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Research Article

Characterizations of Bertrand curve pairs via new Frenet formulas

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ABSTRACT

In this article, we first introduce new Frenet formulas by making use of the properties of connected curves. Then applying these formulas we show that some decisive properties of Bertrand partner curve can be given in terms of a Bertrand curve. More precisely, we offer differential equations of the Bertrand partner curve with respect to both Levi-Civita and normal Levi-Civita connections in terms of the Bertrand curve. We also give harmonicity conditions of the partner curve of Bertrand curve pair by the same method. We obtain some new results and finally we give an example to support our allegations.

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INTRODUCTION

Bertrand curve pair which is one of the commonly used example of connected curves that we may constitute a relationship between two curve pairs. In the present paper we also use this fact to give the characterizations of a Bertrand partner curve by means of Frenet apparatus of the Bertrand curve. By making sense of the relationship between the Frenet frames of Bertrand curve pairs we first write new Frenet formulae. We show that all main properties of the Bertrand partner curve can be given through the medium of the Bertrand curve. Here from we give the harmonicity conditions of Bertrand partner curve by means of the Bertrand curve. Referring this formula we also write differential equations which represent the Bertrand partner curve through the main curve. It follows that we get smooth differential equations. Also this method made it easier for us to interpret the harmonicity of the Bertrand partner curve. We can now mention some noticeable works related to Bertrand curve pairs and applications of differential equations. Babaarslan and Yayli [1] research main characters of Bertrand curves in Euclidean space. Ekmekci and Ilarslan [2] add some contributions to Bertrand curve pairs. They give the essential situations in order to be a general helix in Lorentzian space. Okuyucu et al [3] describe some properties of the Bertrand curve couples, with giving some main properties, in a three dimensional Lie groups. Chen and Ishikawa introduce the classifications of biharmonic curves [4]. Senyurt and Cakir [5] demonstrate a new method to classify a given curve through the medium of an another curve. Kocayigit and Hacisalihoglu [6] investigate the properties of 1-type harmonic curves by making use of the mean curvature vector of the curve itself. Also Senyurt and Cakir [7] examine the biharmonic curves whose mean curvature vector is the core of the Laplace. Yokus et al [8] analyze some new type analytical answers to the well-known

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nonlinear evolutionary equations. Sulaiman et al [9] survey the coupled Boussinesq equations that lift off in water waves for two-layered fluid flow. Din and Li [10] obtain the adjoint equations using the techniques of weak derivatives and sensitivities. Din et al [11] plan a pattern and by this method verify some simple results for the well-posedness in terms of boundedness and positivity. Li et al [12] use a stochastic Markovian dynamics oncoming to define the spreading of dengue and the brink of the disease. We can now have a look at some basic notions of fundamental differential geometry. Frenet formulas along a differentiable curve α are given by

$$T' = \vartheta\kappa N,\ \ N' = -\vartheta\kappa T + \vartheta\tau B,\ \ \ B' = -\vartheta\tau N. \tag{1}$$

where ϑ is the norm of α, see [14].

Definition 1.1 Suppose that α and β are two differentiable curves whose Frenet apparatus are $\{T, N, B, \kappa, \tau\}$ and $\{T_\beta, N_\beta, B_\beta, \kappa_\beta, \tau_\beta\}$ respectively .If the curve pair (α, β) shares a mutual principal normal, then we call α as a Bertrand curve and β as a Bertrand partner of α. It is obvious from this statement that

$$\beta(s) = \alpha(s) + \lambda(s)N(s),\ \ \lambda(s)\ \in \mathbb{R} \tag{2}$$

here s is the arc length parameter of α, see [14].

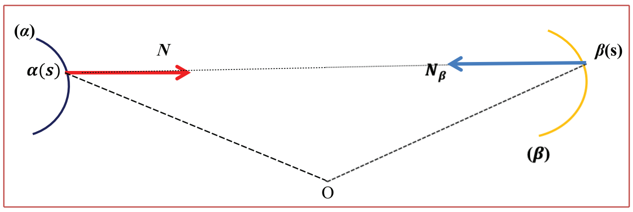

Figure 1. (α, β) Bertrand curve pairs.

When we look at the Figure 1, we realize the relationship between the Frenet vectors of α and β as

$$T_\beta = cos\theta T + sin\theta B,\ N_\beta = N,\ B_\beta = -sin\theta T + cos\theta B \tag{3}$$

where θ is the angle between the tangent vectors of α and β. Also the relation between the curvatures of α and β is stated as follows

$$\kappa_\beta(s) = \frac{\lambda\kappa - sin^2\theta}{\lambda(1-\lambda\kappa)}\ \text{ and }\ \tau_\beta(s) = \frac{1}{\lambda^2\tau}sin^2\theta\ . \tag{4}$$

We also see that the curve pairs (α, β) create a Bertrand couple under the condition

$$\lambda\kappa + \mu\tau = 1 \tag{5}$$

where λ and $\mu \in \mathbb{R}$, see [14].

Definition 1.2 Suppose that α is an arc-length curve. We constitute the mean curvature H of α as

$$H = D_{\alpha'}\alpha' = \kappa N \tag{6}$$

here D represents the Levi-Civita connexion and from here we give the Laplace operator as follows

$$\Delta: \chi^\perp(\alpha(I)) \longrightarrow \chi(\alpha(I))$$

$$\Delta H = -D_T^2 H. \tag{7}$$

Indicating the normal bundle of α by $x^\perp(\alpha\ (s))$ it is posterior to the normal connection $D^\perp$ as

$$D_T^\perp: \chi^\perp(\alpha(I)) \rightarrow \chi^\perp(\alpha(I))$$

$$D_T^\perp X = D_T X - < D_T X, T > T \tag{8}$$

and maintaining the same idea the normal Laplace operator $\Delta^\perp$ is given by

$$\Delta_T^\perp X = -D_T^\perp D_T^\perp X,\ \ \forall X \in \chi^\perp(\alpha(I)), \tag{9}$$

see [4].

Theorem 1.1 *Suppose that α is a smooth curve and H is the mean curvature vector of it. We clearly have following statements.*

i) $\Delta H = 0$ in this case α is named a biharmonic curve.
ii) $\Delta H = \lambda H$ in this case α is named a 1-type of harmonic curve, $\lambda \in \mathbb{R}$.
iii) $\Delta^\perp H = 0$ in this case α is named a weak biharmonic curve.
iv) $\Delta^\perp H = \lambda H$ in this case α is named a 1-type of harmonic curve, $\lambda \in \mathbb{R}$,
see [4].

Theorem 1.2 *Suppose that α is any differentiable curve. Differential equation which represents the curve α according to unit tangent T can be given*

$$D_T^3 T + \lambda_2 D_T^2 T + \lambda_1 D_T T + \lambda_0 T = 0$$

with the coefficients λ_0, λ_1 and λ_2

$$\lambda_0 = \vartheta^2\kappa\tau(\tfrac{\kappa}{\tau})',$$

$$\lambda_1 = \vartheta^2(\kappa^2+\tau^2) - \frac{\vartheta''}{\vartheta} - \frac{\kappa''}{\kappa} + (\frac{\vartheta'}{\vartheta} + \frac{\kappa'}{\kappa})(3\frac{\vartheta'}{\vartheta} + \frac{\tau'}{\tau}) + 2(\frac{\kappa'}{\kappa})^2,$$

$$\lambda_2 = -(3\frac{\vartheta'}{\vartheta} + 2\frac{\kappa'}{\kappa} + \frac{\tau'}{\tau}),$$

see [7].

Characterizations of Bertrand Curve Pairs Via New Frenet Formulas

In this part of the study the set $\{T, N, B, \kappa, \tau\}$ denotes the elements of Frenet of α and the set $\{T_\beta, N_\beta, B_\beta, \kappa_\beta, \tau_\beta\}$

represents the Frenet elements of the Bertrand partner curve β provided that the norm of the Bertrand partner curve is $\vartheta = \| \beta'(s) \|$.

Theorem 2.1 *Suppose that the ordered pair* (α, β) *is a Bertrand curve. Covariant derivative of* α *according to* β *is*

$$D_B T = (\frac{1-cos\theta}{sin\theta})\kappa N,$$

$$D_B N = -(\frac{1-cos\theta}{sin\theta})\kappa T + (\frac{1-cos\theta}{sin\theta})\tau B,$$

$$D_B B = -(\frac{1-cos\theta}{sin\theta})\tau N. \tag{10}$$

Proof. Applying the equalities (1.1) , (1.2) we have a norm of Bertrand partner curve $\vartheta = \| \beta'(s) \| = \tau\sqrt{\lambda^2+\mu^2}$ and it follows from the equalities (1.3) and (1.4) that

$$D_{T_\beta} T_\beta = \vartheta \kappa_\beta N_\beta$$

$$= \vartheta \frac{\lambda\kappa - sin^2\theta}{\lambda(1-\lambda\kappa)} N \tag{11}$$

We insert the equalities (1.3) , (1.4) to the left side of this equality and we get

$$D_{T_\beta} T_\beta = D_{(cos\theta T + sin\theta B)}(cos\theta T + sin\theta B)$$

$$= cos\theta D_T(cos\theta T + sin\theta B) + sin\theta D_B(cos\theta T + sin\theta B)$$

$$= cos^2\theta\kappa N - cos\theta sin\theta \tau N + cos\theta sin\theta D_B T + sin^2\theta D_B B. \tag{12}$$

By (2.2) and (2.3) we obtain

$$cos\theta sin\theta D_B T + sin^2\theta D_B B = (\vartheta \frac{\lambda\kappa - sin^2\theta}{\lambda(1-\lambda\kappa)} - \kappa cos^2\theta + \tau cos\theta sin\theta)N. \tag{13}$$

Using the same idea we may derive the counterpart of $D_{T_\beta} N_\beta$ as follows

$$D_{T_\beta} N_\beta = -\vartheta \kappa_\beta T_\beta + \vartheta \tau_\beta B_\beta$$

$$= -\vartheta \frac{\lambda\kappa - sin^2\theta}{\lambda(1-\lambda\kappa)}(cos\theta T + sin\theta B) + \frac{\vartheta sin^2\theta}{\lambda^2\tau}(-sin\theta T + cos\theta B)$$

$$= (\frac{-\vartheta sin^3\theta}{\lambda^2\tau} - \vartheta \frac{\lambda\kappa - sin^2\theta}{\lambda(1-\lambda\kappa)} cos\theta)T + (\frac{\vartheta sin^2\theta cos\theta}{\lambda^2\tau} - \vartheta \frac{\lambda\kappa - sin^2\theta}{\lambda(1-\lambda\kappa)} sin\theta)B. \tag{14}$$

Making use of the equalities (1.3) , (1.4) to the left side of this equality we get

$$D_{T_\beta} N_\beta = D_{(cos\theta T + sin\theta B)} N$$

$$= cos\theta D_T N + sin\theta D_B N$$

$$= -\kappa cos\theta T + \tau cos\theta B + sin\theta D_B N. \tag{15}$$

From (2.5) and (2.6) we find out

$$D_B N = (\kappa cot\theta - \vartheta \frac{\lambda\kappa - sin^2\theta}{\lambda(1-\lambda\kappa)} cot\theta - \frac{\vartheta sin^2\theta}{\lambda^2\tau})T + (\frac{\vartheta sin\theta cos\theta}{\lambda^2\tau} - \tau cot\theta - \vartheta \frac{\lambda\kappa - sin^2\theta}{\lambda(1-\lambda\kappa)})B.$$

From (1.5), $\lambda\kappa + \mu\tau = 1$ and $\vartheta = \tau\sqrt{\lambda^2+\mu^2}$, so we can complete that

$$D_B N = (\frac{cos\theta - 1}{sin\theta}\kappa)T + (\frac{1-cos\theta}{sin\theta}\tau)B.$$

We use the same procedures and obtain the equivalent of $D_{T_\beta} B_\beta$ as

$$D_{T_\beta} B_\beta = -\vartheta \tau_\beta N_\beta$$

$$= -\frac{\vartheta sin^2\theta}{\lambda^2\tau} N. \tag{16}$$

On the other hand left side of the above equality is

$$D_{T_\beta} B_\beta = D_{(cos\theta T + sin\theta B)}(-sin\theta T + cos\theta B)$$

$$= cos\theta D_T(-sin\theta T + cos\theta B) + sin\theta D_B(-sin\theta T + cos\theta B)$$

$$= -cos\theta sin\theta D_T T + cos^2\theta D_T B - sin^2\theta D_B T + sin\theta cos\theta D_B B$$

$$= (-\kappa cos\theta sin\theta - \tau cos^2\theta)N - sin^2\theta D_B T + sin\theta cos\theta D_B B. \tag{17}$$

From (2.7) and (2.8) we get

$$-sin^2\theta D_B T + cos\theta sin\theta D_B B = (\kappa sin\theta cos\theta - \frac{\vartheta sin^2\theta}{\lambda^2\tau} + \tau cos^2\theta)N. \tag{18}$$

From the equalities (2.4) and (2.9) we obtain $D_B T$ and $D_B B$ as

$$D_B T = (\frac{\vartheta sin^2\theta}{\lambda^2\tau} + \vartheta \frac{\lambda\kappa - sin^2\theta}{\lambda(1-\lambda\kappa)} cot\theta - \kappa cot\theta)N = (\frac{1-cos\theta}{sin\theta}\kappa)N$$

and

$$D_B B = (\vartheta \frac{\lambda\kappa - sin^2\theta}{\lambda(1-\lambda\kappa)} - \frac{\vartheta sin\theta cos\theta}{\lambda^2\tau} + \tau cot\theta)N = (\frac{cos\theta - 1}{sin\theta}\tau)N.$$

Theorem 2.2 *Suppose that* (α, β) *is a Bertrand curve pair. Covariant derivative of* α *according to normal Levi-Civita connection* $D^\perp$ *is given as*

$$D_B^\perp T = (\frac{1-cos\theta}{sin\theta})\kappa N, \quad D_B^\perp N = -(\frac{1-cos\theta}{sin\theta})\kappa T. \tag{19}$$

Proof. It is obvious from (2.1) that we have

$$D_B T = (\frac{1-cos\theta}{sin\theta})\kappa N,$$

$$D_B N = -(\frac{1-cos\theta}{sin\theta})\kappa T + (\frac{1-cos\theta}{sin\theta})\tau B,$$

$$D_B B = -(\frac{1-cos\theta}{sin\theta})\tau N.$$

Now we use the proposition (1.8) and we get

$$D_B^\perp X = D_B X - \langle D_B X, B \rangle B \implies D_B^\perp T = (\frac{1-cos\theta}{sin\theta})\kappa N$$

and

$$D_B^{\perp}X = D_BX - < D_BX, B > B \quad \Rightarrow D_B^{\perp}N = -(\frac{1-cos\theta}{sin\theta})\kappa T.$$

Theorem 2.3 *Suppose that the ordered pair (α, β) is a Bertrand curve. Following statements are true according to connection D*

1) Partner curve β is biharmonic under the condition that

$$(3\lambda\kappa - sin^2\theta)\kappa' = 0,$$
$$(\lambda\kappa - sin^2\theta)(1-cos\theta)^2(\kappa^2+\tau^2) - \lambda\kappa'' sin^2\theta = 0,$$
$$sin^2\theta\tau' - \lambda\kappa\tau' - 2\lambda\kappa'\tau = 0. \quad (20)$$

2) Partner curve β is 1-type of harmonic under the condition that

$$(3\lambda\kappa - sin^2\theta)\kappa' = 0,$$
$$\frac{(\lambda\kappa - sin^2\theta)(1-cos\theta)^2(\kappa^2+\tau^2) - \lambda\kappa'' sin^2\theta}{sin^2\theta} = \lambda\,(\lambda\kappa - sin^2\theta),$$
$$sin^2\theta\tau' - \lambda\kappa\tau' - 2\lambda\kappa'\tau = 0, \quad \lambda \in \mathbb{R}. \quad (21)$$

Proof. Using (1.6) we obviously have $\mathrm{H}_\beta = \vartheta\kappa_\beta N_\beta$ and also using (1.3) , (1.4) we get

$$\mathrm{H}_\beta = \vartheta\frac{\lambda\kappa - sin^2\theta}{\lambda(1-\lambda\kappa)}N = (\frac{\lambda\kappa - sin^2\theta}{\mu\, sin\theta})\,N. \quad (22)$$

If we constrict the coefficients of (2.1) as in the following way

$$D_BT = (\frac{1-cos\theta}{sin\theta})\kappa N = \alpha_1 N,$$
$$D_BN = -(\frac{1-cos\theta}{sin\theta})\kappa T + (\frac{1-cos\theta}{sin\theta})\tau B = -\alpha_1 T - \alpha_2 B,$$
$$D_BB = -(\frac{1-cos\theta}{sin\theta})\tau N = \alpha_2 N$$

and the mean curvature of H_β as

$$\mathrm{H}_\beta = \vartheta\frac{\lambda\kappa - sin^2\theta}{\lambda(1-\lambda\kappa)}N = c_1N$$

we may acquire the Laplace image of mean curvature more clearly. Equalities (1.1) , (1.7) give us

$$\begin{aligned}\Delta\mathrm{H}_\beta &= -D_B^2(\mathrm{H}_\beta) = -D_BD_B(c_1N)\\ &= -D_B(c'_1N + c_1D_BN)\\ &= (c'_1\alpha_1 + (c_1\alpha_1)')T + (c_1(\alpha_1^2+\alpha_2^2) - c''_1)N + (c'_1\alpha_2 + (c_1\alpha_2)')B\\ &= (\frac{(3\lambda\kappa - sin^2\theta)\kappa'}{\mu(1+cos\theta)})T + (\frac{(\lambda\kappa - sin^2\theta)(1-cos\theta)^2(\kappa^2+\tau^2) - \lambda\kappa'' sin^2\theta}{\mu sin^3\theta})N\\ &\quad + (\frac{sin^2\theta\tau' - \lambda\kappa\tau' - 2\lambda\kappa'\tau}{\mu(1+cos\theta)})B.\end{aligned}$$

If we pay attention to the case ΔH = 0 it is clear that (2.11) is satisfied. Taking the condition $\Delta\mathrm{H}_\beta = \lambda\mathrm{H}_\beta$ into consideration it is seen that second case (2.12) is also true.

Theorem 2.4 *Suppose that the ordered pair (α, β) is a Bertrand curve. Following statements are true according to normal connection $D^{\perp}$.*

1) Partner curve β is weak biharmonic under the condition that

$$3\lambda\kappa\kappa' - sin^2\theta\,\kappa' = 0,$$
$$\frac{(\lambda\kappa - sin^2\theta)(1-cos\theta)^2\kappa^2 - \lambda\kappa'' sin^2\theta}{\mu sin^3\theta} = 0. \quad (23)$$

2) Partner curve β is 1-type of harmonic under the condition that

$$3\lambda\kappa\kappa' - sin^2\theta\,\kappa' = 0,$$
$$\frac{(\lambda\kappa - sin^2\theta)(1-cos\theta)^2\kappa^2 - \lambda\kappa'' sin^2\theta}{sin^2\theta} = \lambda(\lambda\kappa - sin^2\theta). \quad (24)$$

Proof. By the Theorem 2.3 we can write

$D_BT = \alpha_1N$, $D_BN = -\alpha_1T - \alpha_2B$, $D_BB = \alpha_2N$ and $\mathbb{H}_\beta = c_1N$.

From (1.1) , (1.9) we may estimate the normal Laplace of mean curvature H_β as

$$\begin{aligned}\Delta^{\perp}\mathrm{H}_\beta &= -D_B^{\perp}D_B^{\perp}(c_1N)\\ &= -D_B^{\perp}(c'_1N + c_1D_B^{\perp}N)\\ &= (\frac{3\lambda\kappa\kappa' - sin^2\theta\,\kappa'}{\mu(1+cos\theta)})T + (\frac{(\lambda\kappa - sin^2\theta)(1-cos\theta)^2\kappa^2 - \lambda\kappa'' sin^2\theta}{\mu sin^3\theta})N.\end{aligned}$$

Paying attention to the case $\Delta^{\perp}\mathrm{H}_\beta = 0$ it is clear that (2.14) is true. By the same way taking the phrase $\Delta^{\perp}\mathrm{H}_\beta = \lambda\mathrm{H}_\beta$ into consideration we see that (2.15) is also fulfilled.

Corollary 2.1 *Suppose that (α, β) is a Bertrand curve pair. It succeeds that partner curve β is 1-type of harmonic curve with respect to connection D if and only if α is a circular helix.*

Proof. Say that a circular helix is given as $\alpha(t) = \frac{1}{\sqrt{2}}(cost, sint, t)$. It follows the Bertrand partner is

$$\beta(t) = \frac{1}{\sqrt{2}}(cost, sint, t) + \lambda(-cost, -sint, 0),\ \lambda \in \mathbb{R}\,.$$

Evaluating the Laplace of $\mathrm{H}_\beta = \frac{\lambda\kappa - sin^2\theta}{\mu sin\theta}N$ from Theorem 2.1 we find

$$\Delta\mathrm{H}_\beta = \frac{\lambda\kappa - sin^2\theta}{\mu\, sin\theta}\ (\frac{cos\theta - 1}{sin\theta})^2 N.$$

Considering the equation (2.12), we conclude that β is 1-type of harmonic curve with $\lambda = (\frac{cos\theta-1}{sin\theta})^2 \in \mathbb{R}$.

Corollary 2.2 *Given that* (α, β) *is a Bertrand curve pair. It succeeds that partner curve* β *is 1-type of harmonic curve according to normal connection* $D^{\perp}$ *if and only if* α *is a circular helix.*

Proof. Suppose that $\alpha(t) = \frac{1}{\sqrt{2}}(cost, sint, t)$ is a circular helix. It follows that the Bertrand partner is

$$\beta(t) = \frac{1}{\sqrt{2}}(cost, sint, t) + \lambda(-cost, -sint, 0), \ \ \lambda \in \mathbb{R}.$$

Evaluating the normal Laplace of $H_\beta = \frac{\lambda\kappa - sin^2\theta}{\mu sin\theta}N$ from Theorem 2.2 we figure out that

$$\Delta^{\perp}H_\beta = \frac{\lambda\kappa - sin^2\theta}{\mu\, sin\theta}\left(\frac{cos\theta - 1}{sin\theta}\right)^2 N.$$

When we consider (2.15), we deduce that β is a 1 type of harmonic curve where $\lambda = \frac{1}{2}\left(\frac{cos\theta-1}{sin\theta}\right)^2 \in \mathbb{R}$.

Theorem 2.5 *Suppose that the ordered pair* (α, β) *is a Bertrand curve. Differential equation which represents the partner curve* β *according to binormal* B *of* α *given by*

$$D_B^3B - \left(\frac{\kappa'}{\kappa} + 2\frac{\tau'}{\tau}\right)D_B^2B + \left(\left(\frac{1-cos\theta}{sin\theta}\right)^2(\kappa^2 + \tau^2) + 2\left(\frac{\tau'}{\tau}\right)^2 + \frac{\kappa'\tau'}{\kappa\tau} - \frac{\tau''}{\tau}\right)D_BB + \left(\kappa\tau\left(\frac{1-cos\theta}{sin\theta}\right)^2\left(\frac{\tau}{\kappa}\right)'\right)B = 0. \quad (25)$$

Proof. From Theorem 2.1 we have

$$D_BT = \alpha_1 N, D_BN = -\alpha_1 T - \alpha_2 B, D_BB = \alpha_2 N \quad (26)$$

with the coefficients α_1, α_2

$$\alpha_1 = \left(\frac{1-cos\theta}{sin\theta}\kappa\right) \quad and \quad \alpha_2 = \left(\frac{cos\theta-1}{sin\theta}\tau\right).$$

Obviously we may write

$$D_BB = \alpha_2 N \ \ \Longrightarrow \ \ N = \frac{1}{\alpha_2}D_BB \quad (27)$$

and

$$D_BT = \alpha_1 N \ and \ D_BB = \alpha_2 N \Longrightarrow D_BT = \frac{\alpha_1}{\alpha_2}D_BB. \quad (28)$$

When we take the successive derivatives of $D_BB = \alpha_2 N$ according to B we acquire

$$D_BB = \alpha_2 N \Longrightarrow D_B(D_BB) = D_B(\alpha_2 N)$$
$$D_B^2B = -\alpha_1\alpha_2 T + \alpha'_2 N - \alpha_2^2 B$$

and

$$D_B(D_B^2B) = D_B(-\alpha_1\alpha_2 T + \alpha'_2 N - \alpha_2^2 B)$$
$$D_B^3B = (-\alpha_1\alpha_2)'T - \alpha_1\alpha_2 D_BT + \alpha''_2 N + \alpha'_2 D_BN - (\alpha_2^2)'B - \alpha_2^2 D_BB.$$

We put the tantamounts of D_BN, N and D_BT respectively from (2.17), (2.18) and (2.19) we have

$$D_B^3B = (-(\alpha_1\alpha_2)' - \alpha_1\alpha'_2)T + \left(\frac{\alpha''_2}{\alpha_2} - \alpha_1^2 - \alpha_2^2\right)D_BB - (\alpha_2\alpha'_2 + (\alpha_2^2)')B. \quad (29)$$

Computing the tantamount of T we acquire that

$$D_BN = -\alpha_1 T - \alpha_2 B \Longrightarrow T = -\frac{1}{\alpha_1}D_BN - \frac{\alpha_2}{\alpha_1}B = \frac{-1}{\alpha_1\alpha_2}D_B^2B + \frac{\alpha'_2}{\alpha_1\alpha_2^2}D_BB - \frac{\alpha_2}{\alpha_1}B.$$

Finally putting this in (2.20) we get

$$D_B^3B + ((\alpha_1\alpha_2)' + \alpha_1\alpha'_2)\left(\frac{-1}{\alpha_1\alpha_2}D_B^2B + \frac{\alpha'_2}{\alpha_1\alpha_2^2}D_BB - \frac{\alpha_2}{\alpha_1}B\right) + \left(\frac{-\alpha''_2}{\alpha_2} + \alpha_1^2 + \alpha_2^2\right)D_BB + (\alpha_2\alpha'_2 + (\alpha_2^2)')B = 0.$$

Or in short

$$D_B^3B - \left(\frac{\alpha'_1}{\alpha_1} + 2\frac{\alpha'_2}{\alpha_2}\right)D_B^2B + \left(\alpha_1^2 + \alpha_2^2 + 2\left(\frac{\alpha'_2}{\alpha_2}\right)^2 + \frac{\alpha'_1\alpha'_2}{\alpha_1\alpha_2} - \frac{\alpha''_2}{\alpha_2}\right)D_BB + \left(\alpha'_2\alpha_2 - \frac{\alpha'_1}{\alpha_1}\alpha_2^2\right)B = 0.$$

Theorem 2.6 *Suppose that the ordered pair* (α, β) *is a Bertrand curve. We write the following differential equations which represent the partner curve* β *with respect to the normal connection* $D^{\perp}$.

i) We write an equation according to unit tangent T as

$$\left(\frac{1-cos\theta}{sin\theta}\kappa\right)D_B^{\perp}D_B^{\perp}T - \left(\frac{1-cos\theta}{sin\theta}\kappa'\right)D_B^{\perp}T + \left(\frac{1-cos\theta}{sin\theta}\kappa\right)^3 T = 0.$$

ii) We write an equation according to principal normal N as

$$\left(\frac{1-cos\theta}{sin\theta}\kappa\right)D_B^{\perp}D_B^{\perp}N - \left(\frac{1-cos\theta}{sin\theta}\kappa'\right)D_B^{\perp}N + \left(\frac{1-cos\theta}{sin\theta}\kappa\right)^3 N = 0.$$

Proof. From Theorem 2.2 we have

$$D_B^{\perp}T = \left(\frac{1-cos\theta}{sin\theta}\kappa\right)N = \alpha_1 N \ and \ D_B^{\perp}N = -\left(\frac{1-cos\theta}{sin\theta}\kappa\right)T = -\alpha_1 T.$$

Taking derivative with respect to normal connection $D^{\perp}$ gives

$$D_B^{\perp}T = \alpha_1 N \ \ \Longrightarrow \ \ D_B^{\perp}(D_B^{\perp}T) = D_B^{\perp}(\alpha_1 N) = \alpha'_1 N + \alpha_1 D_B^{\perp}N = \frac{\alpha'_1}{\alpha_1}D_B^{\perp}T - \alpha_1^2 T.$$

Rewriting the equality we get

$$\alpha_1 D_B^{\perp}D_B^{\perp}T - \alpha'_1 D_B^{\perp}T + \alpha_1^3 T = 0.$$

By the same way

$$D_B^{\perp}N = -\alpha_1 T \quad \Longrightarrow \quad D_B^{\perp}(D_B^{\perp}N) = D_B^{\perp}(-\alpha_1 T)$$

$$= -\alpha'_1 T - \alpha_1 D_B^{\perp}T$$

$$= \frac{\alpha'_1}{\alpha_1} D_B^{\perp}N - \alpha_1^2 N.$$

It follows that

$$\alpha_1 D_B^{\perp} D_B^{\perp} N - \alpha'_1 D_B^{\perp} N + \alpha_1^3 N = 0.$$

Example 2.1 *Given that we have a circular helix with the equation of* $\alpha(t) = \frac{1}{\sqrt{2}}(cost, sint, t)$. *It is clear that the Bertrand partner of* α *is*

$$\beta(t) = \frac{1}{\sqrt{2}}(cost, sint, t) + \lambda(-cost, -sint, 0)\,, \quad \lambda \in \mathbb{R}.$$

If we make some algebraic operations we find out the differential equations which characterize the partner curve β in terms of the main curve as

i) If we consider the Theorem 2.1 we can write

$$D_B^3 B + \left(\frac{1-cos\theta}{sin\theta}\right)^2 D_B B = 0.$$

ii) If we consider the Theorem 2.2 we can write

$$D_B^{\perp} D_B^{\perp} T + \left(\frac{cos\theta - 1}{\sqrt{2}\, sin\theta}\right)^2 T = 0 \text{ and } D_B^{\perp} D_B^{\perp} N + \left(\frac{cos\theta - 1}{\sqrt{2}\, sin\theta}\right)^2 N = 0.$$

CONCLUSION

We derive new Frenet formulas by taking advantage of the properties of connected curves. Using these formulas we give some basic qualifications of Bertrand partner curve through the medium of the Bertrand curve. With the help of this method we obtain elementary differential equations and also this method made it easier for us to expound the harmonicity of the Bertrand partner curve. We conclude that giving the qualifications of partner curve through the medium of the main curve facilitates the calculations. We hope that this paper reveals the geometers to search similar scientific inquiries in non-Euclidean spaces.

AUTHORSHIP CONTRIBUTIONS

Concept: S.Ş., O.Ç.; Design: S.Ş., O.Ç.; Supervision: S.Ş., O.Ç.; Materials: S.Ş., O.Ç.; Analysis: S.Ş., O.Ç.; Literature search: S.Ş., O.Ç.; Writing: S.Ş., O.Ç.; Critical revision: S.Ş., O.Ç.

DATA AVAILABILITY STATEMENT

This publication comprises the graphics and data which are all developed during the present study.

CONFLICT OF INTEREST

We clearly proclaim no potential conflicts of interest with respect to the research, authorship and publication of this article.

ETHICS

There is no ethical problem in publishing this article.

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