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Research Article

New moving frames for the curves lying on a surface

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ABSTRACT

In this article, three new orthogonal frames are defined for the curves lying on a surface. These moving frames, obtained based on the Darboux frame, are called "Osculator Darboux Frame", "Normal Darboux Frame" and "Rectifying Darboux Frame", respectively. Also, the Osculator Darboux Frame components and curvatures are calculated for a presented example.

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INTRODUCTION

The concept of frame is important in the differential geometry of curves. One of the most important tools used to analyze a curve is a moving frame. The relationship of the vector fields forming the frame at the opposite points of two different curves reveals the special curve pairs [1,2]. Curvature functions are defined on the curve using moving frames [3,4]. These curvature functions are called differential invariants of the curve. Curves become special thanks to the relationships between the differential invariants of the curve [5-10]. Many different frames have been defined in different spaces [11-13]. The most commonly used moving frames are the Frenet frame and Bishop frame for the space curves, and the Darboux frame for the surface curves. The Darboux frame is known as the frame of the curve-surface pair [14-17]. Hananoi et al. describe three new vector fields associated with the Darboux frame along the curve on the surface [15]. In addition, Önder defines three new special curves on the surface, taking these three new vectors into account. In this definition, he names these curves as D_i-Darboux slant helices, where the indices $i \in \{o, n, r\}$ represent the osculator, normal, and rectifying planes of the curve on the surface, respectively [18].

In this study, three new moving frames are constructed for the surface curves using these three new vector fields defined in [15]. The curvature functions are calculated for each frame. Relevant theorems are presented with their proofs.

Preliminaries

In this section, some basic concepts related to the subject discussed are presented.

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Let M be an oriented surface in 3-dimensional Euclidean space E^3, and let $\alpha(s): I \subset R \to M$ be a unit speed surface curve on M and s be the arc-length parameter of α. If we denote the Frenet frame of α by $\{T, N, B\}$, then the Frenet equations of α are given by

$$\begin{aligned} T' &= \kappa N \\ N' &= -\kappa T + \tau B \\ B' &= -\tau N \end{aligned}$$

where $\kappa(s)$ is curvature (or first curvature function), $\tau(s)$ is torsion (or second curvature function) and, T, N and B are the unit tangent vector, the principal normal vector and the binormal vector of α, respectively.

On the other hand, since the curve α is a surface curve, it has another orthonormal frame called the Darboux frame and this frame is denoted by $\{T, V, U\}$, where T is the unit tangent of the curve, U is the unit normal of the surface M along the curve α, and V is a unit vector defined by $V = UxT$. Using the fact that the unit tangent T is common in both the Frenet frame and the Darboux frame, the relation between these frames can be given as follows

$$\begin{bmatrix} T \\ V \\ U \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix},$$

where θ is the angle between the vectors V and N. The Darboux equations of α are given by

$$\begin{bmatrix} T' \\ V' \\ U' \end{bmatrix} = \begin{bmatrix} 0 & k_g & k_n \\ -k_g & 0 & \tau_g \\ -k_n & -\tau_g & 0 \end{bmatrix} \begin{bmatrix} T \\ V \\ U \end{bmatrix}, \tag{1}$$

where k_n, k_g and τ_g are called normal curvature, geodesic curvature, and geodesic torsion of α, respectively [19]. The relations between these curvatures and κ, τ are given as follows

$$k_g = \kappa\cos\theta,\ k_n = \kappa\sin\theta,\ \tau_g = \tau + \frac{d\theta}{ds}.$$

Definition 2.1: The osculator Darboux vector field for a unit speed curve on an oriented surface M with the Darboux frame $\{T, V, U\}$ is defined as $D_o = \tau_g(s)T(s) - k_n(s)V(s)$ [15].

Definition 2.2: The normal Darboux vector field for a unit speed curve on an oriented surface M with the Darboux frame $\{T, V, U\}$ is defined as $D_n = -k_n(s)V(s) + k_g(s)U(s)$ [15].

Definition 2.3: The rectifying Darboux vector field for a unit speed curve on an oriented surface M with the Darboux frame $\{T, V, U\}$ is defined as $D_r = \tau_g(s)T(s) + k_g(s)U(s)$ [15].

Definition 2.4: A unit speed curve is called a generalized helix if its unit tangent vector makes a constant angle with a fixed direction [14].

Definition 2.5: Let α be a unit speed curve on an oriented surface M and $\{T, V, U\}$ be the Darboux frame along α. The curve α is called a relatively normal-slant helix if the vector field V of α makes a constant angle with a fixed direction, i.e. there exists a constant unit vector d and a constant angle φ such that $\langle V, d\rangle = \cos\varphi$ [16].

Definition 2.6: Let α be a unit speed curve on an oriented surface M and $\{T, V, U\}$ be the Darboux frame along α. The curve α is called an isophote curve (or U-strip slant helix) if the vector field V of α makes a constant angle with a fixed direction, i.e. there exists a constant unit vector d and a constant angle φ such that $\langle V, d\rangle = \cos\varphi$ [17].

The Osculator Darboux Frame

In this section, a new frame called the "Osculator Darboux Frame" and related theorem are presented.

Let $\alpha: I \subset IR \to M$ be a unit speed curve with the Darboux frame $\{T, V, U\}$ on surface M in E^3. Let k_g, k_n, τ_g be the curvatures of the curve α and $D_o = \tau_g(s)T(s) - k_n(s)V(s)$ be the osculator Darboux vector. It is clear that;

$$\tilde{D}_o(s) = \frac{D_o(s)}{\|D_o(s)\|} = \frac{\tau_g(s)}{\sqrt{\tau_g^2(s) + k_n^2(s)}}T(s) - \frac{k_n(s)}{\sqrt{\tau_g^2(s) + k_n^2(s)}}V(s).$$

Since the vector $\tilde{D}_o$ is in the plane spanned by T and V, the vectors $\tilde{D}_o$ and V are perpendicular to each other. So, $\tilde{D}_o \perp U$. Hence, the unit vector $Y_o = \tilde{D}_o \times U$ can be defined. It is clear that the vectors $\tilde{D}_o, U, Y_o$ are unit vectors and perpendicular to each other. Therefore, using these vectors $\tilde{D}_o, U, Y_o$, a new orthonormal frame can be constructed along the curve α on the surface.

Definition 3.1. The frame constructed with the vector fields $\{\tilde{D}_o, U, Y_o\}$ defined as $\tilde{D}_o = \|D_o\|$, $\tilde{D}_o \perp U$, $Y_o = \tilde{D}_o \times U$ is called the osculator Darboux frame, or OD-frame briefly.

Theorem 3.1. Let $\alpha: I \subset IR \to M$ be a unit speed curve on surface M in E^3. The osculator Darboux frame of the curve α is defined as

$$\begin{aligned} \tilde{D}_o' &= -\delta_o Y_o \\ U' &= \mu_o Y_o \\ Y_o' &= \delta_o \tilde{D}_o - \mu_o U, \end{aligned} \tag{2}$$

where $\delta_o = \left(\frac{\tau_g}{k_n}\right)' \left(\frac{k_n^2}{k_n^2 + \tau_g^2}\right) + k_g$ and $\mu_o = \sqrt{k_n^2 + \tau_g^2}$.

Proof: Firstly; $\tilde{D}_o'$ can be expressed as a linear combination of the vectors $\{\tilde{D}_o, U, Y_o\}$ as follows:

$$\tilde{D}_o' = a_1\tilde{D}_o + a_2 U + a_3 Y_o. \tag{3}$$

By the inner product both sides of the equality (3) with D_o, the equality

$$\left\langle \tilde{D}_o', \tilde{D}_o \right\rangle = a_1$$

is obtained. Since $\left\| \tilde{D}_o \right\| = 1$, $\left\langle \tilde{D}_o, \tilde{D}_o \right\rangle = 1$ and thus

$$a_1 = \left\langle \tilde{D}_o', \tilde{D}_o \right\rangle = 0.$$

By the inner product of both sides of the equality (3) with U, the equality

$$\left\langle \tilde{D}_o', U \right\rangle = a_2$$

is obtained. On the other hand, the vector $\tilde{D}_o$ can be written as

$$\tilde{D}_o = \sin\Phi T - \cos\Phi V, \tag{4}$$

with the help of the equations $\dfrac{\tau_g}{\sqrt{\tau_g^2 + k_n^2}} = \sin\Phi$ and $\dfrac{k_n}{\sqrt{\tau_g^2 + k_n^2}} = \cos\Phi$. If the derivatives of the Darboux frame vectors (1) are used in the derivative of the equality (4),

$$\tilde{D}_o' = (\Phi' + k_g)\cos\Phi T + (\Phi' + k_g)\sin\Phi V. \tag{5}$$

By the inner product of both sides of the equality (5) with U, the equality

$$a_2 = \left\langle \tilde{D}_o', U \right\rangle = 0$$

is obtained. By the inner product of both sides of the equality (3) with Y_o, the equality

$$\left\langle \tilde{D}_o', Y_o \right\rangle = a_3$$

is obtained. On the other hand, it is clear that

$$Y_o = \tilde{D}_o \times U = -\cos\Phi T - \sin\Phi V. \tag{6}$$

By the inner product of both sides of the equality (5) with this Y_o, the equality

$$a_3 = \left\langle \tilde{D}_o', Y_o \right\rangle = -(\Phi' + k_g)$$

is obtained. In here, by arranging the derivative of the expression $\tan\Phi = \dfrac{\tau_g(s)}{k_n(s)}$, the equality

$$\Phi'(s) = \left(\frac{\tau_g(s)}{k_n(s)}\right)' \left(\frac{k_n^2(s)}{k_n^2(s) + \tau_g^2(s)}\right) \tag{7}$$

is reached. If this expression is written in the equality $a_3 = -(\Phi' + k_g)$,

$$a_3 = -\left[\left(\frac{\tau_g}{k_n}\right)' \left(\frac{k_n^2}{k_n^2 + \tau_g^2}\right) + k_g\right].$$

Thus, the equation (3) is written as

$$\tilde{D}_o' = -\left[\left(\frac{\tau_g}{k_n}\right)' \left(\frac{k_n^2}{k_n^2 + \tau_g^2}\right) + k_g\right] Y_o. \tag{8}$$

Secondly; U' can be expressed as a linear combination of the vectors $\{\tilde{D}_o, U, Y_o\}$ as follows:

$$U' = b_1\tilde{D}_o + b_2 U + b_3 Y_o. \tag{9}$$

The equality (9) is inner multiplied with $\tilde{D}_o$ and so

$$\left\langle U', \tilde{D}_o \right\rangle = b_1.$$

By the inner product of both sides of the equality (4) with the third equality $U' = -k_n T - \tau_g V$ of the equation system (1), the equality

$$b_1 = \left\langle U', \tilde{D}_o \right\rangle = 0$$

is found. The equality (9) is inner multiplied with U and so

$$\left\langle U', U \right\rangle = b_2.$$

Since $\left\| U \right\| = 1$, $\left\langle U, U \right\rangle = 1$ and thus

$$b_2 = \left\langle U', U \right\rangle = 0.$$

The equality (9) is inner multiplied with Y_o and so

$$\left\langle U', Y_o \right\rangle = b_3.$$

By the inner product of both sides of the equality (6) with the third equality $U' = -k_n T - \tau_g V$ of the equation system (1), the equality

$$b_3 = \left\langle U', Y_o \right\rangle = \sqrt{\tau_g^2 + k_n^2}$$

is obtained. Thus, the equation (9) is written as

$$U' = \sqrt{\tau_g^2 + k_n^2}\, Y_o. \tag{10}$$

Thirdly; Y_o can be expressed as a linear combination of the vectors $\{\tilde{D}_o, U, Y_o\}$ as follows:

$$Y_o' = c_1 \tilde{D}_o + c_2 U + c_3 Y_o. \tag{11}$$

By the inner product of both sides of the equality (11) with $\tilde{D}_o$, the equality

$$\left\langle Y_o', \tilde{D}_o \right\rangle = c_1$$

is obtained. If the derivatives of the Darboux frame vectors (1) are used in the derivative of the equality (6),

$$Y_o' = (\Phi' + k_g)\sin\Phi T - (\Phi' + k_g)\cos\Phi V - (k_n \cos\Phi + \tau_g \sin\Phi)U. \tag{12}$$

The equality (12) is inner multiplied with the equality (4) and so

$$c_1 = \left\langle Y_o', \tilde{D}_o \right\rangle = \Phi' + k_g.$$

In this here, the equality (7) is used and

$$c_1 = \left(\frac{\tau_g}{k_n}\right)' \left(\frac{k_n^2}{k_n^2 + \tau_g^2}\right) + k_g.$$

By the inner product of both sides of the equality (11) with U, the equality

$$\left\langle Y_o', U \right\rangle = c_2$$

is obtained. Using the equation (12),

$$c_2 = \left\langle Y_o', U \right\rangle = -\sqrt{\tau_g^2 + k_n^2}.$$

By the inner product of both sides of the equality (11) with Y_o, the equality

$$c_3 = \left\langle Y_o', Y_o \right\rangle$$

is obtained. Since $\|Y_o\| = 1$, $\left\langle Y_o, Y_o \right\rangle = 1$ and thus

$$c_3 = \left\langle Y_o', Y_o \right\rangle = 0.$$

Thus, the equation (11) is written as

$$Y_o' = \left[\left(\frac{\tau_g}{k_n}\right)' \left(\frac{k_n^2}{k_n^2 + \tau_g^2}\right) + k_g\right] \tilde{D}_o - \sqrt{\tau_g^2 + k_n^2}\, U. \tag{13}$$

Finally, the expressions (8), (10) and (13) give the matrix equality

$$\begin{pmatrix} \tilde{D}_o' \\ U' \\ Y_o' \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\delta_o \\ 0 & 0 & \mu_o \\ \delta_o & -\mu_o & 0 \end{pmatrix} \begin{pmatrix} \tilde{D}_o \\ U \\ Y_o \end{pmatrix},$$

for $\delta_o = \left(\frac{\tau_g}{k_n}\right)' \left(\frac{k_n^2}{k_n^2 + \tau_g^2}\right) + k_g$, $\mu_o = \sqrt{k_n^2 + \tau_g^2}$.

Theorem 3.2: In E^3, let $\{D_o, U, Y_o\}$ be the OD-frame and α be a unit speed curve. The curve α is the isophote curve relative to the OD-frame if and only if the expression $\frac{\mu_o}{\delta_o}$ is constant, for $\mu_o \neq 0$ and $\delta_o \neq 0$.

Proof: ($\Rightarrow$) Let α be the isophote curve relative to the OD-frame. Let d be the unit, constant direction and $\langle U, d\rangle = \cos\theta = c \neq 0$. So, the vector d can be expressed as

$$d = a_1 \tilde{D}_o + cU + a_2 Y_o,$$

and the derivative of this equality gives the system.

$$\begin{aligned} a_1' + a_2 \delta_o &= 0, \\ a_2 \mu_o &= 0, \\ a_2' - a_1 \delta_o + c\mu_o &= 0. \end{aligned}$$

Since $\mu_o \neq 0$ and $\delta_o \neq 0$, $a_2 = 0$ and $a_1 =$ constant become. As a result, $\frac{\mu_o}{\delta_o} =$ constant.

($\Leftarrow$) Let $\frac{\mu_o}{\delta_o}$ be constant. This constant value can be selected as $\frac{\mu_o}{\delta_o} = \frac{\cos\theta}{\sin\theta}$. Also the vector d can be taken as $d = \cos\theta \tilde{D}_o + \sin\theta U$. Since $\mu_o \sin\theta = \delta_o \cos\theta$,

$$d' = 0$$

is written with the help of the derivatives of the OD-frame vectors. So, the vector d is constant. In addition, the equality $d = \cos\theta \tilde{D}_o + \sin\theta U$ is inner multiplied with U and so $\langle d, U\rangle = \sin\theta$ Consequently, the constant vector d and U make a constant angle, and the curve α is the isophote curve.

The Normal Darboux Frame

In this section, a new frame called the "Normal Darboux Frame" and related theorems are presented.

Let $\alpha: I \subset IR \to M$ be a unit speed curve with the Darboux frame $\{T, V, U\}$ on surface M in E^3. Let k_g, k_n, τ_g be the curvatures of the curve α and $D_n = -k_n(s)V(s) + k_g(s)U(s)$ be the normal Darboux vector. It is clear that;

$$\tilde{D}_n(s) = \frac{D_n(s)}{\|D_n(s)\|} = -\frac{k_n(s)}{\sqrt{k_n^2(s)+k_g^2(s)}}V(s) + \frac{k_g(s)}{\sqrt{k_n^2(s)+k_g^2(s)}}U(s).$$

Since the vector $\tilde{D}_n$ is in the plane spanned by U and V, the vectors $\tilde{D}_n$ and T are perpendicular to each other. So, $\tilde{D}_n \perp T$. Hence, the unit vector $Y_n = \tilde{D}_n \times T$ can be defined. It is clear that the vectors $\tilde{D}_n, T, Y_n$ are unit vectors and perpendicular to each other. Therefore, using these vectors $\tilde{D}_n, T, Y_n$ a new orthonormal frame can be constructed along the curve α on the surface.

Definition 4.1. The frame constructed with the vector fields $\{\tilde{D}_n, T, Y_n\}$ defined as $\tilde{D}_n = \frac{D_n(s)}{\|D_n(s)\|}$, $\tilde{D}_n \perp T$, $Y_n = \tilde{D}_n \times T$ is called the normal Darboux frame or ND-frame briefly.

Theorem 4.1. Let $\alpha: I \subset IR \to M$ be a unit speed curve on surface M in E^3. The normal Darboux frame of the curve α is defined as

$$\begin{aligned} \tilde{D}_n' &= -\delta_n Y_n \\ T' &= \mu_n Y_n \\ Y_n' &= \delta_n \tilde{D}_n - \mu_n T, \end{aligned}$$

where $\delta_n = \left(\frac{k_n}{k_g}\right)' \left(\frac{k_g^2}{k_n^2 + k_g^2}\right) + \tau_g$ and $\mu_n = \sqrt{k_n^2 + k_g^2}$.

Proof: It can be proved in a similar way to Theorem 3.1.

Theorem 4.2: In E^3, let $\{\tilde{D}_n, T, Y_n\}$ be the ND-frame and α be a unit speed curve. The curve α is the helix relative to the ND-frame if and only if the expression $\frac{\mu_n}{\delta_n}$ is constant, for $\mu_n \neq 0$ and $\delta_n \neq 0$.

Proof: ($\Rightarrow$) Let α be the helix relative to the ND-frame. Let d be the unit, constant direction and $\langle T, d\rangle = \cos\theta = c \neq 0$. So, the vector d can be expressed as

$$d = a_1\tilde{D}_n + cT + a_2 Y_n,$$

and the derivative of this equality gives the system.

$$\begin{aligned} a_1' + a_2\delta_n &= 0, \\ -a_2\mu_n &= 0, \\ a_2' - a_1\delta_n + c\mu_n &= 0. \end{aligned}$$

Since $\mu_n \neq 0$ and $\delta_n \neq 0$, $a_2 = 0$ and $a_1 =$ constant become. As a result, $\frac{\mu_n}{\delta_n} =$ constant.

($\Leftarrow$) Let $\frac{\mu_n}{\delta_n}$ be constant. This constant value can be selected as $\frac{\mu_n}{\delta_n} = \frac{\cos\theta}{\sin\theta}$. Also the vector d can be taken as $d = \cos\theta\tilde{D}_n + \sin\theta T$. Since $\mu_n \sin\theta = \delta_n \cos\theta$, $d' = 0$ is written with the help of the derivatives of the ND-frame vectors. So, the vector d is constant. In addition, the equality $d = \cos\theta\tilde{D}_n + \sin\theta T$ is inner multiplied with T and so $\langle d,T\rangle = \sin\theta$. Consequently, the constant vector d and T make a constant angle, and the curve α is the helix.

The Rectifying Darboux Frame

In this section, a new frame called the "Rectifying Darboux Frame" and related theorems are presented. Since these theorems given in this section are proven similar to these theorems in section 3, they will be given without proof.

Let $\alpha: I \subset IR \to M$ be a unit speed curve with the Darboux frame $\{T, V, U\}$ on surface M in E^3. Let k_g, k_n, τ_g be the curvatures of the curve α and $D_r = \tau_g(s)T(s) + k_g(s)U(s)$ be the rectifying Darboux vector. It is clear that;

$$\tilde{D}_r(s) = \frac{D_r(s)}{\|D_r(s)\|} = \frac{\tau_g(s)}{\sqrt{k_g^2(s)+\tau_g^2(s)}}T(s) + \frac{k_g(s)}{\sqrt{k_g^2(s)+\tau_g^2(s)}}U(s).$$

Since the vector $\tilde{D}_r$ is in the plane spanned by T and U, the vectors $\tilde{D}_r$ and V are perpendicular to each other. So, $\tilde{D}_r \perp V$. Hence, the unit vector $Y_r = \tilde{D}_r \times V$ can be defined. It is clear that the vectors $\tilde{D}_r, V, Y_r$ are unit vectors and perpendicular to each other. Therefore, using these vectors $\{\tilde{D}_r, V, Y_r\}$ a new orthonormal frame can be constructed along the curve α on the surface.

Definition 5.1. The frame constructed with the vector fields $\{\tilde{D}_r, V, Y_r\}$ defined as $\tilde{D}_r = \frac{D_r(s)}{\|D_r(s)\|}$, $\tilde{D}_r \perp V$, $Y_r = \tilde{D}_r \times V$ is called the Rectifying Darboux frame or RD-frame briefly.

Theorem 5.1. Let $\alpha: I \subset IR \to M$ be a unit speed curve on surface M in E^3. The Rectifying Darboux frame of the curve α is defined as

$$\begin{aligned} \tilde{D}_r' &= -\delta_r Y_r \\ T' &= \mu_r Y_r \\ Y_r' &= \delta_r \tilde{D}_r - \mu_r V, \end{aligned}$$

where $\delta_r = \left(\frac{\tau_g}{k_g}\right)' \left(\frac{k_g^2}{k_g^2 + \tau_g^2}\right) - k_n$ and $\mu_r = \sqrt{k_g^2 + \tau_g^2}$.

Theorem 5.2: In E^3, let $\{\tilde{D}_r, V, Y_r\}$ be the RD-frame and α be a unit speed curve. The curve α is the relatively slant

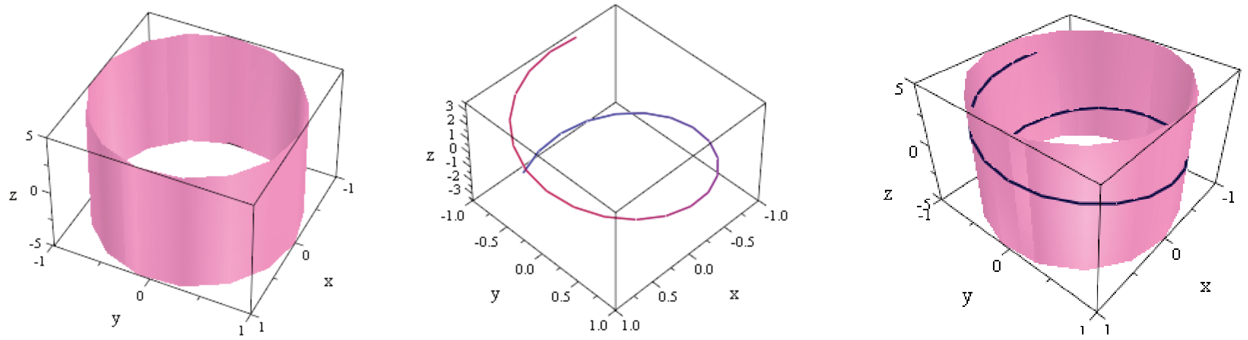

Figure 1. The surface M, The curve $\alpha(s)$, The curve $\alpha(s)$ on the surface M.

helix with respect to the RD-frame if and only if the expression $\dfrac{\mu_r}{\delta_r}$ is constant, for $\mu_r \neq 0$ and $\delta_r \neq 0$.

Example: Let's consider the cylinder surface M given by parameterization $\varphi(u,v) = (\sin u, \cos u, v)$. The curve $\alpha : I \to M$ given by the parametric form $\alpha(s) = \left(\sin\dfrac{s}{\sqrt{2}}, \cos\dfrac{s}{\sqrt{2}}, \dfrac{s}{\sqrt{2}}\right)$ is a helix on the surface M.

The vector fields and curvatures of the Darboux frame for this curve $\alpha(s)$ are calculated as:

$$T(s) = \frac{1}{\sqrt{2}}\left(\cos\frac{s}{\sqrt{2}}, -\sin\frac{s}{\sqrt{2}}, 1\right),$$

$$V(s) = \frac{1}{\sqrt{2}}\left(-\cos\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}}, 1\right),$$

$$U(s) = \left(-\sin\frac{s}{\sqrt{2}}, -\cos\frac{s}{\sqrt{2}}, 0\right),$$

$$k_g = 0,\ k_n = 1/2,\ \tau_g = -1/2.$$

The osculator Darboux vector is calculated as

$$\tilde{D}_o = \frac{\tau_g(s)}{\sqrt{\tau_g^2(s) + k_n^2(s)}} T(s) - \frac{k_n(s)}{\sqrt{\tau_g^2(s) + k_n^2(s)}} V(s) = (0,0,-1).$$

Also, it's clear that

$$Y_o = \tilde{D}_o \times U = \left(-\cos\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}}, 0\right).$$

Finally, the curvatures of the OD-frame for this curve $\alpha(s)$ are found as $\delta_0 = 0$ and $\mu_o = \dfrac{1}{\sqrt{2}}$ where

$$\left\langle \tilde{D}_o, \tilde{D}_o \right\rangle = 1,\ \left\langle U, U \right\rangle = 1,\ \left\langle Y_o, Y_o \right\rangle = 1,$$
$$\left\langle \tilde{D}_o, U \right\rangle = 0,\ \left\langle \tilde{D}_o, Y_o \right\rangle = 0,\ \left\langle Y_o, U \right\rangle = 0.$$

CONCLUSION

In this article, three new orthogonal frames were defined along the curve α lying on the surface, with the help of the Darboux frame.

First, the vector $Y_o = \tilde{D}_o \times U$ was defined because the osculator Darboux vector $\tilde{D}_o$ is perpendicular to the vector U. Therefore, using these vectors $\{\tilde{D}_o, U, Y_o\}$ the osculator Darboux frame was constructed.

After, the vector $Y_n = \tilde{D}_n \times T$ was defined because the normal Darboux vector $\tilde{D}_n$ is perpendicular to the vector T. Therefore, using these vectors $\{\tilde{D}_n, T, Y_n\}$ the normal Darboux frame was constructed.

Finally, the vector $Y_r = \tilde{D}_r \times V$ was defined because the rectifying Darboux vector $\tilde{D}_r$ is perpendicular to the vector V. Therefore, using these vectors $\{\tilde{D}_r, V, Y_r\}$ the normal Darboux frame was constructed.

Also, the Osculator Darboux Frame components and curvatures are calculated for a presented example.

The frame concept is very important in differential geometry. This study has a unique value in terms of putting forward three new frames and it may be a basis for many new studies.

Annotation: This article is prepared from Akın Alkan's doctoral thesis.

AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

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