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Research Article

Some approximations and identities from special sequences for the vertices of suborbital graphs

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ABSTRACT

In this study, we investigate the vertices arising from the action of a suborbital graph, in terms of continued fractions, matrix, and recurrence relations. Using the approximation of Fibonacci sequence by the Binet formula, we demonstrate that the vertices of the suborbital graph are related to Lucas numbers. Then, we provide new identities and approximations regarding Fibonacci, Lucas, Pell, and Pell-Lucas numbers.

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INTRODUCTION

The history of graph theory is based on Plato according to some sources and Euler according to some other sources. However, with the discovery of non-Euclidean geometries in the 19^{th} century, graph theory began to be studied in this area. One of the groups used for transitive action from one point to another is the Modular group and its equivalent subgroups. Generally, action with an element of the Modular group is known as the Mobius transformations. In [1], some new ideas are presented on suborbital motion related with transitive permutations groups (G, Ω). In [2], the authors are used Γ as G and $\widehat{\mathbb{Q}}$ as Ω. Graphs are defined on Hyperbolic geometry, edges and vertices forming graphs in the upper half plane of complex plane. Moreover, the vertices obtained in the suborbital graph $F_{u,N}$ with an element of the Modular group Γ is investigated. In addition, the author is studied the Modular groups on suborbital graphs in [3]. The authors studied suborbital graphs and continued fractions in [4]. In [5], it is indicated that each vertex in the suborbital graph $F_{u,N}$ has a continued fraction structure for $(u, N) = 1$, $u \leq N$ and expressed by the value of any vertex on a path of minimal length with the Fibonacci numbers. Furthermore, paths of minimal lengths and its vertices defined on the suborbital graphs are investigated in [6].

An approximation in mathematics is the process of finding a number that is acceptably close to a certain value. Approximation has always been an important process in experimental sciences and engineering. Because it is impossible to make partially certain solutions [7].

In this study, by using the approximation $F_n \cong \frac{\alpha^n}{\sqrt{5}}$, p_n is expressed with Lucas numbers from $p_n = (-1)^n F_{2n}$, where the n^{th} numerator is p_n of the continued fraction. Then related matrix is obtained with Lucas numbers. From here, the vertices of the suborbital graph $F_{u,N}$ is

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expressed with these numbers. Later, some approximations are discovered with $\frac{-p_n}{p_{n+1}} \cong \frac{L_n}{\alpha L_{n+1}}$ for all $n \in \mathbb{N}$. In addition, some identities and approximations are found by using identities $L_{2n} = (-1)^n tr[S_n]$, $5F_{2n} = (-1)^n tr[H_n]$, $Q_{2n} = \frac{(-1)^n}{2} tr[R_n]$ and, $P_{2n} = \frac{(-1)^n}{16} tr[z_n]$ related with Pell and Pell-Lucas numbers.

Preliminaries

The Fibonacci sequence is known as 1, 1, 2, 3, 5, 8, 13, …. The name of the sequence was given by the French mathematician Lucas in 1876 [8]. The recurrence relation $F_n = F_{n-1} + F_{n-2}$ is defined, where F_n is n^{th} term of the sequence. In other words, the sum of two consecutive terms gives the next term. The Lucas sequence is known as 1, 3, 4, 7, 11, 18, …. The recurrence relation $L_n = L_{n-1} + L_{n-2}$ is defined, where L_n is n^{th} term of the sequence such as the Fibonacci sequence. For more information about Fibonacci, Lucas and Pell numbers, [8] can be examined. The Pell sequence is known as 0, 1, 2, 5, 12, 29, …. The recurrence relation $P_n = 2P_{n-1} + P_{n-2}$ is defined, where P_n is n^{th} term of the sequence. As for, the Pell-Lucas sequence, it is known as 1, 1, 3, 7, 17, 41, …. The recurrence relation $Q_n = 2Q_{n-1} + Q_{n-2}$ is defined, where Q_n is n^{th} term of the sequence.

In recent years, the Graph Theory began to be studied on the hyperbolic plane. $G_{u,N}$, $F_{u,N}$ and especially Farey graph $G_{1,1} = F$ are defined. [1–3] can be examined for $G_{u,N}$, $F_{u,N}$ and Farey graph $G_{1,1} = F$. Here, starting from ∞, the values of the graph at $\widehat{\mathbb{Q}} = \widehat{\mathbb{Q}} \cup \{\pm\infty\}$ are considered as vertices, the Hyperbolic geodesics consisting of the lines perpendicular to $\overline{\mathbb{R}}$ between the vertices and semi-circles whose center is on $\overline{\mathbb{R}}$ are considered as edges. [9–11] can be examined for studies in the Hyperbolic field of the graph theory. For more detailed information about graph theory, [12] and [13] can be examined. Also, the ideas of [1] are studied for finite groups in [14] and [15]. The movement of Γ on $\widehat{\mathbb{Q}}$ is transitive. Therefore, each suborbit contains the pair (∞, v) for $v \in \widehat{\mathbb{Q}}$. If $v = \frac{u}{N}$ for $N \geq 0$ and $(u, N) = 1$, we denote this suborbital by $O_{u,N}$ and the suborbital graph $G(\infty, v)$, corresponding to the suborbital as $G_{u,N}$. $F_{u,N}$ is the subgraph of $G_{u,N}$ consisting of the block $[\infty]$ whose vertices contain ∞. Thus, $G_{u,N}$ consists of discrete copies of $F_{u,N}$. $\frac{r}{s} \rightarrow \frac{x}{y} \in F_{u,N}$ if and only if $x \equiv ur \ (mod\ N)$ and $ry - sx = N$ for right directed graphs and $x \equiv -ur \ (mod\ N)$ and $ry - sx = -N$ for left directed graphs, where $\frac{r}{s}$ and $\frac{x}{y}$ are vertices on $F_{u,N}$. In addition, $\Gamma_0(N) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mid c \equiv 0 \ (mod\ N)\right\}$ transitively permutes the vertices and edges of $F_{u,N}$. For more detailed information, [2] and [3] can be examined.

In [2], motion with an element of the Modular group Γ on the suborbital graph $F_{u,N}$ is studied. In [5] indicated that, $(u, N) = 1$, $u \leq N$, each vertex in the suborbital graph $F_{u,N}$ has a continued fraction structure.

In this regard, let $v_0, v_1, \cdots, v_m$ be a sequence of different vertices of the suborbital graph $F_{u,N}$. If $m \geq 2$, path $v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_m \rightarrow v_0$ is called a directed circuit (or closed path). The path $v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_m$ and $v_0 \rightarrow v_1 \rightarrow \cdots$ is called a path and an infinite path in the graph $F_{u,N}$, respectively. If the graph $F_{u,N}$ does not have a vertex larger (or smaller) than the $\frac{x}{y}$ vertex that connects to the $\frac{r}{s}$ vertex, the $\frac{x}{y}$ vertex is called the farthest (or nearest) vertex. The path $v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_m$ in the graph $F_{u,N}$ to have the minimal length, $v_i \nleftrightarrow v_j$ and vertex v_{i+1} should be the farthest vertex that connects to vertex v_i, where $i < j - 1$, $i \in \{0, 1, 2, \cdots, m\text{-}2\}$, $j \in \{2, 3, 4, \cdots, m\}$ [6].

Lemma 1. [4] If $(u, N)=1$, there is an integer k that satisfies the congruence equation $u^2 + ku + 1 \equiv 0 \pmod{N}$.

For $k \geq 2$ and $k \in \mathbb{Z}$, $\begin{pmatrix} -u & \frac{u^2+ku+1}{N} \\ -N & u+k \end{pmatrix} \in \Gamma_0(N)$, the element of an equivalent subgroup of the Modular group, connects the vertices respectively on an infinite minimal length path in suborbital graph $F_{u,N}$ and each vertex forms a continued fractional structure.

$$\infty \rightarrow \frac{u}{N} \rightarrow \frac{u+\frac{1}{k}}{N} \rightarrow \frac{u+\frac{1}{k-\frac{1}{k}}}{N} \rightarrow \frac{u+\frac{1}{k-\frac{1}{k-\frac{1}{k}}}}{N} \rightarrow \frac{u+\frac{1}{k-\frac{1}{k-\frac{1}{k-\frac{1}{k}}}}}{N} \rightarrow \cdots$$

From the recurrence relations of the continued fractions, the continued fraction is

$$K_{n=1}^{\infty}\left(\frac{-1}{-k}\right) = \frac{u+\frac{1}{k-\frac{1}{k-\frac{1}{k-\ddots}}}}{N}$$

with initial conditions $p_{-1} = 1$, $q_{-1} = 0$, $p_0 = 0$, $q_0 = 1$. If $b_n = -k$ and $a_n = -1$, recurrence relation can be defined as

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \left\{-k\begin{pmatrix} p_{n-1} \\ q_{n-1} \end{pmatrix} - \begin{pmatrix} p_{n-2} \\ q_{n-2} \end{pmatrix}, n = 1,2,3,,\cdots\right\} \tag{1}$$

Also, a continued fraction can be given as

$$b_0 + \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3 + \ddots}}}$$

for $i \in \mathbb{N} \cup \{0\}$, $a_i \in \mathbb{Z} - \{0\}$ and $b_i \in \mathbb{Z}$. Then, the continued fraction is represents with $b_0 + K^{\infty}_{i=1}(a_i/b_i)$. In addition, the n^{th} aproximation of continued fraction is demonstrated with $f_n = b_0 + K^{n}_{i=1}(a_i/b_i)$. Moreover, $i \geq 1$, $a_i \neq 0$ and sequence $(\{a_i\}_{i\in\mathbb{N}}, \{b_i\}_{i\in\mathbb{N}\cup\{0\}})$ and linear fractional transformation sequences $\{t_n(s)\}_{n\in\mathbb{N}\cup\{0\}}$ and $\{T_n(s)\}_{n\in\mathbb{N}\cup\{0\}}$ have formed sequence $\{f_n\}$, where $t_0(s) = s$, $t_n(s) = a_n/(b_n+s)$, $T_0(s) = t_0(s)$, $T_n(s) = T_{n-1}(t_n(s))$ and $f_n = T_n(0) \in \widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ for $n = 1, 2, 3, \cdots$. From here, $((\{a_i\}_{i\in\mathbb{N}}, \{b_i\}_{i\in\mathbb{N}\cup\{0\}}), \{f_n\})$ can be written. This sequence corresponds to the continued fraction. Here, the element a_i is called the partial numerator and the element b_i is called the partial denominator. Furthermore, transformation $T_n(s)$ can be showed as

$$T_n(s) = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ddots + \cfrac{a_n}{b_n + s}}}}.$$

However, $T_n(s)$ can be written as $T_n(s) = (t_0 \circ t_1 \circ t_2 \circ \ldots \circ t_n)(s)$, where "o" is compound function. Then, $(t_0 \circ t_1)(s) = t_0(t_1(s))$ and $t^n(s) = \left(\underbrace{to to \ldots o t}_{n \text{ kere}}\right)(s)$ are found. In addition, $X_n = x_1 x_2 x_3 \cdots x_n = \begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{pmatrix}$ is achieved by using $t_n(s) = \frac{p_n}{q_n + s}$ and $x_n = \begin{pmatrix} 0 & p_n \\ 1 & q_n \end{pmatrix}$, where the element p_n is called the partial numerator and the element q_n is called the partial denominator of continued fraction.

Corollary 1. [6] From the matrix relation of the recurrence relation,

$$\begin{pmatrix} p_{n-1} & p_n \\ -p_n & -p_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & -k \end{pmatrix}^n \tag{2}$$

is obtained.

Theorem 1. [5] If $k = 2$ then $p_n = (-1)^n n$ and $k > 2$ then

$$p_n = (-1)^n 2^{1-n} \sum_{t=1}^{n} (k + \sqrt{k^2 - 4})^{n-t} (k - \sqrt{k^2 - 4})^{t-1}.$$

Lemma 2. [5] $F_{2n} p_{n+1} + F_{2n+2} p_n = 0$, where p_n is the n^{th} numerator of the continued fraction $K_{n=1}^{\infty}\left(\frac{-1}{-3}\right)$.

Theorem 2. [5] Let F_n is the n^{th} Fibonacci number, then for $k = 3$, the following equation is written,

$$S_n = \begin{pmatrix} 0 & -1 \\ 1 & -3 \end{pmatrix}^n = \begin{pmatrix} (-1)^{n-1} F_{2n-2} & (-1)^n F_{2n} \\ (-1)^{n+1} F_{2n} & (-1)^n F_{2n+2} \end{pmatrix} \tag{3}$$

where $p_n = (-1)^n F_{2n}$. Here, it is easily seen that the determinant of the matrix S_n is 1.

From Theorem 2, a vertex can be obtained as $\frac{u + \frac{F_{2n}}{F_{2n+2}}}{N}$ in suborbital graph $F_{u,N}$.

Example 1. If $u = 6$ and $N = 11$, then for $k = 3$ and $u^2 + 3u + 1 \equiv 0 \pmod N$, we obtain path of minimal length in suborbital graph $F_{6,11}$ as

$$\infty \to \frac{6}{11} \to \frac{6 + \frac{1}{3}}{11} \to \frac{6 + \frac{1}{3 - \frac{1}{3}}}{11} \to \frac{6 + \frac{1}{3 - \frac{1}{3 - \frac{1}{3}}}}{11} \to \frac{6 + \frac{1}{3 - \frac{1}{3 - \frac{1}{3 - \frac{1}{3}}}}}{11} \to \cdots.$$

The value of the $(n+1)^{th}$ vertex that n^{th} vertex can be connected as the farthest vertex is $\frac{u + \frac{F_{2n}}{F_{2n+2}}}{N}$. From here, n is taken as 14 to obtain value of the 15^{th} vertex as

$$\frac{6 + \frac{F_{28}}{F_{30}}}{11} = \frac{6 + \frac{710647}{1860498}}{11} = 0.5801787282955228. \tag{4}$$

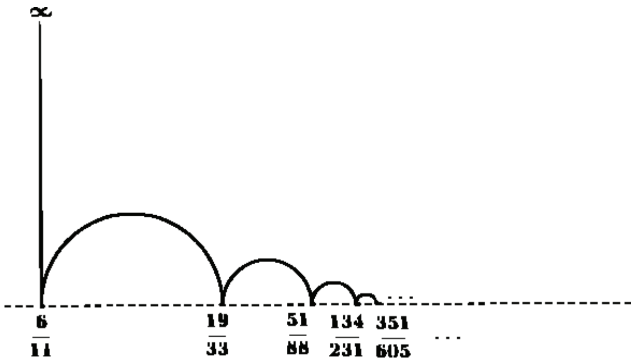

Figure 1. The vertices on the path of minimal length on the suborbital graph $F_{6,11}$

In [16], a new matrix is given which produces Lucas numbers and new identities by using trace of the matrices are obtained.

Theorem 3. [16] If L_n is the n^{th} Lucas number, then

$$\begin{pmatrix} -3 & 2 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & -3 \end{pmatrix}^n = \begin{pmatrix} (-1)^{n-1} L_{2n-2} & (-1)^n L_{2n} \\ (-1)^{n+1} L_{2n} & (-1)^n L_{2n+2} \end{pmatrix} = H_n \tag{5}$$

and

$$det(H_n) = L_{2n}^2 - L_{2n-2} L_{2n+2} = -5 \tag{6}$$

where $n \in \mathbb{Z}^+$

Corollary 2. [16] The following equations is written;

$$L_{2n} = (-1)^n tr[S_n] \tag{7}$$

$$5F_{2n} = (-1)^n tr[H_n]. \tag{8}$$

In addition, the matrix (2) is studied for $k = 6$. In this case, the elements of the matrix as the elements of the Pell sequence are obtained. The identity (10) is discovered by using the identity $P_{n-1} + P_{n+1} = 2Q_n$.

Theorem 4. [16] For Pell sequence P_n and Pell-Lucas sequence Q_n, then we have,

$$\begin{pmatrix} 0 & -1 \\ 1 & -6 \end{pmatrix}^n = \begin{pmatrix} \frac{(-1)^{n-1}}{2} P_{2n-2} & \frac{(-1)^n}{2} P_{2n} \\ \frac{(-1)^{n+1}}{2} P_{2n} & \frac{(-1)^n}{2} P_{2n+2} \end{pmatrix} = R_n \tag{9}$$

$$\begin{pmatrix} -6 & 2 \\ -2 & 6 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & -6 \end{pmatrix}^n = \begin{pmatrix} 2(-1)^{n-1} Q_{2n-2} & 2(-1)^n Q_{2n} \\ 2(-1)^{n+1} Q_{2n} & 2(-1)^n Q_{2n+2} \end{pmatrix} = Z_n$$

Corollary 3. [16] The following equations are valid;

$$Q_{2n} = \frac{(-1)^n}{2} tr[R_n] \tag{10}$$

$$P_{2n} = \frac{(-1)^n}{16} tr[Z_n] \tag{11}$$

RESULTS AND DISCUSSION

In this section, the matrix (2) can be expressed with the Lucas numbers for $k = 3$ and naturally the vertices of the suborbital graph can be found by the Lucas sequence.

Theorem 5. [8] If the identities $F_{2n} = F_n \cdot L_n$ and $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ with $\alpha = \frac{3+\sqrt{5}}{2}$ and $\beta = \frac{3-\sqrt{5}}{2}$ are taken into account for $n \in \mathbb{N}$,

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{\alpha^n}{\sqrt{5}}\left(1 - \left(\frac{\beta}{\alpha}\right)^n\right) \cong \frac{\alpha^n}{\sqrt{5}}$$

is obtained for $n \to \infty$. If the last approximate value is substituted in the identity $F_{2n} = F_n \cdot L_n$, it becomes,

$$F_{2n} = F_n \cdot L_n \cong \frac{\alpha^n}{\sqrt{5}} \cdot L_n.$$

Taking this as our starting point, from $p_n = (-1)^n F_{2n}$ [5]

$$p_n \cong (-1)^n \frac{\alpha^n}{\sqrt{5}} \cdot L_n \tag{12}$$

where n^{th} numerator of the continued fraction is p_n. From here, in the suborbital graph $F_{u,N}$, the approximation representation of the $(n+1)^{th}$ vertex with the Lucas numbers is found as

$$\frac{\frac{-p_n}{p_{n+1}} + u}{N} = \frac{u + \frac{F_{2n}}{F_{2n+2}}}{N} \cong \frac{\frac{(-1)^{n+1}\frac{\alpha^n}{\sqrt{5}} \cdot L_n}{(-1)^{n+1}\frac{\alpha^{n+1}}{\sqrt{5}} \cdot L_{n+1}} + u}{N} = \frac{\frac{L_n}{\alpha L_{n+1}} + u}{N}. \tag{13}$$

In the following graph, v_i, i=1,2,3,4 represents the vertices obtained by Fibonacci numbers. Similarly, v_i^*, i=1,2,3,4 represents the vertices obtained by Lucas numbers. The vertices v_i and v_i^* have been submitted in the following table for i = 1,2,3,4, $u = 6$, $N = 11$ and α =1.6180339887.

Also, it can be seen that $\frac{v_{i+1}}{v_i}$ converges to 1 for $i = 1,2,3,4$.

Now let us investigate the value at which the vertex values obtained with the Fibonacci and Lucas numbers converge on an infinite path of minimal length.

Table 1. The values of vertices v_i and v_i^*, i=1,2,3,4

Vertex	$n = 1$	$n = 2$	$n = 3$	$n = 4$
$\frac{u + \frac{F_{2n}}{F_{2n+2}}}{N}$	v_1 = 0,545454…	v_2 = 0,579545…	v_3 = 0,580086…	v_4 = 0,580165…
$\frac{u + \frac{L_n}{\alpha L_{n+1}}}{N}$	v_1^* = 0,564182…	v_2^* = 0,587593…	v_3^* = 0,577560…	v_4^* = 0,581208…

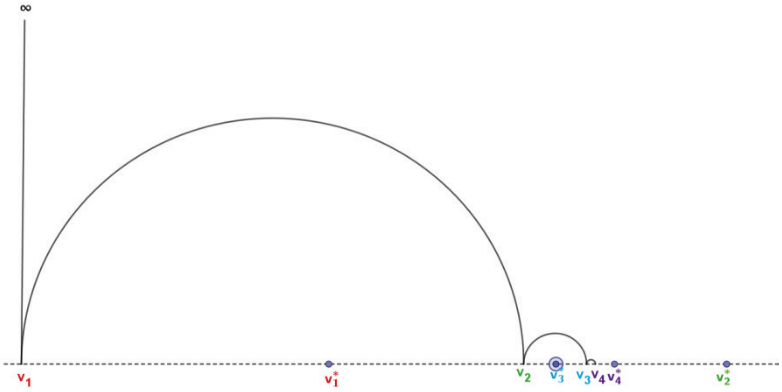

Figure 2. The vertices obtained by Fibonacci and Lucas numbers on the minimal length path on the suborbital graph $F_{6,11}$.

Conclusion 1. The vertices obtained with the Fibonacci and Lucas numbers converge to the same value for $n \to \infty$ on the infinite path of minimal length in $F_{u,N}$.

Proof. For the vertex value obtained with the Fibonacci numbers,

$$\lim_{n\to\infty}\frac{u+\frac{F_{2n}}{F_{2n+2}}}{N}=\frac{u+\lim\limits_{n\to\infty}\frac{F_{2n}}{F_{2n+2}}}{N}$$
$$=\frac{u+\frac{1}{\lim\limits_{n\to\infty}\frac{F_{n+1}L_{n+1}}{F_nL_n}}}{N}$$
$$=\frac{u+\frac{1}{\lim\limits_{n\to\infty}\frac{F_{n+1}}{F_n}\lim\limits_{n\to\infty}\frac{L_{n+1}}{L_n}}}{N}$$
$$=\frac{u+\frac{1}{\alpha^2}}{N}$$
$$=\frac{u+\frac{3-\sqrt{5}}{2}}{N}$$

is obtained. Likewise, for the vertex value obtained with Lucas numbers,

$$\lim_{n\to\infty}\frac{u+\frac{L_n}{\alpha L_{n+1}}}{N}=\frac{u+\lim\limits_{n\to\infty}\frac{L_n}{\alpha L_{n+1}}}{N}$$
$$=\frac{u+\frac{1}{\lim\limits_{n\to\infty}\frac{\alpha L_{n+1}}{L_n}}}{N}$$
$$=\frac{u+\frac{1}{\alpha\lim\limits_{n\to\infty}\frac{L_{n+1}}{L_n}}}{N}$$
$$=\frac{u+\frac{1}{\alpha^2}}{N}$$
$$=\frac{u+\frac{3-\sqrt{5}}{2}}{N}$$

is obtained.

Here we used the approximation (12) approach to construct the matrix M_n, which is approximate to the matrix S_n. With this approximation for an arbitrary value of n, the n^{th} vertex on the path of minimal length in the graph $F_{u,N}$ can be approximated without need for the special periodic continued fraction used previously. Moreover, with the matrix M_n obtained for this arbitrary n value, both the n^{th} approximate vertex value and the $(n+1)^{th}$ approximate vertex value, which is the farthest vertex connected to it in this path, are reached. In previous studies, for the special value $k = 3$, the relevant n^{th} vertex value was reached with the matrix S_n formed by the terms of the Fibonacci number sequence, this approximate vertex value is reached with the M_n matrix whose terms consist of the terms of the Lucas number sequence.

By using (3) and (12), we can easily obtain.

$$S_n \cong \begin{pmatrix} (-1)^{n-1}\frac{\alpha^{n-1}}{\sqrt{5}}\cdot L_{n-1} & (-1)^n\frac{\alpha^n}{\sqrt{5}}\cdot L_n \\ (-1)^{n+1}\frac{\alpha^n}{\sqrt{5}}\cdot L_n & (-1)^n\frac{\alpha^{n+1}}{\sqrt{5}}\cdot L_{n+1}\end{pmatrix} = M_n \quad (14)$$

Making use of the identity

$$L_{-n}=(-1)^nL_n,$$

the matrix obtained in (14) can be written as

$$M_n=\begin{pmatrix}\frac{\alpha^{n-1}}{\sqrt{5}}\cdot L_{1-n} & \frac{\alpha^n}{\sqrt{5}}\cdot L_{-n} \\ -\frac{\alpha^n}{\sqrt{5}}\cdot L_{-n} & -\frac{\alpha^{n+1}}{\sqrt{5}}\cdot L_{-n-1}\end{pmatrix}.$$

From (14), if $n = 14$ and $\alpha = 1.6180339887$,

$$M_{14}=\begin{pmatrix}-\frac{\alpha^{13}}{\sqrt{5}}\cdot L_{13} & \frac{\alpha^{14}}{\sqrt{5}}\cdot L_{14} \\ -\frac{\alpha^{14}}{\sqrt{5}}\cdot L_{14} & -\frac{\alpha^{15}}{\sqrt{5}}\cdot L_{15}\end{pmatrix}=\begin{pmatrix}-121392 & 317811 \\ -317811 & 832039\end{pmatrix}$$

matrix is found. By (13), for $u = 6$, $N = 11$, the value of the 15^{th} vertex in (14) is found as

$$\frac{\frac{-p_n}{p_{n+1}}+u}{N}\cong\frac{\frac{L_n}{\alpha L_{n+1}}+u}{N}$$
$$=\frac{\frac{843}{(1.6180339887)\cdot 1364}+6}{11}$$
$$=0.58017879582326.$$

This value, obtained with Lucas numbers, is very close to the one that is obtained with the Fibonacci numbers in (4).

Conclusion 2. The determinant of matrix M_n is α^{2n} for even numbers and $-\alpha^{2n}$ for odd numbers.

Proof. The determinant of the matrix M_n is found as

$$|M_n|=(-1)^{n-1}\frac{\alpha^{n-1}}{\sqrt{5}}\cdot L_{n-1}(-1)^n\frac{\alpha^{n+1}}{\sqrt{5}}\cdot L_{n+1}-(-1)^n\frac{\alpha^n}{\sqrt{5}}\cdot L_n(-1)^{n+1}\frac{\alpha^n}{\sqrt{5}}\cdot L_n$$
$$=(-1)^{2n-1}\frac{\alpha^{2n}}{5}\cdot L_{n-1}L_{n+1}-(-1)^{2n+1}\frac{\alpha^{2n}}{5}\cdot L_n^2$$
$$=-\frac{\alpha^{2n}}{5}\cdot L_{n-1}L_{n+1}+\frac{\alpha^{2n}}{5}\cdot L_n^2$$
$$=\frac{\alpha^{2n}}{5}(L_n^2-L_{n-1}L_{n+1})$$

is obtained. Since $L_n^2-L_{n-1}L_{n+1}=5(-1)^n$,

$$|M_n|=\frac{\alpha^{2n}}{5}5(-1)^n$$

is achieved. So, the determinant is α^{2n} for even numbers and $-\alpha^{2n}$ for odd numbers.

Conclusion 3. For all $n \in \mathbb{N}$, the approximation

$$\alpha F_{2n}L_{n+1} - F_{2n+2}L_n \cong 0 \tag{16}$$

is obtained.

Proof. From (13), $\frac{F_{2n}}{F_{2n+2}} = \frac{-p_n}{p_{n+1}} \cong \frac{L_n}{\alpha L_{n+1}}$ is obtained. Then, $\alpha F_{2n}L_{n+1} - F_{2n+2}L_n \cong 0$ is found.

Corollary 4. For all $n \in \mathrm{N}$ and p_n is the n^{th} numerator of the continued fraction $K_{n=1}^{\infty}\left(\frac{-1}{-3}\right)$, the following approximations

i. $$L_{2n} \cong \sqrt{5}\left(\frac{p_{2n-1}}{\alpha_{2n-1}} - \frac{p_{2n+1}}{\alpha_{2n+1}}\right)$$

ii. $$F_n \cong -\frac{L_n}{5p_n}\left(\alpha p_{n-1} + \frac{1}{\alpha}p_{n+1}\right)$$

iii. $$L_{n-1}L_{n+1} + (-1)^n \cong \frac{5}{\alpha^{2n}}p_{n-1}p_{n+1} + (-1)^n$$

iv. $$(-1)^{n+1}(\alpha F_n - F_{n+1}) \cong 0$$

v. $$L_n \cong \frac{p_n}{tr[M_n]}\left[-\frac{L_{n-1}}{\alpha} + \alpha L_{n+1}\right]$$

are obtained.

The approximations are submitted in Corollory 4 can be proved easily by using (12) and $p_n = (-1)^n F_{2n}$.

Corollary 5. The following approximations are found from matrices S_n and M_n.

i. $$F_{4n-2} \cong \frac{\alpha^n}{\sqrt{5}}\left(F_{2n}L_n - \frac{F_{2n-2}L_{n-1}}{\alpha}\right)$$

ii. $$F_{2n}L_{2n} \cong \frac{\alpha^n}{\sqrt{5}}(\alpha F_{2n}L_{n+1} - F_{2n-2}L_n)$$

iii. $$F_{2n}L_{2n} \cong \frac{\alpha^n}{\sqrt{5}}\left(F_{2n+2}L_n - \frac{F_{2n}L_{n-1}}{\alpha}\right)$$

iv. $$F_{4n+2} \cong \frac{\alpha^n}{\sqrt{5}}(\alpha F_{2n+2}L_{n+1} - F_{2n}L_n)$$

v. $$\alpha L_{n+1} + \frac{L_{n-1}}{\alpha} = 3L_n$$

Proof. The multiplication of matrices S_n and M_n is obtained as

$$\begin{pmatrix} 0 & -1 \\ 1 & -3 \end{pmatrix}^{2n} \cong \begin{pmatrix} (-1)^{n-1}F_{2n-2} & (-1)^n F_{2n} \\ (-1)^{n+1}F_{2n} & (-1)^n F_{2n+2} \end{pmatrix} \begin{pmatrix} (-1)^{n-1}\frac{\alpha^{n-1}}{\sqrt{5}} \cdot L_{n-1} & (-1)^n \frac{\alpha^n}{\sqrt{5}} \cdot L_n \\ (-1)^{n+1}\frac{\alpha^n}{\sqrt{5}} \cdot L_n & (-1)^n \frac{\alpha^{n+1}}{\sqrt{5}} \cdot L_{n+1} \end{pmatrix}.$$

Let us found both sides of the approximation. The right side of the approximation is found as

$$\begin{pmatrix} (-1)^{n-1}F_{2n-2} & (-1)^n F_{2n} \\ (-1)^{n+1}F_{2n} & (-1)^n F_{2n+2} \end{pmatrix} \begin{pmatrix} (-1)^{n-1}\frac{\alpha^{n-1}}{\sqrt{5}} \cdot L_{n-1} & (-1)^n \frac{\alpha^n}{\sqrt{5}} \cdot L_n \\ (-1)^{n+1}\frac{\alpha^n}{\sqrt{5}} \cdot L_n & (-1)^n \frac{\alpha^{n+1}}{\sqrt{5}} \cdot L_{n+1} \end{pmatrix}$$
$$= \begin{pmatrix} \frac{\alpha^{n-1}}{\sqrt{5}}F_{2n-2}L_{n-1} - \frac{\alpha^n}{\sqrt{5}}F_{2n}L_n & -\frac{\alpha^n}{\sqrt{5}}F_{2n-2}L_n + \frac{\alpha^{n+1}}{\sqrt{5}}F_{2n}L_{n+1} \\ \frac{\alpha^{n-1}}{\sqrt{5}}F_{2n}L_{n-1} - \frac{\alpha^n}{\sqrt{5}}F_{2n+2}L_n & -\frac{\alpha^n}{\sqrt{5}}F_{2n}L_n + \frac{\alpha^{n+1}}{\sqrt{5}}F_{2n+2}L_{n+1} \end{pmatrix}.$$

Similarly, left side of approximation is found as follows:

$$\begin{pmatrix} 0 & -1 \\ 1 & -3 \end{pmatrix}^{2n} = \begin{pmatrix} (-1)^{n-1}F_{2n-2} & (-1)^n F_{2n} \\ (-1)^{n+1}F_{2n} & (-1)^n F_{2n+2} \end{pmatrix} \begin{pmatrix} (-1)^{n-1}F_{2n-2} & (-1)^n F_{2n} \\ (-1)^{n+1}F_{2n} & (-1)^n F_{2n+2} \end{pmatrix}$$
$$= \begin{pmatrix} F_{2n-2}^2 - F_{2n}^2 & -F_{2n-2}F_{2n} + F_{2n}F_{2n+2} \\ F_{2n-2}F_{2n} - F_{2n}F_{2n+2} & F_{2n+2}^2 - F_{2n}^2 \end{pmatrix}.$$

From identities $F_{n+2k}^2 - F_n^2 = F_{2k}F_{2n+2k}$ and $F_{m+n} - F_{m-n}$ is $L_m F_n$ when n is odd otherwise $L_m F_n$ in [8],

$$F_{2n-2}^2 - F_{2n}^2 = -F_{4n-2}$$

$$-F_{2n-2}F_{2n} + F_{2n}F_{2n+2} = F_{2n}(-F_{2n-2} + F_{2n+2}) = F_{2n}L_{2n}F_2 = F_{2n}L_{2n}$$

$$F_{2n-2}F_{2n} - F_{2n}F_{2n+2} = F_{2n}(F_{2n-2} - F_{2n+2}) = -F_{2n}L_{2n}F_2 = -F_{2n}L_{2n}$$

$$F_{2n+2}^2 - F_{2n}^2 = F_{4n+2}.$$

Then, approximations i., ii., iii. and iv. are

$$F_{4n-2} \cong \frac{\alpha^n}{\sqrt{5}}F_{2n}L_n - \frac{\alpha^{n-1}}{\sqrt{5}}F_{2n-2}L_{n-1} = \frac{\alpha^n}{\sqrt{5}}\left(F_{2n}L_n - \frac{F_{2n-2}L_{n-1}}{\alpha}\right)$$

$$F_{2n}L_{2n} \cong -\frac{\alpha^n}{\sqrt{5}}F_{2n-2}L_n + \frac{\alpha^{n+1}}{\sqrt{5}}F_{2n}L_{n+1} = \frac{\alpha^n}{\sqrt{5}}(\alpha F_{2n}L_{n+1} - F_{2n-2}L_n)$$

$$F_{2n}L_{2n} \cong \frac{\alpha^n}{\sqrt{5}}F_{2n+2}L_n - \frac{\alpha^{n-1}}{\sqrt{5}}F_{2n}L_{n-1} = \frac{\alpha^n}{\sqrt{5}}\left(F_{2n+2}L_n - \frac{F_{2n}L_{n-1}}{\alpha}\right)$$

$$F_{4n+2} \cong -\frac{\alpha^n}{\sqrt{5}}F_{2n}L_n + \frac{\alpha^{n+1}}{\sqrt{5}}F_{2n+2}L_{n+1} = \frac{\alpha^n}{\sqrt{5}}(\alpha F_{2n+2}L_{n+1} - F_{2n}L_n)$$

attained. Also, by using ii. and iii.,

$$\alpha F_{2n}L_{n+1} - F_{2n-2}L_n = F_{2n+2}L_n - \frac{F_{2n}L_{n-1}}{\alpha}$$

$$\alpha F_{2n}L_{n+1} + \frac{F_{2n}L_{n-1}}{\alpha} = F_{2n+2}L_n + F_{2n-2}L_n$$

$$F_{2n}\left(\alpha L_{n+1} + \frac{L_{n-1}}{\alpha}\right) = L_n(F_{2n+2} + F_{2n-2})$$

is found. Then,

$$F_{2n}\left(\alpha L_{n+1} + \frac{L_{n-1}}{\alpha}\right) = L_n 3F_{2n}$$

$$\alpha L_{n+1} + \frac{L_{n-1}}{\alpha} = 3L_n,$$

where $F_{2n+2} + F_{2n-2} = L_2 F_{2n} = 3F_{2n}$.

Some New Identities of Fibonacci, Lucas, Pell and Pell-Lucas Numbers

In this section, some new identities of the special number sequences are obtained.

Theorem 6. For all $n \in \mathbb{N}$ and p_n is the n^{th} numerator of the continued fraction $K_{n=1}^{\infty}\left(\frac{-1}{-3}\right)$, identities

i. $x_n = \dfrac{-p_n}{p_{n+1}} = \dfrac{-tr[H_n]}{tr[H_{n+1}]}$

ii. $F_{2n}tr[H_{n+1}] - F_{2n+2}tr[H_n] = 0$

iii. $F_{4n} = \dfrac{tr[R_n]tr[H_n]}{5}$

are obtained.

Proof. i) From (8), p_n is defined as follows:

$$\begin{aligned} p_n &= (-1)^n F_{2n} \\ &= (-1)^n \frac{(-1)^n tr[H_n]}{5} \\ &= \frac{tr[H_n]}{5}. \end{aligned}$$

From here,

$$\begin{aligned} x_n &= \frac{-p_n}{p_{n+1}} \\ &= \frac{-\frac{tr[H_n]}{5}}{\frac{tr[H_{n+1}]}{5}} \\ &= \frac{-tr[H_n]}{tr[H_{n+1}]} \end{aligned}$$

is obtained.

Thus, in the suborbital graph $F_{u,N}$, the representation of the $(n+1)^{th}$ vertex by using trace of matrix H_n is obtained as

$$\begin{aligned} \frac{\frac{-p_n}{p_{n+1}} + u}{N} &= \frac{\frac{-\frac{tr[H_n]}{5}}{\frac{tr[H_{n+1}]}{5}} + u}{N} \\ &= \frac{\frac{-tr[H_n]}{tr[H_{n+1}]} + u}{N}. \end{aligned} \tag{17}$$

ii) From (16), $\frac{F_{2n}}{F_{2n+2}} = \frac{-p_n}{p_{n+1}} = \frac{tr[H_n]}{tr[H_{n+1}]}$ is obtained. Then, $F_{2n}tr[H_{n+1}] - F_{2n+2}tr[H_n] = 0$ is found.

iii) Using the identities given in (7) and (8),

$$5F_{2n}L_{2n} = (-1)^n tr[S_n](-1)^n tr[H_n] = tr[S_n]tr[H_n]$$

is obtained. Then,

$$F_{4n} = \frac{tr[R_n]tr[H_n]}{5}$$

is found.

Theorem 7. For all $n \geq 2$,

$$2Q_{2n} + 16P_{2n} = \frac{1}{2}[P_{2n+2} - P_{2n-2}] + 2[Q_{2n+2} + Q_{2n-2}]$$

is found.

Proof. Let us obtain matrix $R_n + Z_n$.

$$\begin{aligned} R_n + Z_n &= \begin{pmatrix} \frac{(-1)^{n-1}}{2}P_{2n-2} & \frac{(-1)^n}{2}P_{2n} \\ \frac{(-1)^{n+1}}{2}P_{2n} & \frac{(-1)^n}{2}P_{2n+2} \end{pmatrix} + \begin{pmatrix} 2(-1)^{n-1}Q_{2n-2} & 2(-1)^n Q_{2n} \\ 2(-1)^{n+1}Q_{2n} & 2(-1)^n Q_{2n+2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{(-1)^{n-1}}{2}P_{2n-2} + 2(-1)^{n-1}Q_{2n-2} & \frac{(-1)^n}{2}P_{2n} + 2(-1)^n Q_{2n} \\ \frac{(-1)^{n+1}}{2}P_{2n} + 2(-1)^{n+1}Q_{2n} & \frac{(-1)^n}{2}P_{2n+2} + 2(-1)^n Q_{2n+2} \end{pmatrix} \end{aligned}$$

The trace of matrix $R_n + Z_n$ is found as

$$\frac{(-1)^{n-1}}{2}P_{2n-2} + 2(-1)^{n-1}Q_{2n-2} + \frac{(-1)^n}{2}P_{2n+2} + 2(-1)^n Q_{2n+2}.$$

$tr[R_n] = 2(-1)^n Q_{2n}$ is obtained from (10). Similarly, $tr[Z_n] = 16(-1)^n P_{2n}$ is obtained from (11). From $tr[R_n] + tr[Z_n] = tr[R_n + Z_n]$,

$$\begin{aligned} 2(-1)^n Q_{2n} + 16(-1)^n P_{2n} &= \frac{(-1)^{n-1}}{2}P_{2n-2} + 2(-1)^{n-1}Q_{2n-2} \\ &\quad + \frac{(-1)^n}{2}P_{2n+2} + 2(-1)^n Q_{2n+2} \\ 2Q_{2n} + 16P_{2n} &= -\frac{1}{2}P_{2n-2} - 2Q_{2n-2} + \frac{1}{2}P_{2n+2} + 2Q_{2n+2} \\ &= \frac{1}{2}[P_{2n+2} - P_{2n-2}] + 2[Q_{2n+2} - Q_{2n-2}] \end{aligned}$$

is achieved.

Corollary 6. The identities

i. $L_{2n+1} = (-1)^{n+1}(tr[S_{n+1}] + tr[S_n])$

ii. $F_{2n+1} = \dfrac{(-1)^{n+1}}{5}(tr[H_{n+1}] + tr[H_n])$

are obtained, where identity $L_{2n} = (-1)^n tr[S_n]$ is obtained with trace of matrix S_n, the identity $5F_{2n} = (-1)^n tr[H_n]$ is obtained with the trace of matrix H_n

Proof. i) (7) is found as $L_{2n+2} = (-1)^{n+1}tr[S_{n+1}]$ for $n \to n+1$. Hence,

$$\begin{aligned} L_{2n+2} - L_{2n} &= (-1)^{n+1}tr[S_{n+1}] - (-1)^n tr[S_n] \\ &= (-1)^{n+1}tr[S_{n+1}] + (-1)^{n+1}tr[S_n] \\ &= (-1)^{n+1}(tr[S_{n+1}] + tr[S_n]). \end{aligned}$$

ii) (8) is found as $5F_{2n+2} = (-1)^{n+1} tr[H_{n+1}]$ for $n \to n+1$. Hence,

$$F_{2n+2} - F_{2n} = \frac{(-1)^{n+1} tr[H_{n+1}]}{5} - \frac{(-1)^{n} tr[H_{n}]}{5}$$
$$= \frac{(-1)^{n+1} tr[H_{n+1}]}{5} + \frac{(-1)^{n+1} tr[H_{n}]}{5}$$
$$= \frac{(-1)^{n+1}}{5} (tr[H_{n+1}] + tr[H_n]).$$

CONCLUSION

In the suborbital graph $F_{u,N}$, it is given that the condition $\frac{r}{s} \to \frac{x}{y} \in F_{u,N}$ if and only if $x \equiv ur \ (mod\ N)$ and $ry - sx = N$ for right directed graphs and $x \equiv -ur \ (mod\ N)$ and $ry - sx = -N$ for left directed graphs must be satisfied in [2] in order to connect two consecutive vertices together. In addition, for $k \geq 2$ and $k \in \mathbb{Z}$, $\begin{pmatrix} -u & \frac{u^2+ku+1}{N} \\ -N & u+k \end{pmatrix} \in \Gamma_0(N)$, the element of an equivalent subgroup of the Modular group, connects the vertices respectively on an infinite path of minimal length in suborbital graph $F_{u,N}$ and each vertex forms a continued fractional structure. [4], [5] and [6] can be given as references.

$$\infty \to \frac{u}{N} \to \frac{u+\frac{1}{k}}{N} \to \frac{u+\frac{1}{k-\frac{1}{k}}}{N} \to \frac{u+\frac{1}{k-\frac{1}{k-\frac{1}{k}}}}{N} \to \frac{u+\frac{1}{k-\frac{1}{k-\frac{1}{k-\frac{1}{k}}}}}{N} \to \cdots$$

The vertices of a suborbital graph by relating it to a value of k can be achieved much more easier. Especially, these vertices are associated Fibonacci numbers for $k = 3$ and Pell numbers for $k = 6$.

The matrices formed with the elements of the Pell and Pell-Lucas number sequences and the identities have been obtained with the traces of these matrices were examined. Also, identities related to these special number sequences have been achieved. Moreover, by these special number sequences, approximations were obtained, which are used extensively in engineering and applied sciences to approximate a real value. As a result, the vertices of the suborbital graphs have been examined with different number sequences and contributed to the graph theory. Studies with number sequences such as Fibonacci, Lucas, Pell and Pell-Lucas have been increased intensively in recent years. The studies [5], [6], [8], [16], [17] and [18] can be given as references. Literature contribution to sequence spaces and number theory with the help of obtained identities and approximations has been provided.

AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

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