On a non-riemannian quantity of (α,β)-metrics

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ABSTRACT

In this paper, we study a non-Riemannian quantity χ-curvature of (α, β)-metrics, a special class of Finsler metrics with Riemannian metric α and a 1-form β. We prove that every (α, β)-metric has a vanishing χ-curvature under certain conditions.

Keywords: *(α,β)- Finsler metric, vanishing χ-curvature, non-Riemannian quantity.*

INTRODUCTION

M. Matsumoto defined (α,β)-metrics [1] as a generalization of the Randers metric. One can find applications of (α,β)-metrics in physics and biology [2-4] in 1972. We understand geometric properties of Finsler metrics in the general case better as we study (α,β)-metrics more. Although it is more difficult to study (α,β)-metrics, when compared to studying Randers metric, we see good results with full of geometric properties appearing for (α,β)-metrics in recent years, [5-10].

An (α,β)-metric is a scalar function on TM defined by $F = \alpha\emptyset(s)$, $s = \frac{\beta}{\alpha}$ where $\emptyset = \emptyset(s)$ is a C^∞ function on $\left(-b_0, b_0\right)$, $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form. It can be shown that for any Riemannian metric α and any 1-form β on M with $b = ||\beta_x||_\alpha < b_0$ the function $F = \alpha\emptyset\left(\frac{\beta}{\alpha}\right)$ is a (positive definite) Finsler metric if and only if $\emptyset$ satisfies

$$\emptyset(s) > 0, \quad \emptyset(s) - s\emptyset'(s) + (b^2 - s^2)\emptyset''(s) > 0, \quad (|s| \leq \rho < b_0).$$

Such (α, β)-metrics are said to be regular. Randers metrics are special (α, β)- metrics defined by $\emptyset = 1 + s$, i.e., F = α + β.

Non-Riemannian quantities play a quite important role in Finsler geometry [11-15]. They all vanish for Riemannian metrics [13, 16-17]. In this paper we consider a few non-Riemannian quantities. The χ-curvature, H- curvature, S-curvature are some of the non-Riemannian quantities in Finsler geometry. The Riemann curvature

$$R = R^i_{\ k}\frac{\partial}{\partial x^i} \otimes dx^k$$

is defined by

$$R^i_{\ k} := 2\frac{\partial G^i}{\partial x^k} - y^j\frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j\frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}$$

where $G^i = G^i(x, y)$ are the spray coefficients. The χ-curvature is related to the Riemannian curvature as shown below,

$$\chi_i = -\frac{1}{3}\{2R^m_{\ i\cdot m} + R^m_{\ m\cdot i}\}.$$

The χ-curvature, H-curvature, S-curvature are related to each other as shown below.

$$\chi_i = S_{.i|m}y^m - S_{|i},$$
$$H_{ij} = \frac{1}{4}\{\chi_{j\cdot i} + \chi_{i\cdot j}\},$$

$$H_{ij} = \frac{1}{2} S_{.i.j|m} y^m,$$

where $\chi := \chi_j \, dx^j$ and $H := H_{ij} dx^i \otimes dx^j$ denote the χ- and H -curvatures of F on the tangent bundle TM, respectively, S denotes the S -curvature of F. The notations " · " and " | " denote the vertical and horizontal covariant derivatives with respect to the Chern connection of F, respectively, [14]. The Chern connection solves the equivalence problem for Finsler structure, like many other connections, [18], and gives rise to a list of criteria to decide when two such structures differ only by a change of coordinates, [19].

MAIN THEOREM AND SOME APPLICATIONS

In this paper, we study a non-Riemannian quantity, the χ-curvature of (α, β)-metrics, a special and a large class of Finsler metrics with Riemannian metric α and a 1-form β. We prove that every (α, β)-metric has a vanishing χ-curvature under certain conditions. We give the following theorem and its corollary below.

Theorem 1.1 Let F be an (α, β)-metric on n-dimensional manifold M. Assume that β is a closed 1-form and $r_{ij} = K(b^2 a_{ij} - b_i b_i)$, where K is a constant. Then the χ-curvature vanishes.

One can see that the following corollary can easily be proven when we apply the given conditions to the equation in (3.27).

Corollary 1.2 Let F be an (α, β)-metrics on n-dimensional manifold M. β is a closed 1-form and $r_{ij} = 0$ if and only if the χ-curvature vanishes.

As an application we give two examples using a Randers metric defined on S^3 and an (α, β)-metric defined on an open subset in R^3, respectively. These examples show that under the given conditions, namely, when β is a closed 1-form and $r_{ij} = K(b^2 a_{ij} - b_i b_i)$, where K is a constant, we get that the χ- curvature vanishes.

Randers metrics were first introduced by the physicist G. Randers in 1941 in the theory of general relativity where R. S. Ingarden, was the one who named it as Randers Metrics for the first time in an application in his thesis [20] in the theory of the electron microscope. They are among the simplest Finsler metrics, expressed in the form $F = \alpha + \beta$, where $\alpha = \sqrt{a_{ij}(x) y^i y^j}$ is a Riemannian metric on a differentiable manifold M and $\beta = b_i(x) y^i$ is a 1-form with $||\beta_x||_\alpha := \sup_{y \in T_x M} \frac{|\beta(y)|}{\alpha(y)} < 1$ for any point x ∈ M. Many Finsler geometers have studied the geometric properties of Randers metrics and have obtained many important and interesting results, [21-26].

Example 1 [23] Let $F = \alpha + \beta$ be the family of Randers metrics on S^3 constructed in [27]. It is shown that $r_{ij} = 0$ and $s_j = 0$. Thus for any C^∞ positive function $\varphi = \varphi(s)$ satisfying the following

$$\varphi(s) - s\varphi'(s) + (\rho^2 - s^2)\varphi''(s) > 0, \quad (|s| \leq \rho < b_0),$$

the (α, β)-metric $F = \alpha\varphi\left(\frac{\beta}{\alpha}\right)$ has vanishing S -curvature. This implies that the χ-curvature vanishes.

Example 2 [23] Let $F = \alpha\varphi\left(\frac{\beta}{\alpha}\right)$ be an (α, β)-metric defined on an open subset in $\mathbb{R}^3$. At a point x = (x, y, z) in $\mathbb{R}^3$ and in the direction y = (u, v, w) in $T_x\mathbb{R}^3$, $\alpha = \alpha(x, y)$ and $\beta = \beta(x, y)$ are given by $\alpha := \sqrt{u^2 + e^{2x}(v^2 + w^2)}$, $\beta := u$. Then β satisfies $r_{ij} = \varepsilon(b^2 a_{ij} - b_i b_j)$, $s_j = 0$ with $\varepsilon = 1$, $b = 1$. Thus if $\varphi = \varphi(s)$ satisfies

$$\phi = -2(n+1)k \frac{\varphi\Delta^2}{(b^2 - s^2)},$$

for some constant k, then $F = \alpha\varphi\left(\frac{\beta}{\alpha}\right)$ is of constant S -curvature, namely S = (n + 1), hence (1.3) implies that the χ-curvature vanishes.

PROOF OF THE MAIN THEOREM

In this section we prove the main Theorem and the given Proposition below. Before the proofs, we introduce some facts first. Let F be an (α,β)-metric on a manifold M^n defined in the previous section and the spray coefficients G^i of F has the following formula given below

$$G^i = \overline{G}^i + H^i,$$

where $\overline{G}^i$ denote the spray coefficients of α and

$$H^i = \alpha Q s_0^i + \{-2Q\alpha s_0 + r_{00}\}\left\{\theta\frac{y^i}{\alpha} + \Psi b^i\right\} \qquad (3.6)$$

with

$$Q = \frac{\varphi'}{\varphi - s\varphi'}; \quad \Delta = 1 + sQ + (b^2 - s^2)Q'; \quad \psi = \frac{Q'}{2\Delta}; \quad \theta = \frac{Q - sQ'}{2\Delta}; \qquad (3.7)$$

$$b = \|\beta_x\|_\alpha, \ s_j^i = a^{ik}s_{kj}, \ s_0^i = s_j^i \ y^j, \ s_0 = s_i \ y^i, \ r_{00} = r_{ij} \ y^i y^j.$$

Note that the index '0' means 'contracting with respect to y'. We also express

$$\Pi = \overline{\Pi} + \Gamma,$$

where $\Pi = [G^m]_{y^m}$, $\overline{\Pi} = [\overline{G}^m]_{y^m}$, and $\Gamma = [H^m]_{y^m}$. Using the above identities, we get

$$\Gamma = 2\Xi s_0 + 2r_0\Psi - \alpha^{-1}\overline{\phi}r_{00}, \qquad (3.8)$$

where

$$\Xi = \Psi + \Phi Q, \qquad (3.9)$$

$$\Phi = \frac{1}{2\Delta^2}\left\{-(Q - sQ')\{n\Delta + 1 + sQ\} - (b^2 - s^2)(1 + sQ)Q''\right\}. \qquad (3.10)$$

The χ-curvature is given by

$$X_i = \frac{1}{2}\left\{\Pi_{y^i x^m} y^m - \Pi_{x^i} - 2\Pi_{y^i y^m}G^m\right\}, \qquad (3.11)$$

where Π and G^m are as defined above. In the next Proposition 3.1, we compute χ_i by using (3.11).

Proposition 3.1 Let F be an (α, β)-metrics on n-dimensional manifold M. Then the χ-curvature is given below,

$$X_i = C + D + E \qquad (3.12)$$

where

$$
\begin{aligned}
C &= \alpha^{-1}\left\{2\Xi_s s_i + (2r_i\Psi_s - \alpha^{-1}r_{i0}\bar{\Phi}_s)\right\}r_{00} \\
&+ 2\alpha^{-1}\left\{2\Xi_s s_0 + (2r_0\Psi_s - \alpha^{-1}r_{00}\bar{\Phi}_s)\right\}s_{i0} \\
&+ \alpha^{-2}\left\{2\Xi_s s_{0;0} + (2r_{0;0}\Psi_s - \alpha^{-1}r_{00;0}\bar{\Phi}_s)\right\}h_i \\
&+ 2\alpha^{-2}\left\{2\Xi_{s\rho} s_0 + (2r_0\Psi_{s\rho} - \alpha^{-1}r_{00}\bar{\Phi}_{s\rho})\right\}(r_0 + s_0)h_i \\
&- 2\alpha^{-2}\left\{2\Xi_s s_0 + (2r_0\Psi_s - \alpha^{-1}r_{00}\Phi_s)\right\}h_{i\cdot m}H^m \\
&- 2\alpha^{-2}\left\{2\Xi_s s_i + (2r_i\Psi_s - \alpha^{-1}r_{i0}\Phi_s)\right\}h_m H^m \\
&- 2\alpha^{-2}\left\{2\Xi_s s_m + (2r_m\Psi_s - \alpha^{-1}r_{m0}\Phi_s)\right\}h_i H^m \\
&- \alpha^{-3}\left\{2\Xi_s s_0 + (2r_0\Psi_s - \alpha^{-1}r_{00}\Phi_s)\right\}r_{00}y_i, \\
D &= \alpha^{-3}\left\{2\Xi_{ss} s_0 + (2r_0\Psi_{ss} - \alpha^{-1}r_{00}\Phi_{ss})\right\}r_{00}h_i \\
&+ \alpha^{-3}\Phi(r_{00;0}y_i - \alpha^2 r_{i0;0}) \\
&+ 2\alpha^{-3}\Phi_\rho(r_0 + s_0)(r_{00}y_i - \alpha^2 r_{i0}) \\
&- 2\alpha^{-3}\Phi\left(r_{00}a_{im} - r_{i0}y_m\right)H^m \\
&- 4\alpha^{-3}\Phi\left(r_{m0}y_i - \alpha^2 r_{im}\right)H^m, \\
&+ \alpha^{-4}\Phi_s r_{00}(r_{00}y_i - \alpha^2 r_{i0}) \\
&- 2\alpha^{-4}\left\{2\Xi_{ss} s_0 + (2r_0\Psi_{ss} - \alpha^{-1}r_{00}\bar{\Phi}_{ss})\right\}h_i h_m H^m \\
&+ 4\alpha^{-4}\left\{2\Xi_s s_0 + (2r_0\Psi_s - \alpha^{-1}r_{00}\bar{\Phi}_s)\right\}h_i y_m H^m, \\
E &= 2\alpha^{-5}\bar{\Phi}_s\left(r_{00}y_i - \alpha^2 r_{i0}\right)h_m H^m \\
&- 2\alpha^{-5}\bar{\Phi}_s\left(r_{00}y_m - \alpha^2 r_{m0}\right)h_i H^m \\
&+ 6\alpha^{-5}\Phi\left(y_i r_{00} - r_{i0}\alpha^2\right)y_m H^m \\
&- 2\left\{2\Xi_\rho s_0 + (2r_0\Psi_\rho - \alpha^{-1}r_{00}\Phi_\rho)\right\}(r_i + s_i) \\
&+ 2\left\{2\Xi_\rho s_i + (2r_i\Psi_\rho - \alpha^{-1}r_{i0}\Phi_\rho)\right\}(r_0 + s_0) \\
&+ \left\{2\Xi s_{i;0} + (2r_{i;0}\Psi - \alpha^{-1}r_{i0;0}\Phi)\right\} \\
&- \left\{2\Xi s_{0;i} + (2r_{0;i}\Psi - \alpha^{-1}r_{00;i}\Phi)\right\}.
\end{aligned}
$$

where $H^m, Q, \theta, \psi, \Delta, \chi$ and $\bar{\Phi}$ are given above in (3.6), (3.7), and (3.10), respectively.

PROOF OF THE PROPOSITION

We recall Γ defined in (3.8), and we calculate the following terms $\Gamma_{;i}$, $\Gamma_{.i}$, $\Gamma_{.i;m}y^m$, $\Gamma_{.i.m}H^m$ below. When we let $\rho := b^2$, we get $\rho_{,m}y^m = 2(r_0 + s_0)$ We first need the following useful identities for $h_i = \alpha b_i - s y_i$ below in (3.13),

$$
\begin{aligned}
s_{\cdot i} &= \alpha^{-2}h_i; \ s_{;m} = \alpha^{-1}(r_{m0} - s_{m0}); \ \alpha_{\cdot i} = \alpha^{-1}y_i; \ \alpha_{;i} = 0; \\
h_{i;m}y^m &= \alpha(r_{i0} + s_{i0}) - \alpha^{-1}r_{00}y_i; \ s_{;m}y^m = \alpha^{-1}r_{00}, \ y_{i;m}y^m = 0.
\end{aligned}
$$

$$(3.13)$$

$$
\begin{aligned}
\Gamma_{;i} = {} & \alpha^{-1}\Big\{2\Xi_s s_0 + (2r_0\Psi_s - \alpha^{-1}r_{00}\Phi_s)\Big\}r_{i0} \\
& - \alpha^{-1}\Big\{2\Xi_s s_0 + (2r_0\Psi_s - \alpha^{-1}r_{00}\bar\Phi_s)\Big\}s_{i0} \\
& + 2\Big\{2\Xi_\rho s_0 + (2r_0\Psi_\rho - \alpha^{-1}r_{00}\bar\Phi_\rho)\Big\}r_i \\
& + 2\Big\{2\Xi_\rho s_0 + (2r_0\Psi_\rho - \alpha^{-1}r_{00}\bar\Phi_\rho)\Big\}s_i \\
& + \Big\{2\Xi s_{0;i} + (2r_{0;i}\Psi - \alpha^{-1}r_{00;i}\bar\Phi)\Big\},
\end{aligned}
$$

$$(3.14)$$

$$
\begin{aligned}
\Gamma_{\cdot i} = {} & \alpha^{-2}\Big\{2\Xi_s s_0 + (2r_0\Psi_s - \alpha^{-1}r_{00}\Phi_s)\Big\}h_i \\
& + \Big\{2\Xi s_i + (2r_i\Psi - \alpha^{-1}r_{i0}\Phi)\Big\} \\
& + \alpha^{-3}\Phi(r_{00}y_i - \alpha^2 r_{i0}),
\end{aligned}
$$

$$(3.15)$$

$$
\Gamma_{\cdot i;m}y^m = J + L,
$$

$$(3.16)$$

where

$$
\begin{aligned}
J = {} & \alpha^{-3}r_{00}\Big\{2\Xi_{ss}s_0 + (2r_0\Psi_{ss} - \alpha^{-1}r_{00}\Phi_{ss})\Big\}h_i \\
& + 2\alpha^{-2}(r_0 + s_0)\Big\{2\Xi_{s\rho}s_0 + (2r_0\Psi_{s\rho} - \alpha^{-1}r_{00}\Phi_{s\rho})\Big\}h_i \\
& + \alpha^{-2}\Big\{2\Xi_s s_{0;0} + (2r_{0;0}\Psi_s - \alpha^{-1}r_{00;0}\Phi_s)\Big\}h_i \\
& + \alpha^{-1}\Big\{2\Xi_s s_0 + (2r_0\Psi_s - \alpha^{-1}r_{00}\Phi_s)\Big\}r_{i0} \\
& + \alpha^{-1}\Big\{2\Xi_s s_0 + (2r_0\Psi_s - \alpha^{-1}r_{00}\Phi_s)\Big\}s_{i0} \\
& - \alpha^{-3}r_{00}\Big\{2\Xi_s s_0 + (2r_0\Psi_s - \alpha^{-1}r_{00}\Phi_s)\Big\}y_i
\end{aligned}
$$

and

$$
\begin{aligned}
L = {} & \alpha^{-1}r_{00}\Big\{2\Xi_s s_i + (2r_i\Psi_s - \alpha^{-1}r_{i0}\Phi_s)\Big\} \\
& + \Big\{2\Xi s_{i;0} + (2r_{i;0}\Psi - \alpha^{-1}r_{i0;0}\bar\Phi)\Big\} \\
& + 2(r_0 + s_0)\Big\{2\Xi_\rho s_i + (2r_i\Psi_\rho - \alpha^{-1}r_{i0}\bar\Phi_\rho)\Big\} \\
& + \alpha^{-4}\Phi_s r_{00}(r_{00}y_i - \alpha^2 r_{i0}) \\
& + 2\alpha^{-3}\bar\Phi_\rho(r_0 + s_0)(r_{00}y_i - \alpha^2 r_{i0}) \\
& + \alpha^{-3}\Phi(r_{00;0}y_i - \alpha^2 r_{i0;0}),
\end{aligned}
$$

$$
\Gamma_{\cdot i;m}y^m - \Gamma_{;i} = M + N + P,
$$

$$(3.17)$$

where

$$
\begin{aligned}
M &= \alpha^{-3} r_{00}\Big\{2\Xi_{ss} s_0 + (2r_0\Psi_{ss} - \alpha^{-1} r_{00}\bar\Phi_{ss})\Big\} h_i \\
&+ 2\alpha^{-2}(r_0 + s_0)\Big\{2\Xi_{s\rho} s_0 + (2r_0\Psi_{s\rho} - \alpha^{-1} r_{00}\bar\Phi_{s\rho})\Big\} h_i \\
&+ \alpha^{-2}\Big\{2\Xi_s s_{0;0} + (2r_{0;0}\Psi_s - \alpha^{-1} r_{00;0}\bar\Phi_s)\Big\} h_i \\
&+ \alpha^{-1}\Big\{2\Xi_s s_0 + (2r_0\Psi_s - \alpha^{-1} r_{00}\Phi_s)\Big\} r_{i0} \\
&+ \alpha^{-1}\Big\{2\Xi_s s_0 + (2r_0\Psi_s - \alpha^{-1} r_{00}\Phi_s)\Big\} s_{i0},
\end{aligned}
$$

$$
\begin{aligned}
N &= \alpha^{-3} r_{00}\Big\{2\Xi_s s_0 + (2r_0\Psi_s - \alpha^{-1} r_{00}\Phi_s)\Big\} y_i \\
&+ \alpha^{-1} r_{00}\Big\{2\Xi_s s_i + (2r_i\Psi_s - \alpha^{-1} r_{i0}\Phi_s)\Big\} \\
&+ \Big\{2\Xi s_{i;0} + (2r_{i;0}\Psi - \alpha^{-1} r_{i0;0}\Phi)\Big\} \\
&+ 2(r_0 + s_0)\Big\{2\Xi_\rho s_i + (2r_i\Psi_\rho - \alpha^{-1} r_{i0}\Phi_\rho)\Big\} \\
&- \alpha^{-1}\Big\{2\Xi_s s_0 + (2r_0\Psi_s - \alpha^{-1} r_{00}\Phi_s)\Big\} r_{i0} \\
&+ \alpha^{-1}\Big\{2\Xi_s s_0 + (2r_0\Psi_s - \alpha^{-1} r_{00}\Phi_s)\Big\} s_{i0},
\end{aligned}
$$

$$
\begin{aligned}
P &= 2\Big\{2\Xi_\rho s_0 + (2r_0\Psi_\rho - \alpha^{-1} r_{00}\Phi_\rho)\Big\} r_i \\
&- 2\Big\{2\Xi_\rho s_0 + (2r_0\Psi_\rho - \alpha^{-1} r_{00}\Phi_\rho)\Big\} s_i \\
&- \Big\{2\Xi s_{0;i} + (2r_{0;i}\Psi - \alpha^{-1} r_{00;i}\Phi)\Big\} \\
&+ \alpha^{-4}\Phi_s r_{00}(r_{00}y_i - \alpha^2 r_{i0}) \\
&+ 2\alpha^{-3}\bar\Phi_\rho(r_0 + s_0)(r_{00}y_i - \alpha^2 r_{i0}) \\
&+ \alpha^{-3}\bar\Phi(r_{00;0}y_i - \alpha^2 r_{i0;0}),
\end{aligned}
$$

$$
\begin{aligned}
\Gamma_{\cdot i\cdot m}H^m &= -2\alpha^{-4}\Big\{2\Xi_s s_0 + (2r_0\Psi_s - \alpha^{-1} r_{00}\bar\Phi_s)\Big\} h_i y_m H^m \\
&+ \alpha^{-4}\Big\{2\Xi_{ss} s_0 + (2r_0\Psi_{ss} - \alpha^{-1} r_{00}\bar\Phi_{ss})\Big\} h_i h_m H^m \\
&+ \alpha^{-2}\Big\{2\Xi_s s_m + (2r_m\Psi_s - \alpha^{-1} r_{m0}\bar\Phi_s)\Big\} h_i H^m \\
&+ \alpha^{-2}\Big\{2\Xi_s s_0 + (2r_0\Psi_s - \alpha^{-1} r_{00}\bar\Phi_s)\Big\} h_{i\cdot m} H^m \\
&+ \alpha^{-2}\Big\{2\Xi_s s_i + (2r_i\Psi_s - \alpha^{-1} r_{i0}\Phi_s)\Big\} h_m H^m \\
&+ \alpha^{-5}\Phi_s\Big(r_{00}y_m - \alpha^2 r_{m0}\Big) h_i H^m \\
&+ \alpha^{-5}\Phi_s\Big(r_{00}y_i - \alpha^2 r_{i0}\Big) h_m H^m \\
&- 3\alpha^{-5}\Phi\Big(y_i r_{00} - r_{i0}\alpha^2\Big) y_m H^m \\
&+ \alpha^{-3}\Phi\Big(r_{00}a_{im} - r_{i0}y_m\Big) H^m \\
&+ 2\alpha^{-3}\Phi\Big(r_{m0}y_i - \alpha^2 r_{im}\Big) H^m,
\end{aligned}
$$

$$\tag{3.18}$$

where

$$
\begin{aligned}
y_m H^m &= \alpha(r_{00} - 2\alpha Q s_0)(\Theta + s\Psi), \\
s_{\cdot m} H^m &= Q s_0 + \alpha^{-1}(b^2 - s^2)(r_{00} - 2\alpha Q s_0)\Psi, \\
r_m H^m &= \alpha Q r_m s_0^m + (r_{00} - 2\alpha Q s_0)(\alpha^{-1} r_0 \Theta + r\Psi), \\
s_m H^m &= \alpha Q s_m s_0^m + \alpha^{-1} s_0 (r_{00} - 2\alpha Q s_0)\Theta, \\
r_{m0} H^m &= \alpha Q r_{m0} s_0^m + (r_{00} - 2\alpha Q s_0)(\alpha^{-1} r_{00}\Theta + r_0\Psi), \\
r_{im} H^m &= \alpha Q r_{im} s_0^m + (r_{00} - 2\alpha Q s_0)(\alpha^{-1} r_{i0}\Theta + r_i\Psi), \\
a_{im} H^m &= \alpha Q s_{i0} + (r_{00} - 2\alpha Q s_0)(\alpha^{-1} y_i \Theta + b_i\Psi), \\
h_{i\cdot m} H^m &= -Q(\alpha s s_{i0} + y_i s_0) - \alpha^{-1}(b^2 - s^2)(r_{00} - 2\alpha Q s_0)\Psi y_i, \\
&+ \alpha^{-1}(r_{00} - 2\alpha Q s_0)\Theta h_i.
\end{aligned}
\tag{3.19}
$$

Plugging the equations (3.14)-(3.18) into (3.11), we obtain (3.12).

PROOF OF THEOREM 2.1.

We assume that β is a closed 1-form and we have the following further assumptions

$$
r_{ij} = K(b^2 a_{ij} - b_i b_j), \tag{3.20}
$$

$$
s_{ij} = 0. \tag{3.21}
$$

We obtain the following terms.

$$
\begin{aligned}
r_{00} &= K(b^2 - s^2)\alpha^2; \quad r_i = 0; \quad r_0 = 0; \\
r_{i0} &= K(y_i b^2 - b_i s\alpha) = K(b^2 - s^2)y_i - K s h_i; \\
r_{i0} - \alpha^{-2} r_{00} y_i &= -K s h_i; \quad r_{i0;0} = -K\left(r_{i0} s\alpha + b_i r_{00}\right); \\
r_{00;i} &= -2K r_{i0} s\alpha; \quad r_{00;0} = -2K r_{00} s\alpha.
\end{aligned}
$$

We let $r_0 = 0$, $r_i = 0$, $s_{ij} = 0$ in the expression of χ_i in (3.11), then χ_i becomes as follows.

$$
\begin{aligned}
\chi_i &= -\alpha^{-4}\Phi_{ss} r_{00}^2 h_i + 2\alpha^{-5}\Phi_{ss} r_{00} h_m H^m h_i + 2\alpha^{-4}\Phi_s r_{00}^2 y_i \\
&\quad -6\alpha^{-5}\Phi_s r_{00} y_m H^m h_i + 2\alpha^{-3}\Phi_s r_{00} h_{i\cdot m} H^m - 2\alpha^{-2}\Phi_s r_{00} r_{i0} \\
&\quad +4\alpha^{-3}\Phi_s h_m H^m r_{i0} + 4\alpha^{-3}\Phi_s r_{m0} H^m h_i \\
&\quad -\alpha^{-3}\Phi_s r_{00;0} h_i - 2\alpha^{-1}\Phi r_{i0;0} + \alpha^{-1}\Phi r_{00;i} + \alpha^{-3}\Phi r_{00;0} y_i \\
&\quad -4\alpha^{-3}\Phi y_m H^m r_{i0} - 2\alpha^{-5}\Phi_s r_{00} h_m H^m y_i + 6\alpha^{-5}\Phi r_{00} y_m H^m y_i \\
&\quad -2\alpha^{-3}\bar{\Phi} r_{00} a_{im} H^m - 4\alpha^{-3}\bar{\Phi} r_{m0} H^m y_i + 4\alpha^{-1}\bar{\Phi} r_{im} H^m.
\end{aligned}
$$

We compute the following terms we need, and express them in terms of Θ, Ψ and others.

$$
\begin{aligned}
y_m H^m &= \alpha r_{00}(\Theta + s\Psi), \\
r_{m0} H^m &= \alpha^{-1} r_{00}^2 \Theta, \\
r_{im} H^m &= \alpha^{-1} r_{i0} r_{00}\Theta, \\
a_{im} H^m &= r_{00}(\alpha^{-1} y_i \Theta + b_i\Psi), \\
h_{i\cdot m} H^m &= -\alpha^{-1}(b^2 - s^2) r_{00}\Psi y_i + \alpha^{-1} r_{00}\Theta h_i, \\
h_m H^m &= \Psi r_{00}\alpha(b^2 - s^2).
\end{aligned}
$$

We rewrite χ_i by using the above calculations,

$$
\chi_i = -\alpha^{-4}\left\{\bar{\Phi}_{ss} r_{00}^2 - 2\bar{\Phi}_{ss} r_{00}^2 \Psi(b^2 - s^2) + 6s\bar{\Phi}_s \Psi r_{00}^2 + 2\bar{\Phi}\Psi r_{00}^2 + \alpha\bar{\Phi}_s r_{00;0}\right\} h_i
$$

$$+2\alpha^{-4}\left\{\bar{\Phi}_s - 2\bar{\Phi}_s\Psi(b^2-s^2)+2s\bar{\Phi}\Psi\right\}A+\alpha^{-3}\bar{\Phi}B,\tag{3.22}$$

where

$$A = r_{00}(r_{00}y_i - \alpha^2 r_{i0}); \qquad B = (r_{00;0}y_i + \alpha^2 r_{00;i} - 2\alpha^2 r_{i0;0}).\tag{3.23}$$

Moreover, when we use (3.20), we get the following simplified terms,

$$A = Ks\alpha^2 r_{00}h_i; \qquad B = 2Kr_{00}\alpha h_i.\tag{3.24}$$

Using (3.24) into (3.22), we obtain the following compact expression for χ_i.

$$\chi_i = K\left\{-\bar{\Phi}_{ss}(b^2-s^2)+2\bar{\Phi}_{ss}\Psi(b^2-s^2)^2-10s\bar{\Phi}_s\Psi(b^2-s^2)\right.$$
$$\left.-2\bar{\Phi}\Psi(b^2-3s^2)+4s\bar{\Phi}_s+2\bar{\Phi}\right\}r_{00}\alpha^{-2}h_i.\tag{3.25}$$

It is clear that $\bar{\Phi}$ in (3.10), could be rewritten as follows

$$\bar{\Phi} = \frac{\Phi}{2\Delta^2},$$

where

$$\Phi = -(Q-sQ')(n\Delta+1+sQ)-(b^2-s^2)(1+sQ)Q''.$$

We have the following fact that

$$\Phi = 2(n+1)k\frac{\phi\Delta^2}{b^2-s^2} \quad\Leftrightarrow\quad \bar{\Phi} = -(n+1)k\frac{\phi}{b^2-s^2}.$$

We have the following derivatives for $\bar{\Phi}$.

$$\bar{\Phi} = -(n+1)k\frac{\phi}{b^2-s^2},$$
$$\bar{\Phi}_s = -(n+1)k\frac{\phi_s}{(b^2-s^2)}-2(n+1)k\frac{s\phi}{(b^2-s^2)^2},$$
$$\bar{\Phi}_{ss} = -2(n+1)k\frac{\phi}{(b^2-s^2)^2}-8(n+1)k\frac{s^2\phi}{(b^2-s^2)^3}.\tag{3.26}$$

If we use $\bar{\Phi}$ in (3.10), we obtain

$$\chi_i = Kr_{00}\alpha^{-2}h_i\left\{\phi_{ss}-2\phi_{ss}(b^2-s^2)\Psi-2\phi\Psi+2s\phi_s\Psi\right\}(n+1)k.\tag{3.27}$$

Hence, using the equations below, we get $\chi_i = 0$

$$\Psi = \frac{Q'}{2\Delta}; \qquad Q = \frac{\phi'}{\phi-s\phi'},$$
$$\Delta = 1+sQ+(b^2-s^2)Q'.$$

CONCLUSION

Although this paper has a useful result expressing how a non-Riemannian quantity χ-curvature vanishes for (α, β)-metrics with Riemannian metric α and a 1-form β when β is a closed 1-form and $r_{ij} = K(b^2 a_{ij} - b_i b_i)$, where K is a constant as it is stated in the main theorem. We believe one can do more. One may consider doing computations

to see what happens to the non-Riemannian quantity χ-curvature when K is a scalar function. This would allow us to see and explore the geometrical variations for β 1-form with respect to the Riemannian metric α. Next study will basically focus on this open problem and we hope to get a good result after rigorous and somewhat harder and more time consuming computations to be done on Maple.

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