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Research Article

A study on *I*-lacunary statistical convergence of multiset sequences

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ABSTRACT

In classical set theory, elements of the set are written once but the sets in which the same item is repeated several times in daily life are in all areas of our lives. These sets are called multisets and are studied in many fields such as Mathematics, Physics, Chemistry, and Computer Sciences. Sequences consisting of elements of these sets are called multiset sequences. In this paper, we study the concept of *I*-lacunary statistical convergence of multiset sequences and investigate some important results.

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INTRODUCTION

In this section, the subjects that will form the basis of the study are given. In order to better understand the introduction, we choose to give this section under four subsections.

Multiset Sequences

In classical set theory, the elements of a set are written only once but multiset is a collection of objects in which elements are allowed to repeat. In fact, it is possible to see multisets in many areas of our lives. For example:

Telephone numbers: 0 535 713…

Computer codes: 1110001101101…

Water molecules: H_2O

Coincident roots of equations: $(x-3)^2=0$

In each example, there are same numbers and same molecules that play different roles. If these numbers are used once rather than multiple times, it is clear that there will be problems. Hence, multisets are very interesting in mathematics, physics, philosophy, logic, linguistics, computer science, etc.

It is observed from the survey of literature, multisets have been studied under the names of bags, occurrence set, weighted set, sample in the past years. Since the 1970s, Bender, Hickman, Lake, Meyer and Monro investigated some important properties of multisets in [1-5]. In 1981, Knuth studied computer programing and multisets in [6]. Blizard studied on multisets in his doctoral thesis in [7-9]. In multisets, the order of the elements is not important. So, $\{1,3,5,3,4,1,1\}$ and $\{1,1,1,3,3,4,5\}$ sets are same. On the other hand, it is very important how many times the elements are repeated in the set. We denote $\{1,3,5,3,4,1,1\}$ multiset by $\{1,3,4,5\}_{3,2,1,1}$ or $\{1|3,\ 3|2,\ 4|1,\ 5|1\}$ and it means 1 appearing 3 times, 3 appearing 2 times, 4 appearing 1 times and 5 appearing 1 times. The cardinality of a multiset is the sum of the multiplicities of its elements.

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Studies on multisets continued in the 2000s and the studies titled “Mathematics of multisets” published by Syropoulos in 2001 in [10]. “Multigroup actions on multisets” published by İbrahim in 2017 [11], “Soft multisets” published by Majumdar in 2012 [12], “On multisets and multigroups” published by Nazmul in 2013 [13] and “An overview of the application of multiset” published by Singh in 2007 [14] took its place among the important studies.

After these studies on multisets, the multiset sequences and their properties have started to be the subject of research and usual convergence of multiset sequences was studied by Pachilongode and John in 2021 in [15].

Definition 1.1. [15] Let X be a set. A sequence in which all the terms are multiset is known as a multiset sequence. For any sequence $x = (x_i) \in X$ a multiset sequence is defined by

$M = \{x_i|c_i: x_i \in X, c_i \in \mathbb{N}_0\}$.

Example 1.1. [15] Let $N = \{1|1, 2|2, \ldots, n|n\}$. Then, $\{N\}$ is an multiset sequence and n^{th} terms has $\frac{n(n+1)}{2}$ elements.

Example 1.2. [15] The prime factorises n completely, and let Fn be the mset of these factors, including 1. Then, $F_1 = \{1\}$, $F_2 = \{1,2\}$, $F_3 = \{1,3\}$, $F_4 = \{1,2,2\}$ and $F_{36} = \{1,2,2,3,3\}$. In this case $\{F_n\}$ is an mset sequence.

Statistical Convergence

Statistical convergence was first mentioned by Zygmund in her monograph in 1935 in Warsaw [16] and it was formally introduced by Fast and Steinhaus independently in [17-18]. Later on, this concept was studied by Schoenberg as a summability method and various properties were investigated [19]. After the 1950s, studies on the concept of statistical convergence made rapid progress and many studies were conducted on this subject. The most well-known of these areas are number theory by Erdös and Tenenbaum; summability theory by Freedman and Sember and measure theory by Miller in [20-22]. Fridy has an important study in which he studied the properties of statistical convergence [23]. This concept was also studied with ideals, weak convergence, modulus functions, complex uncertain sequences in [24-27]. Statistical convergence is based on the definition of natural density of the $A \subseteq \mathbb{N}$ set such as $d(A) = \lim_{n\to\infty} \frac{A_n}{n}$ where $\mathbb{N}$ is the set of all natural numbers, $A_n = \{k \in A: k \leq n\}$ and $|A_n|$ gives the cardinality of A_n gives the cardinality of A_n.

Definition 1.2. [17] A number sequence (x_i) is statistically convergent to L provided that for every $\varepsilon > 0$,

$$d|\{i \leq n: |x_i - L| \geq \varepsilon\}| = 0$$

In this case we write $st - lim x_i = L$ and usually the set of statistically convergent sequences is denoted by S. Considering the definition of natural density, this definition can also be expressed as for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} |\{i \leq n: |x_i - L| \geq \varepsilon\}| = 0$$

Lacunary statistical convergence was defined by Fridy and Orhan in 1993 in [28]. Before giving this definition, let's remind the definition of a lacunary sequence.

Definition 1.3. A lacunary sequence is an increasing integer sequence $\theta = (i_r)$ such that $i_0 = 0$ and $h_r = i_r - i_{r-1} \to \infty$ as $r \to \infty$. The intervals $J_r = (i_{r-1}, i_r]$ are determined by θ and the ratio is determined $q_r = \frac{i_r}{i_{r-1}}$.

Example 1.3. $\theta = (r^2)$ is a lacunary sequence because $i_0 = 0$ and $h_r = i_r - i_{r-1} \to \infty$ as $r \to \infty$.

Example 1.4. $\theta = (r)$ is not a lacunary sequence because $i_0 = 0$ but $h_r = i_r - i_{r-1} = 1$ for all $r = 0,1, \ldots$

Definition 1.4. [28] Let $\theta = (i_r)$ be a lacunary sequence. The number sequence $x = (x_i)$ is lacunary statistically convergent (or $S_\theta -$ convergent) to L if for every $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{h_r} |\{i \in J_r: |x_i - L| \geq \varepsilon\}| = 0.$$

In this case we write $S_\theta - lim x_i = L$ and usually the set of lacunary statistically convergent sequences is denoted by S_θ.

Another concept closely related to statistical convergence is strong Cesáro summability:

$$|C_1| := \left\{x: for\ some\ L, \lim_n \left(\frac{1}{n} \sum_{i=1}^n |x_i - L|\right) = 0\right\}.$$

Similarly, there is a close relationship between strong Cesáro summability and N_θ space:

$$N_\theta := \left\{x: for\ some\ L, \lim_r \left(\frac{1}{h_r} \sum_{i \in J_r} |x_i - L|\right) = 0\right\}.$$

Statistical convergence of multiset sequences

Debnath and Debnath studied statistical convergence of multiset sequences on $\mathbb{R}$ real numbers and definitions of statistical convergence of multiset sequences are basis of our study in [29].

Definition 1.5. Let $\mathbb{N}_0$ is the set of non-negative integers. The set

$$m\mathbb{R} = \{mx = x_i|c_i: x_i \in \mathbb{R}\ and\ c_i \in \mathbb{N}_0\}$$

is called multiset of real numbers.

Definition 1.6. [29] Let $x = (x_i)$ be a real sequence and $c = (c_i)$ be a sequence of $\mathbb{N}_0$ A multiset sequence $mx = x_i|c_i$ of $m\mathbb{R}$ is statistically convergent to $l|c$ of $m\mathbb{R}$ if given for any $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} \left|\left\{i \leq n: \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \varepsilon\right\}\right| = 0.$$

Example 1.5. [29] Consider a multisequence $mx = x_i|c_i$ given by

$$x_i = \begin{cases} i, & i = n^2; n = 1,2,3, \ldots \\ 1, & otherwise \end{cases} \text{ and } c_i = \begin{cases} i, & i = n^3; n = 1,2,3, \ldots \\ 5, & otherwise \end{cases}$$

Then for any $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} \left|\left\{i \leq n: \sqrt{(x_i - 1)^2 + (c_i - 5)^2} \geq \varepsilon\right\}\right|$$

$$= Density\ of\ perfect\ squared\ or\ perfect\ cubic\ positive\ integers\ or\ bot$$

$$\leq \lim_{n\to\infty} \frac{1}{n}\left(n^{\frac{1}{2}} + n^{\frac{1}{3}} - n^{\frac{1}{6}}\right) = 0$$

Therefore, the multisequence mx statistically converges to $1|5$.

I −convergence

I −convergence has emerged as a generalized form of many types of convergences. This means that, if we choose different ideals we will have different convergences. Kostyrko et al. introduced this concept in a metric space in [30]. We will explain this situation with two examples later. Before defining I −convergence, the definitions of ideal and filter will be needed.

Definition 1.7. A family of sets $I \subset 2^{\mathbb{N}}$ is an ideal if the following properties are provided:

i) $\emptyset \in I$,
ii) $A, B \in I$ implies $A \cup B \in I$,
iii) For each $A \in I$ and each $B \subseteq A$ implies $B \in I$.

We say that I is non-trivial if $\mathbb{N} \notin I$ and I is admissible if $\{n\} \in I$ for each $n \in \mathbb{N}$.

Definition 1.8. A family of the sets $F \subset 2^{\mathbb{N}}$ is a filter if the following properties are provided:

i) $\emptyset \notin F$,
ii) If $A, B \in F$ then we have $A \cap B \in F$,
iii) For each $A \in I$ and each $A \subseteq B$ we have $B \in F$.

Proposition 1.1. If I is an ideal in $\mathbb{N}$ then the collection,

$F(I) = \{A \subseteq \mathbb{N} : \mathbb{N}\backslash A \in I\}$

forms in a filter in $\mathbb{N}$ which is called the filter associated with I.

Definition 1.9. A sequence of reals $x = (x_i)$ is I −convergent to $L \in \mathbb{R}$ if and only if the set

$A_\varepsilon = \{i \in \mathbb{N} : |x_i - L| \geq \varepsilon\} \in I$

for each $\varepsilon > 0$, In this case, we say that L is the I −limit of the sequence x. Following the statistical convergence and I −convergence located an important role in this area, in 2011, Savaş and Das have introduced the concept I −statistical convergence [31].

Definition 1.10. [31] A sequence $x = (x_i)$ is said to be I −statistically convergent to L for each $\varepsilon > 0$, and $\delta > 0$

$$\left\{n \in \mathbb{N} : \tfrac{1}{n} |\{i \leq n : |x_i - L| \geq \varepsilon\}| \geq \delta\right\} \in I.$$

Afterwards, I −statistical convergence was studied in various spaces in [32-33].

MAIN RESULTS

Demir and Gümüş defined I −convergence of multiset sequences in [34]. Now, our aim is to define I −lacunary statistical convergence for multiset sequences. For this purpose, we need following definitions.

Definition 2.1. Let $mx = x_i|c_i$ be a multiset sequence and $I \subset 2^{\mathbb{N}}$ be an admissible ideal. $mx = x_i|c_i$ is said to be I −statistically convergent to $l|c$ if for each $\varepsilon > 0$,

$$\left\{n \in \mathbb{N} : \tfrac{1}{n} \left|\left\{i \leq n : \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \varepsilon\right\}\right| \geq \delta\right\} \in I.$$

In this case, we write $mx \to l|c\ (S(I))$. The set of all I −statistically convergent multiset sequences is symbolized as $S^{l|c}(I)$.

Definition 2.2. Let θ be a lacunary sequence, $mx = x_i|c_i$ be a multiset sequence of $m\mathbb{R}$ and $I \subset 2^{\mathbb{N}}$ be an admissible ideal. $mx = x_i|c_i$ is said to be I −lacunary statistically convergent to $l|c$, if for each $\varepsilon > 0$,

$$\left\{r \in \mathbb{N} : \tfrac{1}{h_r} \left|\left\{i \in J_r : \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \varepsilon\right\}\right| \geq \delta\right\} \in I.$$

In this case, we write $mx \to l|c\ (S_\theta(I))$. The set of all I −lacunary statistically convergent multiset sequences is symbolized as ${S_\theta}^{l|c}(I)$.

Definition 2.3. Let $mx = x_i|c_i$ be a multiset sequence and $I \subset 2^{\mathbb{N}}$ be an admissible ideal. $mx = x_i|c_i$ is said to be I −statistically Cesáro summable to $l|c$ if for each $\varepsilon > 0$,

$$\left\{n \in \mathbb{N} : \tfrac{1}{n} \textstyle\sum_{i=1}^{n} \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \varepsilon\right\} \in I.$$

In this case, we write $mx \to l|c\ (\sigma(I))$. The set of all I −statistically Cesáro summable multiset sequences is symbolized as $\sigma^{l|c}(I)$.

Definition 2.4. Let θ be a lacunary sequence, $mx = x_i|c_i$ be a multiset sequence of $m\mathbb{R}$ and $I \subset 2^{\mathbb{N}}$ be an admissible ideal. $mx = x_i|c_i$ is said to be strongly I −lacunary summable to $l|c$ if for each $\varepsilon > 0$,

$$\left\{r \in \mathbb{N} : \tfrac{1}{h_r} \textstyle\sum_{i \in J_r} \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \varepsilon\right\} \in I.$$

In this case, we write $mx \to l|c\ (N_\theta(I))$. The set of all strongly I −lacunary summable multiset sequences is symbolized as ${N_\theta}^{l|c}(I)$.

In the next theorems, the relationships between these sets are examined. Throughout the paper, $mx = x_i|c_i$ denotes a multiset sequence of $m\mathbb{R}$, $I \subset 2^{\mathbb{N}}$ denotes an admissible ideal and $\theta = (i_r)$ denotes a lacunary sequence.

Theorem 2.1. For any multiset sequence $mx = x_i|c_i$, $mx \in {N_\theta}^{l|c}(I)$ implies $mx \in {S_\theta}^{l|c}(I)$.

Proof Let $mx \in {N_\theta}^{l|c}(I)$ and $\varepsilon > 0$, be given. Then,

$$\sum_{i \in J_r} \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \sum_{\substack{i \in J_r \\ \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \varepsilon}} \sqrt{(x_i - l)^2 + (c_i - c)^2}$$

$$\geq \varepsilon \left|\left\{i \in J_r : \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \varepsilon\right\}\right|$$

and so,

$$\frac{1}{\varepsilon h_r} \sum_{i \in J_r} \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \frac{1}{h_r} \left|\left\{i \in J_r : \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \varepsilon\right\}\right|.$$

Then, for any $\delta > 0$ we have,

$$\left\{r \in \mathbb{N} : \frac{1}{h_r} \left|\left\{i \in J_r : \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \varepsilon\right\}\right| \geq \delta\right\}$$

$$\subseteq \left\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{i \in J_r} \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \varepsilon\delta\right\}$$

belongs to l and proof is completed.

The following theorem shows in which case the inverse of this theorem is valid.

Definition 2.5. A multiset sequence $mx = x_i|c_i$ is said to be bounded provided that there exists a non-negative real number B such that $\sqrt{x_i^2 + (c_i - 1)^2} \leq B$.

Theorem 2.2. Let $mx = x_i|c_i$ be a bounded multiset sequence and $I \subset 2^{\mathbb{N}}$ be an admissible ideal. Then, $mx \in S_\theta^{l|c}(I)$ implies $mx \in N_\theta^{l|c}(I)$.

Proof Let $mx \in S_\theta^{l|c}(I)$ and mx be bounded. Then, there exists a non-negative real number B such that $\sqrt{x_i^2 + (c_i - 1)^2} \leq B$ for all $i \in \mathbb{N}$. At the same time, from the fact that $c, c_i \in \mathbb{N}_0$ and $x_i \to l$, we have,

$$\sqrt{(x_i - l)^2 + (c_i - c)^2} \leq \sqrt{x_i^2 + (c_i - 1)^2} \leq B$$

According to this information, for each $\varepsilon > 0$,

$$\frac{1}{h_r}\sum_{i\in J_r}\sqrt{(x_i - l)^2 + (c_i - c)^2} = \frac{1}{h_r}\sum_{\substack{i\in J_r \\ \sqrt{(x_i-l)^2+(c_i-c)^2}\geq\frac{\varepsilon}{2}}}\sqrt{(x_i - l)^2 + (c_i - c)^2}$$

$$+\frac{1}{h_r}\sum_{\substack{i\in J_r \\ \sqrt{(x_i-l)^2+(c_i-c)^2}<\frac{\varepsilon}{2}}}\sqrt{(x_i - l)^2 + (c_i - c)^2}$$

$$\leq \frac{B}{h_r}\left|\left\{i \in J_r: \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \frac{\varepsilon}{2}\right\}\right| + \frac{\varepsilon}{2}$$

is obtained. Hence, for any $\delta > 0$.

$$\left\{r \in \mathbb{N}: \frac{1}{h_r}\sum_{i\in J_r}\sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \frac{\varepsilon}{2}\right\}$$

$$\subseteq \left\{r \in \mathbb{N}: \frac{1}{h_r}\left|\left\{i \in J_r: \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \frac{\varepsilon}{2}\right\}\right| \geq \frac{\varepsilon}{2B}\right\}.$$

Since the right side belongs to the ideal, we have the proof.

Now, lets investigate the relation between the spaces $S^{l|c}(I)$ and $S_\theta^{l|c}(I)$ with the following two theorems.

Theorem 2.3. If $liminf\, q_r > 1$ then, $mx \in S^{l|c}(I)$ implies $mx \in S_\theta^{l|c}(I)$.

Proof Assume that $liminf\, q_r > 1$. Then, for sufficiently large r there exists a $\lambda > 0$ such that $q_r \geq 1 + \lambda$. This implies

$$\frac{h_r}{i_r} \geq \frac{\lambda}{1+\lambda}.$$

Since $mx \in S^{l|c}(I)$, for each $\varepsilon > 0$, and sufficiently large r, we have,

$$\frac{1}{i_r}\left|\left\{i \leq i_r: \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \varepsilon\right\}\right|$$

$$\geq \frac{1}{i_r}\left|\left\{i \in J_r: \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \varepsilon\right\}\right|$$

$$\geq \frac{\lambda}{1+\lambda}\frac{1}{h_r}\left|\left\{i \in J_r: \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \varepsilon\right\}\right|$$

is obtained. Hence for each $\delta > 0$,

$$\left\{r \in \mathbb{N}: \frac{1}{h_r}\left|\left\{i \in J_r: \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \varepsilon\right\}\right| \geq \delta\right\}$$

$$\subseteq \left\{r \in \mathbb{N}: \frac{1}{i_r}\left|\left\{i \leq i_r: \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \varepsilon\right\}\right| \geq \frac{\delta\lambda}{1+\lambda}\right\}$$

is obtained. From the definition of ideal, we have the proof

Theorem 2.4. Let $\theta = (i_r)$ be a lacunary sequence satisfying the condition $limsup q_r < \infty$ and $I \subset 2^{\mathbb{N}}$ be an admissible ideal of subsets of $\mathbb{N}$. In this case, $mx \in S_\theta^{l|c}(I)$ implies $mx \in S^{l|c}(I)$.

Proof If $limsup q_r < \infty$ then there is a $K > 0$ such that $q_r < K$ for all r. Suppose that $mx \in S_\theta^{l|c}(I)$ and for $\varepsilon, \delta, \eta > 0$ define the sets

$$C = \left\{r \in \mathbb{N}: \frac{1}{h_r}\left|\left\{i \in J_r: \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \varepsilon\right\}\right| < \delta\right\}$$

and for

$$T = \left\{n \in \mathbb{N}: \frac{1}{n}\left|\left\{i \leq n: \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \varepsilon\right\}\right| < \eta\right\}.$$

It is obvious from our assumption $C \in F(I)$ the filter associated with the ideal I. If we can show that $T \in F(I)$ then, we have the proof. For all $j \in C$ let

$$A_j = \frac{1}{h_j}\left|\left\{i \in J_j: \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \varepsilon\right\}\right|.$$

Choose $n \in \mathbb{N}$ such that $i_{r-1} < n < i_r$ for some $r \in C$. Now,

$$\frac{1}{n}\left|\left\{i \leq n: \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \varepsilon\right\}\right|$$

$$\leq \frac{1}{i_{r-1}}\left|\left\{i \leq i_r: \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \varepsilon\right\}\right|$$

$$= \frac{1}{i_{r-1}}\left|\left\{i \in J_1: \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \varepsilon\right\}\right|$$

$$+ \cdots + \frac{1}{i_{r-1}}\left|\left\{i \in J_r: \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \varepsilon\right\}\right|$$

$$= \frac{1}{i_{r-1}}\frac{1}{h_1}\left|\left\{i \in J_1: \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \varepsilon\right\}\right|$$

$$+ \cdots + \frac{1}{i_{r-1}}\frac{1}{h_r}\left|\left\{i \in J_r: \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \varepsilon\right\}\right|$$

$$= \frac{i_r}{i_{r-1}}A_1 + \frac{i_2 - i_1}{i_{r-1}}A_1 + \cdots + \frac{i_r - i_{r-1}}{i_{r-1}}A_1$$

$$\leq \left(\sup_{j\in C} A_j\right)\frac{i_r}{i_{r-1}} < K\delta.$$

Choosing $\eta = \frac{\delta}{K}$ and in view of the fact that $\cup\{n: i_{r-1} < n < i_r, r \in C\} \subset T$ where $C \in F(I)$ it follows from our assumption Talso belongs to $F(I)$.

The following two theorems give us the relationship between $\sigma^{l|c}(I)$ and $N_\theta{}^{l|c}(I)$.

Theorem 2.5. Let $\theta = (i_r)$ be a lacunary sequence satisfying the condition $liminf q_r > 1$ and $I \subset 2^{\mathbb{N}}$ be an admissible ideal. In this case, $mx \in \sigma^{l|c}(I)$ implies $mx \in N_\theta{}^{l|c}(I)$.

Proof If $liminf q_r > 1$ then, for sufficiently large r there exists a $\delta > 0$ such that $q_r > 1 + \delta$. Since

$h_r = i_r - i_{r-1}$ we have $\frac{i_r}{h_r} \le \frac{1+\delta}{\delta}$ and $\frac{i_{r-1}}{h_r} \le \frac{1}{\delta}$. Let $\varepsilon > 0$, and define the set

$$E = \left\{ i_r \in \mathbb{N} : \frac{1}{i_r} \sum_{i=1}^{i_r} \sqrt{(x_i - l)^2 + (c_i - c)^2} < \varepsilon \right\}.$$

We can easily say that E $\in F(I)$ which is the filter of the ideal I.

$$\frac{1}{h_r} \sum_{i \in J_r} \sqrt{(x_i - l)^2 + (c_i - c)^2} = \frac{1}{h_r} \sum_{i=1}^{i_r} \sqrt{(x_i - l)^2 + (c_i - c)^2}$$
$$- \frac{1}{h_r} \sum_{i=1}^{i_{r-1}} \sqrt{(x_i - l)^2 + (c_i - c)^2} = \frac{i_r}{h_r} \frac{1}{i_r} \sum_{i=1}^{i_r} \sqrt{(x_i - l)^2 + (c_i - c)^2}$$
$$- \frac{i_{r-1}}{h_r} \frac{1}{i_{r-1}} \sum_{i=1}^{i_{r-1}} \sqrt{(x_i - l)^2 + (c_i - c)^2} \le \left(\frac{1+\delta}{\delta} \right) \varepsilon - \frac{1}{\delta} \varepsilon'$$

for each $i_r \in E$. Choose $\eta = \left(\frac{1+\delta}{\delta}\right) \varepsilon - \left(\frac{1}{\delta}\right) \varepsilon'$.
Therefore,

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{i \in J_r} \sqrt{(x_i - l)^2 + (c_i - c)^2} < \eta \right\} \in F(I)$$

and it completes the proof.

Theorem 2.6. Let $\theta = (i_r)$ be a lacunary sequence satisfying the condition $limsup q_r < \infty$ and $I \subset 2^{\mathbb{N}}$ be an admissible ideal. In this case, $mx \in N_\theta{}^{l|c}(I)$ implies $mx \in \sigma^{l|c}(I)$.

Proof If $limsup q_r < \infty$ then there exist $K > 0$ such that $q_i < K$ for all $r \ge 1$.

Let $mx \in N_\theta{}^{l|c}(I)$ and define the sets T and R such that

$$L = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{i \in J_r} \sqrt{(x_i - l)^2 + (c_i - c)^2} < \varepsilon_1 \right\}$$

and

$$R = \left\{ n \in \mathbb{N} : \tfrac{1}{n} \textstyle\sum_{i=1}^{n} \sqrt{(x_i - l)^2 + (c_i - c)^2} < \varepsilon_2 \right\}.$$

Let

$$D_j = \frac{1}{h_j} \sum_{i \in J_r} \sqrt{(x_i - l)^2 + (c_i - c)^2} < \varepsilon_1$$

for all $j \in L$. It is obvious that $L \in F(I)$. Choose n is any integer with $i_{r-1} < n < i_r$, where $r \in L$.

$$\frac{1}{n} \sum_{i=1}^{n} \sqrt{(x_i - l)^2 + (c_i - c)^2}$$
$$\le \frac{1}{i_{r-1}} \sum_{i=1}^{i_r} \sqrt{(x_i - l)^2 + (c_i - c)^2}$$
$$= \frac{1}{i_{r-1}} \left(\sum_{i \in J_1} \sqrt{(x_i - l)^2 + (c_i - c)^2} \right.$$
$$+ \sum_{i \in J_2} \sqrt{(x_i - l)^2 + (c_i - c)^2}$$
$$\left. + \cdots + \sum_{i \in J_r} \sqrt{(x_i - l)^2 + (c_i - c)^2} \right)$$
$$= \frac{i_1}{i_{r-1}} \left(\frac{1}{h_1} \sum_{i \in J_1} \sqrt{(x_i - l)^2 + (c_i - c)^2} \right)$$
$$+ \frac{i_2 - i_1}{i_{r-1}} \left(\frac{1}{h_2} \sum_{i \in J_2} \sqrt{(x_i - l)^2 + (c_i - c)^2} \right)$$
$$+ \cdots \frac{i_r - i_{r-1}}{i_{r-1}} \left(\frac{1}{h_r} \sum_{i \in J_r} \sqrt{(x_i - l)^2 + (c_i - c)^2} \right)$$
$$= \frac{i_1}{i_{r-1}} D_1 + \frac{i_2 - i_1}{i_{r-1}} D_2 + \cdots + \frac{i_r - i_{r-1}}{i_{r-1}} D_r$$
$$\le \left(\sup_{j \in T} D_j \right) \frac{i_1}{i_{r-1}} < \varepsilon_1 K.$$

Choose $\varepsilon_2 = \frac{\varepsilon_1}{K}$ and view of the fact that $\cup \{n : i_{r-1} < n < i_r, r \in L\} \subset R$ where $L \in F(I)$ it follows from our assumption on θ that the set R also belongs to $F(I)$ and this completes the proof of the theorem.

CONCLUSIONS

The fact that multisets are encountered in many areas in daily life makes multiset sequences very important. On the other hand, I −convergence is a type of convergence that generalizes many convergence types, and it is quite interesting how this convergence type can be defined for multiset sequences and what properties it will provide. For this purpose, in this paper we introduce the lacunary I −statistical convergence of multisequences and we investigate some important relations.

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ETHICS

There are no ethical issues with the publication of this manuscript.

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