	{0.3,0.5,0.6}	{0.3,0.5,0.7}	{0.4,0.5,0.6}

Table 7. Decision Set D3

	B_1	B_2	B_3	B_4
A_1	{0.2,0.7,0.9}	{0.4,0.5,0.7}	{0.2,0.4,0.7}	{0.2,0.3,0.4}
A_2	{0.6,0.7,0.9}	{0.3,0.7,0.8}	{0.5,0.8,0.9}	{0.3,0.4,0.6}
A_3	{0.6,0.8,0.9}	{0.4,0.6,0.9}	{0.3,0.6,0.9}	{0.1,0.2,0.7}
A_4	{0.1,0.5,0.9}	{0.2,0.3,0.6}	{0.2,0.5,0.7}	{0.3,0.4,0.7}
A_5	{0.5,0.6,0.7}	{0.4,0.5,0.7}	{0.3,0.4,0.6}	{0.1,0.2,0.5}

$$r_{P_{11}} = \min\{\min(0.5,0.7,0.9), \max(0.2,0.1)\}$$

$$= \min\{0.5,0.2\} = 0.2$$

$$r_{q_{11}} = \max\{\min(0.5,0.7,0.9), \max(0.2,0.1)\}$$

$$= \max\{0.5,0.2\} = 0.5$$

Other elements belong to x^l_{11}, (l=1,2,3) between 0.2 and 0.5 will also belong to x_{11}. Here is only one such element i.e. 0.4, therefore x_{11} ={0.2,0.4,0.5}.

Table 8. Collective decision Set X

	B_1	B_2	B_3	B_4
A_1	{0.2,0.4,0.5}	{0.4,0.5,0.6}	{0.3,0.4}	{0.4}
A_2	{0.6}	{0.3,0.4,0.5,0.6}	{0.5,0.6}	{0.5,0.6}
A_3	{0.4,0.5,0.6}	{0.5}	{0.4,0.6}	{0.4,0.5,0.7}
A_4	{0.2,0.4,0.5}	{0.2,0.3}	{0.7}	{0.3,0.4,0.5,0.7}
A_5	{0.5,0.6,0.7}	{0.4,0.5,0.6}	{0.5,0.6}	{0.4,0.5}

Step 3: Respective induced fuzzy soft sets $D^I{}_1$, $D^I{}_2$, $D^I{}_3$ of D_1, D_2, D_3 are shown as Tables 9-11:

Table 9. Induced Set $D^I{}_1$

	B_1	B_2	B_3	B_4
A_1	{0.37}	{0.43}	{0.23}	{0.63}
A_2	{0.43}	{0.4}	{0.43}	{0.6}
A_3	{0.17}	{0.47}	{0.43}	{0.53}
A_4	{0.37}	{0.2}	{0.8}	{0.5}
A_5	{0.43}	{0.7}	{0.7}	{0.53}

Table 10. Induced Set $D^I{}_2$

	B_1	B_2	B_3	B_4
A_1	{0.4}	{0.43}	{0.53}	{0.5}
A_2	{0.4}	{0.47}	{0.63}	{0.63}
A_3	{0.3}	{0.33}	{0.63}	{0.6}
A_4	{0.47}	{0.5}	{0.6}	{0.47}
A_5	{0.57}	{0.47}	{0.5}	{0.5}

Table 11. Induced Set $D^I{}_3$

	B_1	B_2	B_3	B_4
A_1	{0.6}	{0.53}	{0.43}	{0.3}
A_2	{0.73}	{0.6}	{0.73}	{0.43}
A_3	{0.5}	{0.5}	{0.5}	{0.33}
A_4	{0.37}	{0.25}	{0.7}	{0.47}
A_5	{0.6}	{0.53}	{0.43}	{0.27}

Hesitant fuzzy soft positive Ideal Solution $\tilde{A}^+$ and the hesitant fuzzy soft negative Ideal Solution $\tilde{A}^-$ are shown in Table 12

Table 12. PIS and NIS

	B_1	B_2	B_3	B_4
$\tilde{A}^+$	{0.6,0.7,0.9}	{0.1,0.2,0.3}	{0.7,0.8,0.9}	{0.1,0.2,0.5}
$\tilde{A}^-$	{0.1,0.4}	{0.4,0.8,0.9}	{0.1,0.2,0.4}	{0.3,0.7,0.9}

Step 4: Positive and negative ideal separation matrices are:

$$D^+ = \begin{bmatrix} d(x_{11},\tilde{V}_1^+) + d(x_{12},\tilde{V}_2^+) + d(x_{13},\tilde{V}_3^+) + d(x_{14},\tilde{V}_4^+) \\ (x_{21},\tilde{V}_1^+) + d(x_{22},\tilde{V}_2^+) + d(x_{23},\tilde{V}_3^+) + d(x_{24},\tilde{V}_4^+) \\ (x_{31},\tilde{V}_1^+) + d(x_{32},\tilde{V}_2^+) + d(x_{33},\tilde{V}_3^+) + d(x_{34},\tilde{V}_4^+) \\ (x_{41},\tilde{V}_1^+) + d(x_{42},\tilde{V}_2^+) + d(x_{43},\tilde{V}_3^+) + d(x_{44},\tilde{V}_4^+) \\ (x_{51},\tilde{V}_1^+) + d(x_{52},\tilde{V}_2^+) + d(x_{53},\tilde{V}_3^+) + d(x_{54},\tilde{V}_4^+) \end{bmatrix} = \begin{bmatrix} d(A_1,\tilde{A}^+) \\ d(A_2,\tilde{A}^+) \\ d(A_3,\tilde{A}^+) \\ d(A_4,\tilde{A}^+) \\ d(A_5,\tilde{A}^+) \end{bmatrix}$$

and

$$D^- = \begin{bmatrix} d(x_{11},\tilde{V}_1^-) + d(x_{12},\tilde{V}_2^-) + d(x_{13},\tilde{V}_3^-) + d(x_{14},\tilde{V}_4^-) \\ (x_{21},\tilde{V}_1^-) + d(x_{22},\tilde{V}_2^-) + d(x_{23},\tilde{V}_3^-) + d(x_{24},\tilde{V}_4^-) \\ (x_{31},\tilde{V}_1^-) + d(x_{32},\tilde{V}_2^-) + d(x_{33},\tilde{V}_3^-) + d(x_{34},\tilde{V}_4^-) \\ (x_{41},\tilde{V}_1^-) + d(x_{42},\tilde{V}_2^-) + d(x_{43},\tilde{V}_3^-) + d(x_{44},\tilde{V}_4^-) \\ (x_{51},\tilde{V}_1^-) + d(x_{52},\tilde{V}_2^-) + d(x_{53},\tilde{V}_3^-) + d(x_{54},\tilde{V}_4^-) \end{bmatrix} = \begin{bmatrix} d(A_1,\tilde{A}^-) \\ d(A_2,\tilde{A}^-) \\ d(A_3,\tilde{A}^-) \\ d(A_4,\tilde{A}^-) \\ d(A_5,\tilde{A}^-) \end{bmatrix}$$

Consider the weight vectors $W_1 = [0.3\ \ 0.2\ \ 0.3\ \ 0.2]$ and $W_2 = [0.3\ \ 0.1\ \ 0.4\ \ 0.2]$ for Mr. Ali and Mr. Amir according to their personal choices towards industry selections. Then calculated matrices for Mr. Ali are

$$D^+ = \begin{bmatrix} 0.11 + 0.06 + 0.14 + 0.04 \\ 0.04 + 0.005 + 0.08 + 0.05 \\ 0.07 + 0.06 + 0.1 + 0.05 \\ 0.11 + 0.007 + 0.03 + 0..05 \\ 0.04 + 0.06 + 0.08 + 0.033 \end{bmatrix} = \begin{bmatrix} 0.35 \\ 0.175 \\ 0.28 \\ 0.197 \\ 0.213 \end{bmatrix}$$

and

$$D^- = \begin{bmatrix} 0.05 + 0.04 + 0.03 + 0.06 \\ 0.105 + 0.046 + 0.09 + 0.046 \\ 0.07 + 0.053 + 0.09 + 0.033 \\ 0.05 + 0.093 + 0.14 + 0.025 \\ 0.12 + 0.066 + 0.09 + 0.053 \end{bmatrix} = \begin{bmatrix} 0.18 \\ 0.287 \\ 0.246 \\ 0.308 \\ 0.329 \end{bmatrix}$$

Similarly, the calculated matrices for Mr. Amir are

$$D^+ = \begin{bmatrix} 0.11 + 0.03 + 0.187 + 0.04 \\ 0.04 + 0.03 + 0.107 + 0.05 \\ 0.07 + 0.03 + 0.133 + 0.05 \\ 0.11 + 0.003 + 0.04 + 0..067 \\ 0.04 + 0.03 + 0.107 + 0.033 \end{bmatrix} = \begin{bmatrix} 0.367 \\ 0.227 \\ 0.283 \\ 0.22 \\ 0.21 \end{bmatrix}$$

and

$$D^- = \begin{bmatrix} 0.05 + 0.02 + 0.04 + 0.06 \\ 0.105 + 0.018 + 0.12 + 0.046 \\ 0.07 + 0.027 + 0.093 + 0.033 \\ 0.05 + 0.047 + 0.187 + 0.025 \\ 0.12 + 0.02 + 0.12 + 0.053 \end{bmatrix} = \begin{bmatrix} 0.17 \\ 0.289 \\ 0.223 \\ 0.309 \\ 0.313 \end{bmatrix}$$

Step 5: Calculate the relative closeness RC of each alternative to the ideal solution as follows:

$$RC(A_i) = \frac{d(A_i,\tilde{A}^-)}{d(A_i,\tilde{A}^-)+d(A_i,\tilde{A}^+)}, \text{ i=1,2,3,4}$$

Note that, $RC(A_i)$ for Mr. Ali are:

$RC(A_1) = 0.34$, $RC(A_2) = 0.62$, $RC(A_3) = 0.4677$, $RC(A_4) = 0.61$, $RC(5) = 0.607$

Similarly, $RC(A_i)$ for Mr. Amir are:

$RC(A_1) = 0.316$, $RC(A_2) = 0.56$, $RC(A_3) = 0.44$, $RC(A_4) = 0.584$, $RC(A_5) = 0.598$

Step 6: Ranking the alternatives for Mr. Ali is given by: $A_1 < A_3 < A_5 < A_4 < A_2$. Ranking the alternatives for Mr. Amir is given by: $A_1 < A_3 < A_2 < A_4 < A_5$. Therefore A_2 (food industry) is the best option for Mr. Ali and A_5 (sugar industry) is the best option for Mr. Amir to invest the money.

Step 7: Since both persons select different industries so they may not be satisfied with each other's decision. We now calculate optimal fuzzy soft constants given as:

$$a_{P_1}^1 = 0.62, a_{P_2}^1 = -0.684 = -(1-0.316),$$

$$a_{P_1}^2 = -0.66 = -(1-0.34), a_{P_2}^2 = 0.598$$

Substitute these values in system (1) to obtain the following system of equations

$$\frac{dP_1}{dt} = 0.62P_1 - 0.684P_2$$
$$\frac{dP_2}{dt} = -0.66P_1 + 0.598P_2 \quad (3)$$

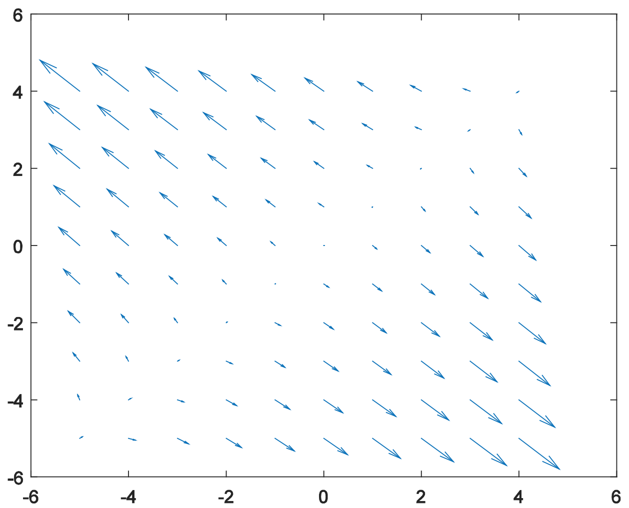

Figure 3. Phase Portrait for $P_1 P_2$-plane and phase portrait for differential equations (3).

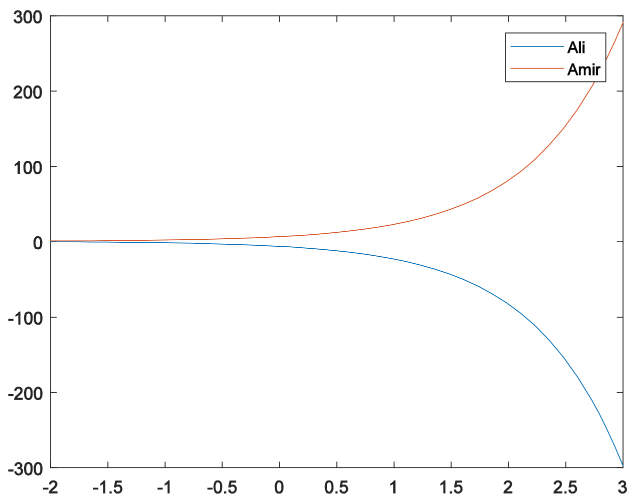

Figure 4. Line graph for differential equation (3).

Figure 3 and Figure 4 show that both persons disagree with each others' feelings over the time t.

Through VIKOR

Now, step-wise calculations through VIKOR are as under:

Step 1 - Step 3: Same as through TOPSIS.

Step 4: By consider the weight vectors $W_1 = [0.3 \;\; 0.2 \;\; 0.3 \;\; 0.2]$ and $W_2 = [0.3 \;\; 0.1 \;\; 0.4 \;\; 0.2]$ for Mr. Ali and Mr. Amir respectively, calculate $S_i\, R_i$ and Q_i which are shown in Tables 13 and 14 respectively.

Table 13. Information table for Mr. Ali

i	S_i	R_i	Q_i
1	$0.6562=S^-$	$0.2625=R^-$	1
2	$0.3281=S^+$	$0.15=R^+$	0
3	0.5250	0.1875	0.4666
4	0.3693	0.2062	0.3124
5	0.3993	0.15	0.1085

Table 14. Information table for Mr. Amir

i	S_i	R_i	Q_i
1	$0.6881=S^-$	$0.3506=R^-$	1
2	0.4256	$0.2006=R^+$	0.0541
3	0.5306	0.2493	0.3948
4	0.4125	0.2062	0.0485
5	$0.3937=S^+$	0.2006	0

Step 5: Ranking of alternatives according to the values $S_i\, R_i$ and Q_i with decreasing order is shown in Table 15.

Table 15. Sorting $S_i\, R_i$ and Q_i with decreasing order

For Mr Ali			For Mr. Amir		
S_i	R_i	Q_i	S_i	R_i	Q_i
$\mathbf{A_1}$	A_1	A_1	$\mathbf{A_1}$	A_1	A_1
$\mathbf{A_3}$	A_4	A_3	$\mathbf{A_3}$	A_3	A_3
$\mathbf{A_5}$	A_3	A_4	$\mathbf{A_2}$	A_4	A_2
$\mathbf{A_4}$	A_5	A_5	$\mathbf{A_4}$	A_2	A_4
$\mathbf{A_2}$	A_2	A_2	$\mathbf{A_5}$	A_5	A_5

Step 6: Find ranking of alternatives for Mr. Ali is: $A_1 < A_3 < A_4 < A_5 < A_2$ and for Mr. Amir is: $A_1 < A_3 < A_2 < A_4 < A_5$.

Step 7: Obtain a system of linear differential equations with $a^i_{P_j}$ i,j=1,2 derived from Q_i

$$\frac{dP_1}{dt} = P_1 - P_2$$
$$\frac{dP_2}{dt} = -P_1 + P_2 \quad (4)$$

Phase portrait and line graph for system (4) are same as for system (3), shown in Figure 3 and Figure 4 respectively. Therefore, two persons will disagree with each other in future.

Through HFBM

Step 1: Obtain the decision sets, shown in tables 16-18, from Tables 5-7 respectively by taking the complements of the values belong to the cost attributes (B_2 and B_4).

Table 16. Decision Set D_1

	B_1	B_2	B_3	B_4
A_1	{0.2,0.4,0.5}	{0.4,0.5,0.8}	{0.1,0.2,0.4}	{0.1,0.3,0.6}
A_2	{0.1,0.4,0.8}	{0.4,0.6,0.8}	{0.2,0.5,0.6}	{0.3,0.4,0.5}
A_3	{0.1,0.4}	{0.1,0.5}	{0.3,0.4,0.6}	{0.3,0.5,0.6}
A_4	{0.2,0.4,0.5}	{0.7,0.8,0.9}	{0.7,0.8,0.9}	{0.1,0.5,0.9}
A_5	{0.1,0.4,0.8}	{0.1,0.2,0.6}	{0.5,0.7,0.9}	{0.1,0.6,0.7}

Table 17. Decision Set D_2

	B_1	B_2	B_3	B_4
A_1	{0.1,0.4,0.7}	{0.3,0.6,0.8}	{0.3,0.5,0.8}	{0.4,0.5,0.6}
A_2	{0.2,0.4,0.6}	{0.3,0.5,0.8}	{0.5,0.6,0.8}	{0.1,0.3,0.7}
A_3	{0.1,0.3,0.5}	{0.4,0.6,0.9}	{0.4,0.7,0.8}	{0.2,0.3,0.7}
A_4	{0.2,0.4,0.8}	{0.1,0.6,0.8}	{0.3,0.7,0.8}	{0.3,0.4,0.7}
A_5	{0.2,0.7,0.8}	{0.4,0.5,0.7}	{0.3,0.5,0.7}	{0.4,0.5,0.6}

Table 18. Decision Set D_3

	B_1	B_2	B_3	B_4
A_1	{0.2,0.7,0.9}	{0.3,0.5,0.6}	{0.2,0.4,0.7}	{0.6,0.7,0.8}
A_2	{0.6,0.7,0.9}	{0.2,0.3,0.7}	{0.5,0.8,0.9}	{0.4,0.6,0.7}
A_3	{0.6,0.8,0.9}	{0.1,0.4,0.6}	{0.3,0.6,0.9}	{0.3,0.8,0.9}
A_4	{0.1,0.5,0.9}	{0.4,0.7,0.8}	{0.2,0.5,0.7}	{0.3,0.4,0.7}
A_5	{0.5,0.6,0.7}	{0.3,0.5,0.6}	{0.3,0.4,0.6}	{0.5,0.8,0.9}

Step 2: Aggregated values are obtained (shown in table 19) by using definition 7 with $l = m = 1$. Average of those values is shown in Table 19.

Table 19. Aggregated Set through BM

	B_1	B_2	B_3	B_4
A_1	{0.6144}	{0.7049}	{0.5385}	{0.6644}
A_2	{0.6909}	{0.6817}	{0.7745}	{0.5971}
A_3	{0.5602}	{0.5897}	{0.7313}	{0.6793}
A_4	{0.6075}	{0.8242}	{0.7992}	{0.6995}
A_5	{0.7108}	{0.5826}	{0.7126}	{0.7396}

Step 3: WBM is obtained by definition 5 for two persons and shown in Table 20.

Table 20. Weighted Aggregated Values

Alternatives	For Mr. Ali	For Mr. Amir
A_1	0.1542	0.1471
A_2	0.1715	0.1677
A_3	0.1585	0.1562
A_4	0.1802	0.1737
A_5	0.1705	0.1692

Step 4: Based on Table 20, rank the alternatives. An alternative with maximum value is considered as the best one.

Ranking for Mr. Ali: $A_1 < A_3 < A_5 < A_2 < A_4$.

Ranking for Mr. Amir: $A_1 < A_3 < A_2 < A_5 < A_4$.

Step 5: Obtain a system of linear differential equations

$$\frac{dP_1}{dt} = -(0.8198P_1 + 0.8263P_2)$$
$$\frac{dP_2}{dt} = -(0.8198P_1 + 0.8263P_2) \quad (5)$$

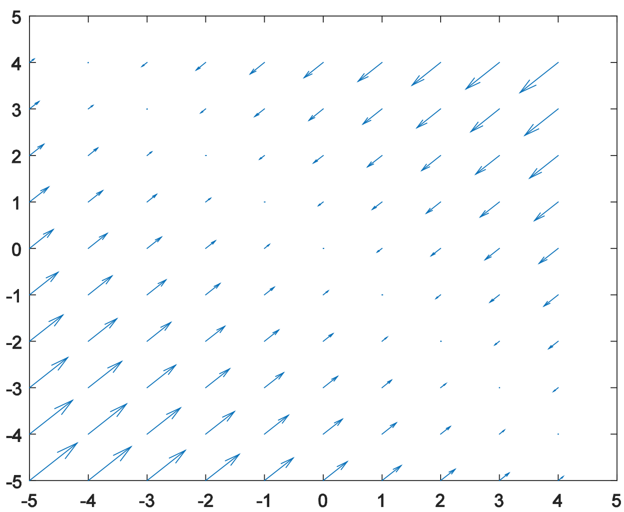

Figure 5. Phase Portrait for P_1 P_2 -plane and phase portrait for differential equations (5).

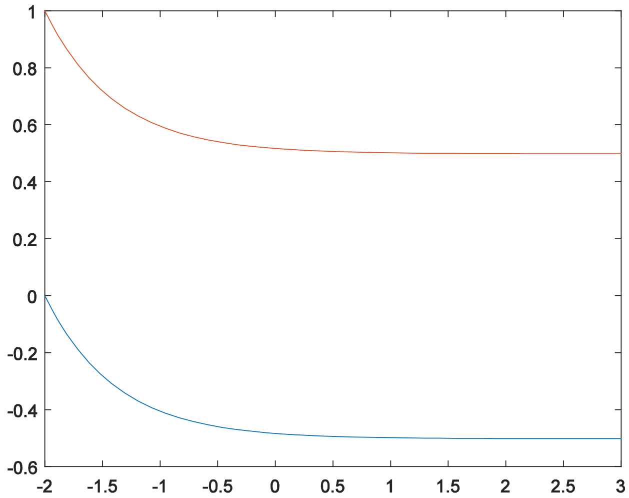

Figure 6. Line graph for differential equation (5).

Figure 5 and Figure 6 show that two persons agree with each others over the time *t*.

It can be observed that TOPSIS and VIKOR show the same result but HFBM shows a different result.

CONCLUSION

This research work is about the analysis of human attitude which changes or remains the same after the decision. It depends upon the interpersonal influences of experts and public opinions after their decisions. Weight vectors play a vital role in the selection of alternatives. Usually different weights from different experts result the different choices of alternatives but it is also possible to have the same choice. The first method described in section 3.1 is the procedure presented by Beg et al. [4] with the difference of initial data. We are taking hesitant fuzzy soft sets instead of fuzzy soft sets as decision sets. Aggregation formulas and distance measures have been used as defined for hesitant fuzzy soft elements. The second and third methods described in sections 3.2 and 3.3 is the utilization of TOPSIS and VIKOR respectively in human attitude analysis. The common strategy between the application of three approaches is to make use of constants which rank the alternatives and provide a base for the construction of the system of differential equations. The basic advantage of the utilization of TOPSIS and VIKOR in this procedure is the preparation of ground for other MCDM methods in this analysis.

AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

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