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Research Article

Operational matrix for multi-order fractional differential equations with hermite polynomials

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ABSTRACT

In this article, a new operational matrix of fractional integration of Hermite polynomials is derived to solve multi-order linear fractional differential equations (FDEs) with spectral tau approach. We firstly convert the FDEs into an integrated-form through multiple fractional integration in association with the Riemann-Liouville sense. This integral equation is then formulated as an algebraic equation system with Hermite polynomials. Finally, linear multi-order FDEs with initial conditions are solved with this method. We present exact and approximated solutions for a number of representative examples. Numerical results indicate that the proposed method provides a high degree of accuracy to solve the linear multi-order FDEs.

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INTRODUCTION

The calculus of fractional order can be considered as a generalization of ordinary differentiation

and integration to arbitrary order. The fractional calculus was born in 1695 with G.W.Leibniz's question arising the uncertainty of the rational order of the derivation [1]. Fractional calculus' history can be found in [1-3]. Fractional differential equations (FDEs) have gained an increasing interest with a wide range of significant applications within science domain [4-7]. Examples of application areas are mechanics [8], biology [9], signal processing [10], economics [11] and control theory [12]. The main motivation behind the research of FDEs is its high accuracy when compared with integer order models, providing a high level of flexibility for choosing degree of derivation. This is because FDEs ensure more realistic models for complex real-world problems. In order to solve the FDEs, efficient solutions are required accurately in which different methods have attempted to solve FDEs. In recent years, spectral methods have been an effective method for numerical solutions of FDEs, particularly in the area of computational fluid dynamics. A typical example of spectral methods attempts to formulate Jacobi pseudospectral scheme to solve multi-dimensional fractional Schrodinger equations in association with various boundary conditions [13]. It solves the variable-order FDEs by deriving the operational matrices for fractional variable-order of the derivative and integral with Jacobi polynomials. A series of attempts

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This paper was recommended for publication in revised form by Editor in-Chief Ahmet Selim Dalkilic employing Laguerre polynomials have been carried out to solve the FDEs with different spectral methods in the scope of numerical methods [14]. The work conducted in [15] resulted in the development of a novel algorithm targeting time-dependent problems under the basis of spectral Laguerre approximations. A recent work considered the modified Laguerre functions by proposing a novel tau method [16]. It is based upon the operational matrix of fractional integration (OMFI) inspired by Riemann-Liouville paradigm, highlighting the efficiency of the proposed idea using illustrative examples. Another work exploited the Chebyshev polynomials in order to present a fractional radiative transfer equation, whereby the multi-dimensional issue is approximated by FDEs system [17]. In [18], a new explicit solution, which is targetted for shifted Chebyshev polynomials with flexible degree and fractional order, is formed to figure out multi-term FDEs. To solve the same linear problem, the work in [19] combined the shifted Chebyshev polynomials and extended spectral operational tau approach.

Jacobi polynomials have recently gained an important interest in both theory and practice. A derivation of shifted Jacobi operational matrix of fractional derivatives and spectral tau approach are applied together for solving the multi-term FDEs [20].To solve the nonlinear Langevin equation, on the other hand, a jacobi Gauss Lobatto collocation method is proposed in [21]. Authors in [22] included shifted Jacobi polynomials for a derivation of an OMFI using Riemann-Liouville, resulting in a direct solution of FDEs. An operational version of Legendre- tau technique to numerically solve the multi-term FDEs is also proposed in [23]. An extended work of Legendre polynomials presents an implementation of operational matrix with the sense of Riemann-Liouville [24]. Recently, a numerical solution for solving linear and non-linear FDEs is presented using Bernoulli polynomials with the methods of tau and collocation [25]. In this study, we attempt to present a new solution for the integrated form of FDEs with Hermite polynomials along with the fractional integration' operational matrix with the sense of Riemann Liouville. To do this, FDEs are initially re-written in the integral form which is then converted into an algebraic equation system with the introduction of the OMFI of Hermite polynomials. Upon the solution of the algebraic equations with initial conditions, we obtain exact and approximated solutions for a number of illustrative problems. The organization of the paper is as follows. Section II introduces the required notations and preliminaries, particularly Riemann-Liouville. The derivation of the Hermite OMFI is presented in section III. The operational matrix derived in the previous section is applied to solve linear FDEs in section IV. In section V, the proposed methods are implemented to various representative examples. Finally, the paper is concluded in section VI.

PRELIMINARIES AND NOTATION

The Fractional Integration in Rieman-Liouville Sense

The most common definition of Rieman-Liouville integration is:

$$J^{v} f(x) = \frac{1}{\Gamma(v)} \int_0^x (x-t)^{v-1} f(t)\,dt \quad v > 0, x > 0 \tag{2.1}$$

and if $v = 0$, then

$$J^{0} f(x) = f(x) \tag{2.2}$$

A significant property of Rieman-Liouville integration part is:

$$J^{v} x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1+v)} x^{\beta+v}. \tag{2.3}$$

The definition of Rieman-Liouville fractional derivation of order v is:

$$D^{v} f(x) = \frac{d^m}{dx^m}(J^{m-v} f(x)), \tag{2.4}$$

where $m-1 < v \le m, m \in N$ and m is the smallest integer greater than v.

Lemma1. If $m-1 < v \le m, m \in N$ then,

$$D^{v} J^{v} f(x) = f(x) \text{ and}$$
$$J^{v} D^{v} f(x) = f(x) - \sum_{i=0}^{m-1} f^{(i)}(0^+) \frac{x^i}{i!}, \quad x > 0. \tag{2.5}$$

The Properties of Hermite Polynomials

Let $\Lambda = (-\infty, \infty)$ and $w(x) = e^{-x^2}$ be the weight function on Λ. The analytic form of Hermite polynomials of degree i is defined [26]

$$H_i(x) = \sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} \frac{(-1)^k i! (2x)^{i-2k}}{k!(i-2k)!} \tag{2.6}$$

where $H_0(x) = 1$ and $H_1(x) = 2x$.

Hermite polynomials satisfy this recurrence relation

$$H_{i+1}(x) = 2xH_i(x) + 2iH_{i-1}(x). \tag{2.7}$$

The set of Hermite polynomials are orthogonal polynomials is an orthogonal system, namely

$$\int_{-\infty}^{\infty} H_i(x) H_j(x) w(x)\,dx = h_j \delta_{ij} \tag{2.8}$$

where $\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$ indicates the function of Kronecker and $h_j = 2^j j!\sqrt{\pi}$.

HERMITE OPERATIONAL MATRIX of FRACTIONAL INTEGRATION

In this section, we aim to derive an OMFI for Hermite polynomials. Let $u(x) \in L^2(\Lambda)$, then $u(x)$ can be defined in terms of Hermite polynomials as

$$u(x) = \sum_{j=0}^{\infty} a_j H_j(x) \tag{3.1}$$

Then, coefficient a_j can be written as

$$a_j = \frac{1}{2^j j!\sqrt{\pi}} \int_{-\infty}^{\infty} u(x) H_j(x) w(x) dx\,,\ j = 0,1,\dots \tag{3.2}$$

The initial $(N + 1)$ terms of Hermite polynomials are only taken into consideration, such that

$$u_N(x) = \sum_{j=0}^{N} a_j H_j(x) = A^T \phi(x) \tag{3.3}$$

where

$$A^T = [a_0 \quad a_1 \ \dots \ a_N] \text{ and}$$
$$\phi(x) = [H_0(x) \ \ H_1(x) \ \dots \ H_N(x)]^T \tag{3.4}$$

When we define q-step repeating integration of Hermite vector $\phi(x)$ by $J^q\phi(x)$ it will be

$$J^q \phi(x) = \boldsymbol{P}^{(q)} \phi(x) \tag{3.5}$$

where q indicates a fixed integer value and $\boldsymbol{O}^{(q)}$ represents the actual operational matrix of integration of $\phi(x)$.

Theorem 1. Let $\phi(x)$ be Hermite vector and $v > 0$ then

$$J^v(x) \cong O^{(v)} \phi(x) \tag{3.6}$$

where $O^{(v)}$ shows $(N + 1) \times (N + 1)$ OMFI of order v in the Rieman-Liouville sense which can be given as follows:

$$\boldsymbol{O}^{(v)} = \begin{bmatrix} \psi_v(0,0) & \psi_v(0,1) & \psi_v(0,2) & \dots & \psi_v(0,n) \\ \psi_v(1,0) & \psi_v(1,1) & \psi_v(1,2) & \dots & \psi_v(1,n) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \psi_v(i,0) & \psi_v(i,1) & \psi_v(i,2) & \cdots & \psi_v(i,n) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \psi_v(N-1,0) & \psi_v(N-1,1) & \psi_v(N-1,2) & \cdots & \psi_v(N-1,N) \\ \psi_v(N,0) & \psi_v(N,1) & \psi_v(N,2) & \cdots & \psi_v(N,N) \end{bmatrix} \tag{3.7}$$

where

$$\psi_v(n,j) = \sum_{i=0}^{\lfloor n/2 \rfloor} \sum_{r=0}^{\lfloor j/2 \rfloor} \frac{1}{2^j j!\sqrt{\pi}} \frac{(-1)^{i+r} n! j! 2^{n-2i+j-2r} . \Gamma(^{n-2i+v+j-2r+1}/_2)}{2.i!r!(j-2r)!\Gamma(n-2i+v+1)}, \quad j = 0,1,2 \dots N \tag{3.8}$$

Proof: We will apply Rieman-Liouville integration to the analytic form of Hermite polynomials as:

$$J^v(H_n(x)) = \sum_{i=0}^{[n/2]} \frac{(-1)^i J^v[(2x)^{n-2i}]}{(n-2i)!i!} = \sum_{i=0}^{[n/2]} \frac{(-1)^i 2^{n-2i} x^{n-2i+v}}{i!\Gamma(n-2i+1+v)} \tag{3.9}$$

If we approximate x^{n-2i+v} by $N + 1$ Hermite polynomial series; we obtain

$$x^{n-2i+v} = \sum_{j=0}^{N} c_j H_j(x) \tag{3.10}$$

where c_j is given from (3.2) and it is

$$c_j = \frac{1}{2^j j!\sqrt{\pi}} \sum_{r=0}^{\lfloor j/2 \rfloor} \frac{(-1)^r j! 2^{j-2r} \Gamma(\frac{n-2i+v+j-2r+1}{2})}{2.r!(j-2r)!\Gamma(n-2i+v+1)}. \tag{3.11}$$

Then in virtue of (3.9) and (3.10), we obtain

$$J^v(H_n(x)) = \sum_{j=0}^{N} \psi_v(n,j) H_j(x) \quad i = 0,1,\dots,N, \tag{3.12}$$

where

$$\psi_v(n,j) = \sum_{i=0}^{\lfloor n/2 \rfloor} \sum_{r=0}^{\lfloor j/2 \rfloor} \frac{1}{2^j j!\sqrt{\pi}} \frac{(-1)^{i+r} n! j! 2^{n-2i+j-2r} . \Gamma(n-2i+v+j-2r+1)}{2.i!r!(j-2r)!\Gamma(n-2i+v+1)}, \quad j = 0,1,2,\dots,N \tag{3.13}$$

HERMITE TAU METHOD WITH OPERATIONAL MATRIX

In practice, various problems are driven by initial value conditions of multi-term FDEs. This section modifies the Hermite tau method with the operational matrix for solving the FDEs. Each step of the whole process is given below.

$$D^v u(x) = \sum_{j=1}^{k} \gamma_j D^{\beta_j} u(x) + \gamma_{k+1} u(x) + f(x), \tag{4.1}$$

with initial conditions

$$u^{(i)}(0) = d_i \tag{4.2}$$

where $\gamma_{j,(j=0,1,2,\dots,k+1)}$ are real constants and $m\text{-}1 < v \leq m$, and $0 < \beta_1 < \beta_2 < \cdots < \beta_k < v$ and $f(x)$ is source function [16]. Rieman-Liouville integral of order v is applied to (4.1) after utilization (2.4), an integrated form of (4.1) is obtained, such as

$$u(x) - \sum_{j=0}^{m-1} u^{(j)}(0^+) \frac{x^j}{j!} = \sum_{i=1}^{k} \gamma_i J^{v-\beta_i} \left[u(x) - \sum_{j=0}^{m_i-1} u^{(j)}(0^+) \frac{x^j}{j!} \right] + \gamma_{k+1} J^v u(x) + J^v f(x) \tag{4.3}$$
$$u^{(i)}(0) = d_i, i = 0,\dots,m-1$$

where $m_i - 1 < \beta_i \leq m_i,\ m_i \in N$. This states that

$$u(x) = \sum_{i=1}^{k} \gamma_i J^{v-\beta_i} u(x) + \gamma_{k+1} J^v u(x) + g(x), \tag{4.4}$$
$$u^{(i)}(0) = d_i, i = 0,\dots,m-1$$

where

$$g(x) = J^{v} f(x) + \sum_{j=0}^{m-1} u^{(j)}(0^{+}) \frac{x^{j}}{j!} + \sum_{i=1}^{k} \gamma_i J^{v-\beta_i} \left[u(x) - \sum_{j=0}^{m_i-1} u^{(j)}(0^{+}) \frac{x^{j}}{j!} \right] \quad (4.5)$$

To apply Tau method with OMFI for Hermite polynomials to solve the fully-integrated problem (4.4) given by initial conditions (4.3), $u(x)$ and $g(x)$ are approximated by the Hermite polynomials as

$$u_N(x) \cong \sum_{i=0}^{N} c_i H_i(x) = C^T \phi(x) \quad (4.6)$$

$$g(x) \cong \sum_{i=0}^{N} g_i H_i(x) = G^T \phi(x) \quad (4.7)$$

where the vector $G = [g_0 \ g_1 \ \dots \ g_N]^T$ can be calculated from (4.7), whereas $[c_0 \ c_1 \ \dots \ c_N]^T$ is unkown vector. We then appply Rieman-Liouville integral of order v and $(v - \beta_j)$ of the approximate solution, it is re-written as

$$J^{v} u_N(x) \cong C^T J^{v} \phi(x) \cong C^T \boldsymbol{P}^{(v)} \phi(x) \quad (4.8)$$

and

$$J^{v-\beta_j} u_N(x) \cong C^T J^{v-\beta_j} \phi(x) \cong C^T \boldsymbol{P}^{(v-\beta_j)} \phi(x), \quad j = 1,2,\dots,k. \quad (4.9)$$

The residual $R_N(x)$ will be given as [24-25]

$$R_N(x) = \left(C^T - C^T \sum_{i=1}^{k} \gamma_i J^{v-\beta_i} - \gamma_{k+1} P^{(v)} - G^T \right) \phi(x) \quad (4.10)$$

with Tau method, by applying

$$\langle R_N(x), H_j(x) \rangle = \int_{-\infty}^{\infty} R_N(x). H_j(x)\, w(x) dx = 0, \quad j = 0,1,\dots,N-m. \quad (4.11)$$

$N - m + 1$ linear algebraic equations are generated. Then by using (3.2) and (4.6) for (4.3) we generate m linear equations. Then by solving these two sets of equations, we get the vector C. From the vector C, we obtain the approximate solution $u_N(x)$.

ILLUSTRATIVE EXAMPLES

Example 1. The first example is this following problem

$$D^2 u(x) + D^{3/2} u(x) + u(x) = x^2 + 2 + \frac{4x^{1/2}}{\Gamma(0.5)}, \quad u(0) = 0, u'(0) = 0, x \epsilon \Lambda. \quad (5.1)$$

whose exact solution is $u(x) = x^2$.

By applying our method for $N = 2$, we can write the approximate solution as

$$u_N(x) = \sum_{i=0}^{2} c_i H_i(x) = C^T \phi(x) \text{ and}$$

$$g(x) = \sum_{i=0}^{2} g_i H_i(x) = G^T \phi(x)$$

From (3.8), we can find

$$\boldsymbol{P}^{(2)} = \begin{bmatrix} \mathbf{0.1250} & \mathbf{0.1410} & \mathbf{0.0625} \\ \mathbf{0.0940} & \mathbf{0.1250} & \mathbf{0.0705} \\ \mathbf{-0.1250} & \mathbf{-0.0940} & \mathbf{0} \end{bmatrix} \text{ and}$$

$$\boldsymbol{P}^{0.5} = \begin{bmatrix} \mathbf{0.3901} & \mathbf{0.2885} & \mathbf{0.0488} \\ \mathbf{0.3847} & \mathbf{0.3901} & \mathbf{0.1443} \\ \mathbf{-0.1580} & \mathbf{0.1923} & \mathbf{0.2925} \end{bmatrix} \text{ and } \boldsymbol{G} = \begin{bmatrix} \boldsymbol{g_1} \\ \boldsymbol{g_2} \\ \boldsymbol{g_3} \end{bmatrix}$$

By using equation (4.10) and (4.11) we obtain

$$1.5151c_0 + 0.4787c_1 - 0.2801c_2 = 0.7185 \quad (5.2)$$

Now by applying (4.3) for the initial condition we have

$$c_0 - 2c_2 = 0, \quad 2c_1 = 0 \quad (5.3)$$

By solving linear system (5.2) and (5.3) we get

$$[c_0 \quad c_1 \quad c_2] = [0.5 \quad 0 \quad 0.25]$$

thus our solution $u_N(x) = [c_0 \quad c_1 \quad c_2] \begin{bmatrix} H_0(x) \\ H_1(x) \\ H_2(x) \end{bmatrix} = x^2$

which is the same as the exact solution.

Example 2. We now consider the following initial value problem as follow

$$D^{3/2} u(x) + 3u(x) = 3x^3 + \frac{8x^{3/2}}{\Gamma(0.5)} \quad u(0) = 0, u'(0) = 0 \quad (5.4)$$

whose exact solution is given by $u(x) = x^3$.

For $N = 3$, if we apply our tecnique to this problem, the approximate solution

$$u_N(x) = \sum_{i=0}^{3} c_i H_i(x) = C^T \phi(x)$$

and

$$\boldsymbol{P}^{(\frac{3}{2})} = \begin{bmatrix} \boldsymbol{\psi}_{(0,0)} & \boldsymbol{\psi}_{(0,1)} & \boldsymbol{\psi}_{(0,2)} & \boldsymbol{\psi}_{(0,3)} \\ \boldsymbol{\psi}_{(1,0)} & \boldsymbol{\psi}_{(1,1)} & \boldsymbol{\psi}_{(1,2)} & \boldsymbol{\psi}_{(1,3)} \\ \boldsymbol{\psi}_{(2,0)} & \boldsymbol{\psi}_{(2,1)} & \boldsymbol{\psi}_{(2,2)} & \boldsymbol{\psi}_{(2,3)} \\ \boldsymbol{\psi}_{(3,0)} & \boldsymbol{\psi}_{(3,1)} & \boldsymbol{\psi}_{(3,2)} & \boldsymbol{\psi}_{(3,3)} \end{bmatrix} \text{ and } \boldsymbol{G} = \begin{bmatrix} \boldsymbol{g_0} \\ \boldsymbol{g_1} \\ \boldsymbol{g_2} \\ \boldsymbol{g_3} \end{bmatrix}$$

By using (4.10) and (4.11) we get

$$1.5770c_0 + 0.4681c_1 - 0.4946c_2 - 1.5602c_3 = 0.1560 \quad (5.5)$$

$$0.5851c_0 + 1.5770c_1 - 0.2340c_2 - 1.4838c_3 = 0.9973 \quad (5.6)$$

Now by applying (4.3) we get

$$c_0 - 2c_2 = 0 \quad (5.7)$$

$$2c_1 - 12c_3 = 0 \quad (5.8)$$

By solving Eqs. (5.5)-(5.8) we have 4 unknown coefficients, which are found as

$$[c_0 \quad c_1 \quad c_2 \quad c_3] = [0 \quad 0.75 \quad 0 \quad 0.125]$$

Thereby we can write our solution as

$$u_N(x) = \sum_{i=0}^{3} c_i H_i(x) = C^T \phi(x) = x^3$$

Example 3. Another example is considered

$$D^{0.5}u(x) + u(x) = \frac{2x^{1.5}}{\Gamma(2.5)} - \frac{x^{0.5}}{\Gamma(1.5)} + x^2 - x \quad u(0) = 0,$$

The exact solution of this example is:

$$u(x) = x^2 - x$$

With our method for N=2, we obtain the following equations

$$1.3901c_0 + 0.3847c_1 - 0.1560c_2 = 0.4637$$

$$0.2885c_0 + 1.3901c_1 + 0.1923c_2 = -0.5027$$

$$c_0 - 2c_2 = 0$$

Upon solution of these algebraic equations, we present the following values of C parameters

$$[c_0 \quad c_1 \quad c_2] = [0.5 \quad -0.5 \quad 0.25]$$

Therefore, our proposed method successfully finds the exact solution as

$$u_N(x) = [c_0 \quad c_1 \quad c_2] \begin{bmatrix} H_0(x) \\ H_1(x) \\ H_2(x) \end{bmatrix} = x^2 - x$$

Example 4. Following problem is also considered

$$D^{3/2}u(x) + u(x) = x^2 + \frac{4x^{1/2}}{\Gamma(0.5)}, \quad u(0) = 0, u'(0) = 0 \quad (5.9)$$

whose exact solution is $u(x) = x^2$.

For N=2 if we apply our method and we obtain

$$[c_0 \quad c_1 \quad c_2] = [0.5 \quad 0 \quad 0.25].$$

From here, the approximate solution is obtained the same as the exact solution like

$$u_N = u(x) = x^2.$$

Example 5. Consider this initial value problem

$$D^2u(x) + D^{3/2}u(x) + u(x) = x^2 + 2 + \frac{4x^{1/2}}{\Gamma(1.5)},$$
$$u(0) = 0, u'(0) = 0 \quad (5.11)$$

whose exact solution is $u(x) = x^3$.

After applying our technique for $N = 3$ we get $[c_0 \quad c_1 \quad c_2 \quad c_3] = [0 \quad 0.75 \quad 0 \quad 0.125]$.

The approximate solution is $u_N(x) = u(x) = x^3$.

Example 6. The following inital value problem is considered

$$D^\alpha u(x) + u(x) = 0, \quad (5.12)$$

with conditions

$$0 < \alpha \leq 2, \quad u(0) = 1, u'(0) = 0$$

Also, we have the second initial condition valid for only $\alpha > 1$ [27]. The exact solution is given as:

$$u(x) = \sum_{k=0}^{\infty} \frac{(-x^\alpha)^k}{\Gamma(\alpha k + 1)}$$

The proposed method solves this problem, and the absolute error is given in fig. 1 for α = 0.75, 0.85, 0.95 and N = 4. It is important to note that the exact solution converges to the analytical solution of exp(-x) with α = 1. It can

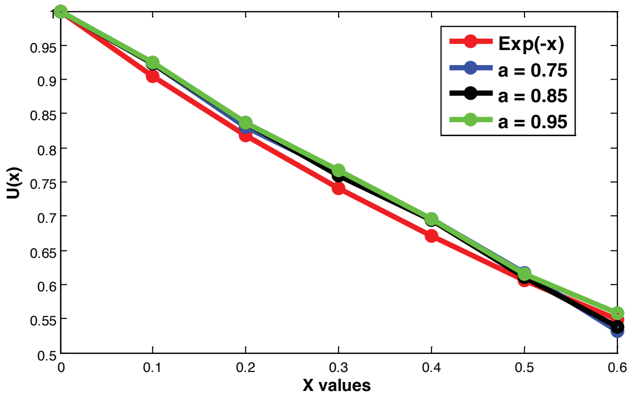

Figure 1. Comparisons of u(x) with varying α = 0.75, 0.85, 0.95.

be seen that our method presents a good approximation in comparison to results given in [23].

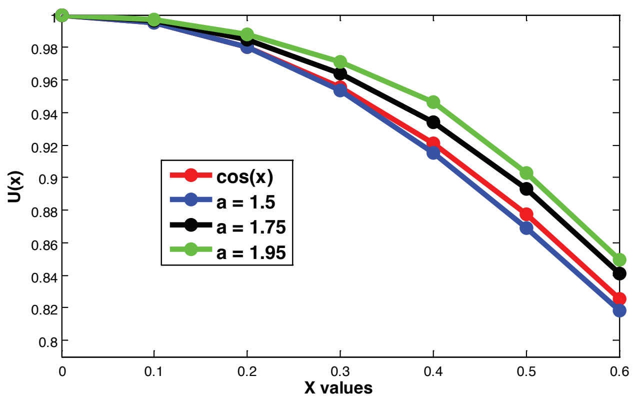

Figure 1. Comparisons of u(x) with varying α = 1.5, 1.75, 1.95.

In order to show the results for α > 1, where exact solution is given as cos(x) with α = 2, we selected three values of α, as indicated in fig. 2 below. A good balance between the exact solutions and obtained solutions is achieved. For α = 1.5, the best results are achieved with a very low error ratio.

Example 7. Consider the following equation

$$D^2u(x) - aD^vu(x) - bu(x) = 8, \qquad (5.13)$$

with conditions

$$0 < v < 2, u(0) = 0, u'(0) = 0$$

We obtain approximated solutions for v = 0.5 and 1.5 with varying N, which are illustrated in Table 1 and Table 2 compared with the exact solution. In this solution, we use a special case for a = b = -1. The results exhibit a satisfactory approximation solution with solutions presented in [24].

Table 1. Numerical results in comparison to exact solution for v = 0.5

x	N=2	N=3	N=4	N=5	Exact Solution
0	0	0	0	0	0
0.1	0.0291	0.0343	0.0378	0.378	0.03975
0.2	0.1164	0.1352	0.1488	0.1492	0.157036
0.3	0.2619	0.2998	0.3293	0.3306,	0.347370
0.4	0.4656	0.5251	0.5746	0.5779	0.604695
0.5	0.7275	0.8080	0.8798	0.8863	0.921768
0.6	1.0476	1.1458	1.2393	1.2502	1.290452
0.7	1.4259	1.5353	1.6468	1.6633	1.702008
0.8	1.8624	1.9736	2.0955	2.1185	2.147287
0.9	2.3571	2.4578	2.578	2.6078	2.617001
1	2.91	2.9848	3.0863	3.1224	3.101906

Table 2. Numerical results in comparison to exact solution for v = 1.5

x	N=2	N=3	N=4	N=5	Exact Solution
0	0	0	0	0	0
0.1	0.0291	0.0304	0.0319	0.0321	0.125221
0.2	0.1164	0.1187	0.1245	0.1246	0.033507
0.3	0.2619	0.2607	0.2731	0.2717	0.267609
0.4	0.4656	0.4518	0.4729	0.4675	0.455435
0.5	0.7275	0.6879	0.7186	0.7061	0.684335
0.6	1.0476	0.9646	1.0049	0.9816	0.950393
0.7	1.4259	1.2775	1.3260	1.2884	1.249959
0.8	1.8624	1.6223	1.6762	1.6213	1.579557
0.9	2.3571	1.9947	2.0491	1.9752	1.935832
1	2.91	2.3903	2.4385	2.3458	2.315528

CONCLUSION

This paper presented a general derivation for the OMFI of the Hermite polynomials. Riemann-Liouville sense is exploited to define the FDE as a form of fully integrated integration. The operational matrix obtained is a key part of the idea, in order to approximate the numerical solutions of the linear FDEs. A number of signals existed in the integrated form equation are treated as linear combinations of the Hermite polynomials. Then, a final algebraic equation is obtained with the integrated form equation introducing the OMFI of the Hermite polynomials. The numerical solutions obtained showed the accuracy of the proposed method.

AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICTS OF INTEREST

The author(s) declare that there are no conflicts of interest regarding the publication of this paper.

ETHICS

There are no ethical issues with the publication of this manuscript.

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