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Research Article

(2+1)-dimensional new bi-hamiltonian integrable system: Symmetries, Noether's theorem and integrals of motion

Salih YAMAN[1], Devrim YAZICI[1,*]

[1]*Department of Physics, Faculty of Arts and Sciences, Yildiz Technical University, İstanbul, 34220, Türkiye*

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ABSTRACT

In this work, we investigate a symmetry reduction of the recently discovered (3 + 1)-dimensional equation of the Monge-Ampère type. This equation forms a bi-Hamiltonian system using Magri's theorem when expressed in the two-component form. We select a particular linear combination of the Lie point symmetries belonging to this system to conduct symmetry reduction, resulting in a new (2 + 1)-dimensional system in two-component form. Lagrangian and first Hamiltonian densities are then calculated. We employ Dirac's theory of constraints to obtain symplectic and first Hamiltonian operators. Subsequently, we transform the symmetry condition of the reduced system into a skew-factorized form to determine the recursion operator. Applying the recursion operator to the first Hamiltonian operator yields the second Hamiltonian operator. We demonstrate that the reduced system is a bi-Hamiltonian integrable system in the sense of Magri. Lie point symmetries of the reduced system are identified. Finally, we calculate integrals of motion using the inverse Noether theorem and prove that they have the total divergence form.

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INTRODUCTION

Evolutionary Hirota type equations in (3 + 1)-dimensions have the form:

$$F = f(u_{ij}) - u_{tt}g(u_{ij}) = 0 \iff u_{tt} = \frac{f}{g}\,,\ g \neq 0, \qquad (1)$$

where u is an unknown that depends on the coordinates (z_1, z_2, z_3, t) and f, g are smooth functions of u_{ij} $(i, j = 1,2,3,t)$. The subscripts i, j of u denote partial derivatives with respect to the designated variables, such as $u_{t2} = \partial^2 u / \partial t \partial z_2$, $u_{t2} = \partial^2 u/\partial t\partial z_2$. In [1], these types of equations were studied extensively and a general equation of the form (1) that possesses a Lagrangian had been derived. All such equations have the Monge-Ampère form, where the only nonlinear terms consist of minors of the Hessian matrix of u. In this paper, it is sufficient for our purposes to restrict ourselves to a particular case of such an equation, namely:

$$\begin{aligned} F = {} & a_{11}(u_{tt}u_{23} - u_{t2}u_{t3}) + c_4(u_{t1}u_{23} - u_{t2}u_{13}) \\ & + c_5(u_{t2}u_{23} - u_{t3}u_{22}) + c_8(u_{t2}u_{13} - u_{t3}u_{12}) \\ & + c_9(u_{11}u_{23} - u_{12}u_{13}) + c_{10}(u_{12}u_{23} - u_{13}u_{22}) = 0. \end{aligned} \qquad (2)$$

***Corresponding author.**
*E-mail address: devrimyazici@gmail.com

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Here, a_{11}, c_4, c_5, c_8 and c_{10} are arbitrary constants. This equation, denoted as *System-I* , is expressed in the two component form:

$$\begin{aligned} u_t &= v \\ v_t &= \frac{1}{u_{23}}\{v_2v_3 - c_4L_{12(3)}[v] - c_5L_{23(2)}[v] - c_8L_{23(1)}[v] \\ &\quad - c_9L_{12(3)}[u_1] - c_{10}L_{23(2)}[u_1]\} = q, \end{aligned} \tag{3}$$

where a_{11} and the condition $c_{10}c_8 = c_5c_9$ is imposed. [1,2]. The operator:

$$L_{ij(k)} = u_{jk}D_i - u_{ik}D_j, \tag{4}$$

is introduced for brevity, where D_i denotes the total derivative with respect to z_i. The explicit form of (3) is given as:

$$\begin{aligned} u_t &= v \\ v_t &= \frac{1}{u_{23}}\{v_2v_3 + c_4(u_{13}v_2 - u_{23}v_1) + c_5(u_{22}v_3 - u_{23}v_2) \\ &\quad + c_8(u_{12}v_3 - u_{13}v_2) + c_9(u_{13}u_{12} - u_{23}u_{11}) \\ &\quad + c_{10}(u_{22}u_{13} - u_{23}u_{12}) = q. \end{aligned} \tag{5}$$

In [1, 2], the bi-Hamiltonian structure of (5) was discovered, demonstrating that this system is integrable in the sense of Magri [3,4]. In four dimensions, the evolutionary Hirota-type equations (1) exhibit the symplectic Monge-Ampère property, as demonstrated in former studies [5,6]. These equations find applications in various fields, particularly in gravitational physics. For instance, they are relevant to Plebanski's so called heavenly equations which simplify the complex Einstein field equations governing self-dual gravitational fields [7].

In this study, we find out if the two-component system (3) could be reduced into a (2 + 1)-dimensional bi-Hamiltonian system. We perform the reduction using the method previously applied in [8,9,10]. We choose a specific linear combination of symmetries that is critical to the success of the reduction. Upon obtaining the (2 + 1)-dimensional system in two-component form, we employ the method used before in [11-16] to construct the bi-Hamiltonian system. In order to obtain the first Hamiltonian structure, we use Dirac's constraint analysis [17]. The skew-factorized form of the symmetry condition is reduced from the (3 + 1)-dimensional system [1] to obtain the recursion operator. The second Hamiltonian operator is obtained by applying the recursion operator to the first. Magri's theorem [3,4] is then employed to determine whether the (2 + 1)-dimensional system forms a bi-Hamiltonian system, indicating its integrability.

Completely integrable systems are intriguing because they present many symmetries and conserved densities in their solutions, although finding them is often challenging. We employ tools of Lie symmetry analysis to conduct symmetry reduction and discover first integrals. Recent papers such as [18-20] have used this powerful approach, where the authors have adopted power series expansion to find exact solutions of some nonlinear equations. In addition to well-known analytical methods like Darboux [21], Bäcklund transformations [22] and the recently discovered Kudryashov method [23], as well as the generalized auxiliary equation technique [24]; numerical methods also play a crucial role in this research field. Historically, the well-known KdV equation was initially solved through a numerical study [25]. Recently, new numerical approaches, such as the Fractional Iteration Algorithm [26] and Variational Iterational Algorithm [27] have been employed to obtain exact solutions for some nonlinear evolution equations. In this paper, we adopt Dirac constraint analysis which is very powerful in handling variational problems when the Lagrangian density is linear in velocity. However, in any other case, such as when the Lagrangian density is quadratic in velocity, this approach is not applicable. Magri made valuable contributions to the field of Hamiltonian systems by proving a theorem stating that evolutionary systems may have a multi-Hamiltonian structure. The Magri theorem, along with Dirac constraint analysis, has paved the way for discovering new integrable Hamiltonian systems, as evidenced in [28-33]. Besides the theoretical realm of science, Hamiltonian systems find utility in applied engineering problems as demonstrated in [34].

This paper is organized as follows: In section 2, we define the symmetries of the system (3) and conduct symmetry reduction to obtain the reduced system in two-component form. In section 3, we verify that the system is in Euler-Lagrange form and determine the degenerate Lagrangian density belonging to the system. Starting from the degenerate Lagrangian density, we construct the first Hamiltonian structure of the reduced system. In section 4, we obtain the recursion operator using the skew-factorized method for the symmetry condition. In section 5, we compose the recursion operator with the first Hamiltonian operator to get the second Hamiltonian operator. Then, we apply Magri's Theorem to establish the second Hamiltonian structure of the reduced system. In section 6, we identify Lie point symmetries and obtain the Lie Algebra of the reduced system. We determine the symmetry characteristics and apply these results in Noether's Theorem to identify new conserved densities of the system. Once we obtain the new conserved densities, we validate their legitimacy by casting them into total divergence form.

SYMMETRY REDUCTION AND THE (2 + 1) -DIMENSIONAL SYSTEM

In [2], the generators of point symmetries for (3) were identified as follows;

$$\begin{gathered} X_1 = \partial_1,\ X_2 = u\partial u + v\partial v,\ X_3 = \partial_t, \\ X_4 = t\partial_t + z_1\partial_1 + z_2\partial_2 + u\partial_u,\ X_a = a(z_3)\partial_u, \\ Y_b = b(z_3)\partial_3,\ X_\infty = X_{(c,e)} = c(\zeta)\partial_2 + e(\zeta)\partial_u, \end{gathered} \tag{6}$$

where a, b, c and e are arbitrary smooth functions, and ζ is defined as $\zeta = c_5z_1 - c_8z_2$. Given these symmetries, we choose the particular combination:

$$X = X_1 + X_{(1,0)} - Y_1, \tag{7}$$

and get the symmetry:

$$X = \partial_1 + \partial_2 - \partial_3, \tag{8}$$

Equation (8) leads to the characteristic equation:

$$\frac{dz_1}{1} = \frac{dz_2}{1} = -\frac{dz_3}{1} \tag{9}$$

[35]. Integrating both sides of the first two equations given in (9) results in an invariant Z_1 as follows:

$$\int dz_1 = \int dz_2 \implies z_1 = z_2 + Z_1. \tag{10}$$

Likewise, integrating both sides of the last two equations given in (9) leads to an invariant Z_2 as follows:

$$\int dz_2 = -\int dz_3 \implies z_2 = -z_3 + Z_2. \tag{11}$$

Therefore, the invariants of X determined by its characteristic equation (9) are:

$$Z_1 = z_1 - z_2,\ Z_2 = z_2 + z_3,\ T = t,\ U = u,\ \ V = v. \tag{12}$$

Consequently, the total derivatives undergo a transformation expressed as:

$$D_{z_1} = D_{Z_1},\ \ D_{z_2} = D_{Z_2} - D_{Z_1},\ \ D_{z_3} = D_{Z_2},\ D_t = D_T. \tag{13}$$

In equation (2), by replacing the derivatives with expressions from (13) and renaming variables:

$$U \to u,\ \ V \to v,\ \ T \to t,\ \ Z_1 \to z_1,\ \ Z_2 \to z_2, \tag{14}$$

we obtain the new (2 + 1)-dimensional evolutionary equation:

$$\begin{aligned} F = u_{tt}\Delta - u_{t2}^2 + u_{t1}u_{t2} + a(u_{t1}u_{12} - u_{t2}u_{11}) \\ + b(u_{t2}u_{12} - u_{t1}u_{22}) + c(u_{12}^2 - u_{11}u_{22}) = 0, \end{aligned} \tag{15}$$

where $a = c_5 - c_8$, $b = c_5 - c_4$, $c = c_{10} - c_9$ are arbitrary constants and $\Delta = u_{22} - u_{12}$. Equation (15) is represented in the two component form:

$$\begin{aligned} u_t^r &= v \\ v_t^r &= \frac{1}{\Delta}\{v_2^2 - v_1 v_2 + a(v_2 u_{11} - v_1 u_{12}) \\ &\quad + b(v_1 u_{22} - v_2 u_{12}) + c(u_{11}u_{22} - u_{12}^2)\} \equiv \text{q}. \end{aligned} \tag{16}$$

The superscript r indicates that the relevant parameter is for the reduced (2 + 1)-dimensional system. Two equations presented in (16) compose the new (2 + 1)-dimensional system.

FIRST HAMILTONIAN STRUCTURE OF THE (2 + 1)-DIMENSIONAL REDUCED SYSTEM

Lagrangian density is the starting point for constructing the Hamiltonian structure of the new system. Thus, it is essential to verify that the reduced equation (15) is an Euler-Lagrange equation. Euler-Lagrange equations must satisfy the Helmholtz condition [35]. We verify that (15) possesses a Lagrangian density by checking the Helmholtz condition. Homotopy formula enables us to obtain the Lagrangian density. We present the result of our calculation after skipping the total derivative terms as follows:

$$\begin{aligned} L^r = \frac{1}{2}u_t^2\Delta + \frac{u_t}{3}[a(u_2u_{11} - u_1u_{12}) + b(u_1u_{22} - u_2u_{12})] \\ + \frac{u}{3}c(u_{12}^2 \text{-} u_{11}u_{22}). \end{aligned} \tag{17}$$

Euler-Lagrange equation using this result yields the reduced equation (15) which is in one component form. However, we want to obtain L^r in two component form so that we can proceed with Dirac constraint analysis. The transformation $u_t = v$ is applied to appropriate terms of (17) so that Euler-Lagrange equation with the new Lagrangian density results in the reduced system (16). Skipping total derivative terms, we present the new Lagrangian density as:

$$\begin{aligned} L^r = \left(u_t v - \frac{1}{2}v^2\right)\Delta + \frac{u_t}{3}\{b(u_2u_{12} - u_1u_{22}) + a(u_1u_{12} - u_2u_{11})\} \\ + \frac{u}{3}c(u_{11}u_{22} - u_{12}^2). \end{aligned} \tag{18}$$

Subsequently, we obtain canonical momenta associated with the coordinates u and v as follows:

$$\begin{aligned} \Pi_u^r &= \frac{\partial L^r}{\partial u_t} = v\Delta + \frac{b}{3}(u_2u_{12} - u_1u_{22}) + \frac{a}{3}(u_1u_{12} - u_2u_{11}), \\ \Pi_v^r &= \frac{\partial L^r}{\partial v_t} = 0\ . \end{aligned} \tag{19}$$

With the results obtained so far, the first Hamiltonian density H_1^r follows directly using the Legendre transformation, which in our case is expressed in the following way:

$$H_1^r = \Pi_u^r u_t^r + \Pi_v^r v_t^r - L^r. \tag{20}$$

Substituting, (16),(18) and (19) into (20), we obtain:

$$H_1^r = \frac{1}{2}\Delta v^2 + \frac{u}{3}c(u_{12}^2 - u_{11}u_{22}). \tag{21}$$

Next, we aim to find the symplectic operator K^r. Lagrangian density (18) is degenerate because it is linear in velocity. Consequently, it is not possible to express velocities as a function of momenta and vice versa, as evident in (19). Dirac successfully developed a theory to analyze such cases [17]. Guided by his work, we define the second-class constraints in terms of canonical momenta (19) as:

$$\begin{aligned} \phi_u^r &= \Pi_u^r - v\Delta - \frac{b}{3}(u_2u_{12} - u_1u_{22}) - \frac{a}{3}(u_1u_{12} - u_2u_{11}), \\ \phi_v^r &= \Pi_v^r \end{aligned} \tag{22}$$

so that $\phi_u^r = 0$ and $\phi_v^r = 0$ are set. The symplectic operator is defined in terms of these constraints as:

$$K^r = \begin{pmatrix} \{\phi_u^r(z), \phi_{u'}^r(z')\} & \{\phi_u^r(z), \phi_{v'}^r(z')\} \\ \{\phi_v^r(z), \phi_{u'}^r(z')\} & \{\phi_v^r(z), \phi_{v'}^r(z')\} \end{pmatrix}, \qquad (23)$$

similarly as in [1, 11, 33, 36]. The Poisson Bracket of two constraints is denoted as $\{\phi_i^r, \phi_j^r\}$, where the following relations hold:

$$\{\Pi_i(z), u^k(z')\} = \delta_i^k \delta(z - z'),\ \{\Pi_i(z), \Pi_k(z')\} = 0,\\ \{u^i(z), u^k(z')\} = 0. \qquad (24)$$

Here, δ_i^k is the discrete Dirac Delta function and $\delta(z - z')$ is the continuous Dirac Delta function. Moreover, we set $\Pi_1 = \Pi_u$, $\Pi_2 = \Pi_v$, $u^2 = v$ and $z = (z_1, z_2)$. Using (22) and (24), we can express, for instance, the K_{12}^r element of the symplectic matrix as:

$$K_{12}^r = \{\phi_u^r(z), \phi_{v'}^r(z')\} = -\Delta\{v, \Pi_{v'}^r\}\\ = -\Delta\delta(z_1 - z_1')\delta(z_2 - z_2'). \qquad (25)$$

Making use of the Dirac Delta function properties, (25) results in:

$$K_{12}^r = -\Delta = -(u_{22} - u_{12}). \qquad (26)$$

Through similar but lengthy calculations, we obtain K_{11}^r in the skew-symmetric form:

$$K_{11}^r = -\tfrac{1}{2}(D_1 v_2 + v_2 D_1) + D_2\left(v_2 - \tfrac{v_1}{2}\right) + \left(v_2 - \tfrac{v_1}{2}\right)D_2\\ + a(D_2 u_{11} - D_1 u_{12}) + b(D_1 u_{22} - D_2 u_{12}). \qquad (27)$$

Since the Poisson Bracket operation is anti-symmetric, K_{12}^r is easily found as:

$$K_{21}^r = \{\phi_v^r(z), \phi_{u'}^r(z')\} = -\Delta\{\Pi_v^r, v'\} = -K_{12}^r\\ = \Delta = u_{22} - u_{12} \qquad (28)$$

and using the property given in (24), K_{22}^r is given as follows:

$$K_{22}^r = \{\phi_v^r(z), \phi_{v'}^r(z')\} = \{\Pi_v^r, \Pi_{v'}^r\} = 0. \qquad (29)$$

With these results, we obtain the symplectic matrix:

$$K^r = \begin{pmatrix} K_{11}^r & -\Delta \\ \Delta & 0 \end{pmatrix}, \qquad (30)$$

where K_{11}^r is given in (27). The differential 2-form associated with K^r is given in the form:

$$\omega = \tfrac{1}{2}\int du_i \Lambda K_{ij}^r du_j. \qquad (31)$$

Here, the summation is taken over the repeated subscripts, while Λ denotes the wedge product. Checking the closeness condition:

$$d\omega = 0, \qquad (32)$$

of the differential 2-form (31) in a similar manner as done before in [1], reveals that the closeness condition (32) is satisfied. Therefore, the K^r matrix is a symplectic operator and its inverse which is given as:

$$J_0^r = (K^r)^{-1} = \frac{1}{\sqrt{\det(K^r)}}\begin{pmatrix} K_{22}^r & K_{21}^r \\ K_{12}^r & K_{11}^r \end{pmatrix}\frac{1}{\sqrt{\det(K^r)}}, \qquad (33)$$

is a Hamiltonian operator [37]. With the use of (30) and (33), we get:

$$J_0^r = \begin{pmatrix} 0 & \frac{1}{\Delta} \\ -\frac{1}{\Delta} & J_{0(22)}^r \end{pmatrix}, \qquad (34)$$

where $J_{0(22)}^r$ in skew-symmetric form is given by:

$$J_{0(22)}^r = \tfrac{1}{\Delta}\Big\{-\tfrac{1}{2}(D_1 v_2 + v_2 D_1) + D_2\left(v_2 - \tfrac{v_1}{2}\right) + \left(v_2 - \tfrac{v_1}{2}\right)D_2\\ + a(D_2 u_{11} - D_1 u_{12}) + b(D_1 u_{22} - D_2 u_{12})\Big\}\tfrac{1}{\Delta}. \qquad (35)$$

Here, J_0^r denotes the first Hamiltonin operator of the reduced system. The first Hamiltonian structure of the system is identified by the matrix equation:

$$\begin{pmatrix} u_t^r \\ v_t^r \end{pmatrix} = J_0^r \begin{pmatrix} \delta_u H_1^r \\ \delta_v H_1^r \end{pmatrix}, \qquad (36)$$

where δ_u and δ_v are variational derivatives with respect to u and v, respectively. By substituting equations (16), (21) and (34) into (36) and performing the calculations, we find out that equation (36) holds for the $(2+1)-$dimensional system. Therefore, the reduced system (16) exhibits a Hamiltonian structure just like the original system (3). L^r, H^r, K^r and J^r are obtained with identical results through direct reduction from the corresponding parameters L, H_1, K, J_0 given in [1] using the transformations (13).

SYMMETRY CONDITION IN A SKEW-FACTORIZED FORM

We define two Lie equations:

$$u_\tau^r = \varphi\,, \quad v_\tau^r = \psi, \qquad (37)$$

where τ is the group parameter; φ and ψ are symmetry characteristics. The symmetry condition of an equation is its differential compatibility with the Lie equations, and it is given as:

$$(u_t^r)_\tau - (u_\tau^r)_t = 0\ ,\ (v_t^r)_\tau - (v_\tau^r)_t = 0. \qquad (38)$$

The symmetry condition of the reduced equation (15) is expressed in the following form:

$$\{L_{2t(t)}D_2 + L_{t2(2)}D_t + L_{t1(t)}D_2 + L_{1t(2)}D_t + a(L_{21(t)}D_1 + L_{12(1)}D_t)\\ + b(L_{12(t)}D_2 + L_{21(2)}D_t) + c(L_{12(1)}D_2 + L_{21(2)}D_1)\}\varphi = 0, \qquad (39)$$

where the operator defined in (4) is used for brevity. If the symmetry condition can be converted to the skew-factorized form:

$$(A_1 B_2 - A_2 B_1)\varphi = 0, \qquad (40)$$

while the commutator relations:

$$[A_1, A_2] = 0,\ \ [A_1 B_2] - [A_2 B_1] = 0,\ \ [B_1, B_2] = 0, \tag{41}$$

are satisfied, Lax pairs and the recursion operator can be obtained. The operators are obtained by reduction from equation (6.6) given in [1] as follows:

$$\begin{aligned} &A_1^r = \tfrac{1}{\Delta}\left(L_{t2(2)} + L_{1t(2)}\right),\ B_1^r = \tfrac{1}{\Delta}\{(c_4 - c_8)\left(L_{2t(1)} + L_{t2(2)}\right) + c\left(L_{12(1)} + L_{21(2)}\right)\} \\ &A_2^r = -\tfrac{1}{\Delta} L_{12(2)}, \qquad B_2^r = \tfrac{1}{\Delta}\{a\left(L_{12(1)} + L_{21(2)}\right) + L_{2t(1)} + L_{t2(2)}\}. \end{aligned} \tag{42}$$

The commutator relations (41) are satisfied with these results. Lax pair is defined by:

$$X_1 = \lambda A_1 + B_1,\ \ X_2 = \lambda A_2 + B_2, \tag{43}$$

where λ is the spectral parameter. This pair yields the following results in our case with the use of (42):

$$\begin{aligned} X_1 &= \frac{\lambda}{\Delta}\left(L_{t2(2)} + L_{1t(2)}\right) + \frac{1}{\Delta}\{(c_4 - c_8)\left(L_{2t(1)} + L_{t2(2)}\right) + c\left(L_{12(1)} + L_{21(2)}\right)\} \\ X_2 &= -\tfrac{\lambda}{\Delta} L_{12(2)} + \tfrac{1}{\Delta}\{a\left(L_{12(1)} + L_{21(2)}\right) + L_{2t(1)} + L_{t2(2)}\}. \end{aligned} \tag{44}$$

We have checked that the commutator condition:

$$[X_1, X_2] = 0, \tag{45}$$

holds. Bringing the symmetry condition into the skew-factorized form (40) also enables us to write the recursion relations for symmetries as:

$$A_1^r \tilde{\varphi} = B_1^r \varphi, \quad A_2^r \tilde{\varphi} = B_2^r \varphi. \tag{46}$$

Using (42) in (46) and noting the relation:

$$\varphi_t = \psi, \tag{47}$$

we transform the two equations in (46) into the matrix form:

$$\begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix} = R^r \begin{pmatrix} \varphi \\ \psi \end{pmatrix}, \tag{48}$$

wherefrom we obtain the recursion operator as

$$R^r = \begin{pmatrix} (L_{21(2)}^{-1}[a\left(L_{12(1)} + L_{21(2)}\right) + (v_2 - v_1)D_2] & L_{21(2)}^{-1}\Delta \\ R_{21}^r & \frac{v_2}{\Delta}(D_2 - D_1)L_{21(2)}^{-1}\Delta + c_4 - c_8 \end{pmatrix}. \tag{49}$$

Here R_{21}^r is given by:

$$\begin{aligned} R_{21}^r = \tfrac{1}{\Delta}\{&(c_4 - c_8)(v_1 - v_2)D_2 + c(L_{12(1)} + L_{21(2)}) \\ &+ v_2(D_2 - D_1)L_{21(2)}^{-1}[a\left(L_{12(1)} + L_{21(2)}\right) + (v_2 - v_1)D_2]\}. \end{aligned} \tag{50}$$

Direct reduction from R given in [1] results in the same R^r (49).

SECOND HAMILTONIAN STRUCTURE OF THE (2 + 1)-DIMENSIONAL REDUCED SYSTEM

The second Hamiltonian operator J_1^r is obtained by applying the recursion operator to the first Hamilton operator as expressed by the equation:

$$J_1^r = R^r J_0^r. \tag{51}$$

The matrix element $J_{1(11)}^r$ of the operator J_1^r is obtained through the matrix multiplication (51), utilizing the properties:

$$\begin{aligned} L_{ij(k)} &= -L_{ji(k)}, \\ L_{ij(k)}^{-1} L_{ij(k)} &= 1, \end{aligned} \tag{52}$$

of the operator $L_{ij(k)}$ given in (4). This leads to the expression:

$$J_{1(11)}^r = 0 - L_{21(2)}^{-1}\Delta\frac{1}{\Delta}, \tag{53}$$

which results in:

$$J_{1(11)}^r = L_{12(2)}^{-1}. \tag{54}$$

The equation for $J_{1(12)}^r$ is given by:

$$\begin{aligned} J_{1(12)}^r = {} & L_{21(2)}^{-1}\left[a\left(L_{12(1)} + L_{21(2)}\right) + (v_2 - v_1)D_2\right]\frac{1}{\Delta} \\ & + L_{21(2)}^{-1}\Delta\tfrac{1}{\Delta}\big(v_{22} + 2v_2 D_2 - v_{12} - v_2 D_1 - v_1 D_2 \\ & + aL_{21(2)} + bL_{12(2)}\big)\tfrac{1}{\Delta}, \end{aligned} \tag{55}$$

resulting in:

$$J_{1(12)}^r = [c_4 - c_8 + L_{21(2)}^{-1}(D_2 - D_1)v_2]\tfrac{1}{\Delta}, \tag{56}$$

utilizing the properties (52). For $J_{1(21)}^r$, the equation is given by:

$$J_{1(21)}^r = 0 + \left[\tfrac{v_2}{\Delta}(D_2 - D_1)L_{21(2)}^{-1}\Delta + c_4 - c_8\right]\left(-\tfrac{1}{\Delta}\right), \tag{57}$$

which simplifies to:

$$J_{1(21)}^r = \tfrac{1}{\Delta}\left[v_2(D_1 - D_2)L_{21(2)}^{-1} + c_8 - c_4\right]. \tag{58}$$

For $J_{1(22)}^r$, the equation is given by:

$$\begin{aligned} J_{1(22)}^r = \tfrac{1}{\Delta}\{&(c_4 - c_8)(v_1 - v_2)D_2 + c(L_{12(1)} + L_{21(2)}) \\ &+ v_2(D_2 - D_1)L_{21(2)}^{-1}[a\left(L_{12(1)} + L_{21(2)}\right) + (v_1 - v_2)D_2]\}\tfrac{1}{\Delta} \\ &+ v_2(D_2 - D_1)L_{21(2)}^{-1}\left[a(L_{12(1)} + L_{21(2)}\right) + (v_1 - v_2)D_2]\}\frac{1}{\Delta} \\ &+ \left[\tfrac{v_2}{\Delta}(D_2 - D_1)L_{21(2)}^{-1}\Delta + c_4 - c_8\right]\tfrac{1}{\Delta}(v_{22} + 2v_2 D_2 - v_{12} \\ &- v_2 D_1 - v_1 D_2 + aL_{21(1)} + bL_{12(2)})\tfrac{1}{\Delta}, \end{aligned} \tag{59}$$

resulting in:

$$\begin{aligned} J_{1(22)}^r = {} & \frac{c_4 - c_8}{\Delta}[v_2(D_2 - D_1) + (D_2 - D_1)v_2 + bL_{12(2)} + aL_{21(1)}]\frac{1}{\Delta} \\ & + \tfrac{1}{\Delta}c(L_{21(2)} + L_{12(1)})\tfrac{1}{\Delta} + \tfrac{v_2}{\Delta}(D_2 - D_1)L_{21(2)}^{-1}(D_2 - D_1)v_2\tfrac{1}{\Delta}, \end{aligned} \tag{60}$$

which is in skew-symmetric form. Equations (54), (56), (58) and (60) constitute the matrix representation of the second Hamiltonian operator obtained as:

$$J_1^r = \begin{pmatrix} L_{12(2)}^{-1} & -\{L_{12(2)}^{-1}(D_2 - D_1)v_2 + c_8 - c_4\}\tfrac{1}{\Delta} \\ \tfrac{1}{\Delta}\{v_2(D_2 - D_1)L_{12(2)}^{-1} + c_8 - c_4\} & J_{1(22)}^r \end{pmatrix}, \tag{61}$$

where $J^r_{1(22)}$ is given in (60). Similar to previous parameters, direct reduction from J_1 given in [1] results in the same J^r_1 (61).

The Hamiltonian operators J^r_0 and J^r_1 form a Hamiltonian pair if their linear combination $\Gamma^r = J^r_0 + J^r_1$ is also a Hamiltonian operator. In this case, the linear combination is obliged to satisfy skew symmetry and the Jacobi Identity properties as it is stated by Definition 7.1 in Olver's book [35]. It is easy to see that skew symmetry is satisfied since both J^r_0 and J^r_1 are obviously skew symmetric, i.e, $J^\dagger = -J$ holds for both, where † denotes the adjoint operator. On the other hand, checking the Jacobi Identity condition is a complicated task. However, Theorem 7.8 suggested by Olver in his book simplifies this task. Therefore, we use Olver's method in a similar fashion that is demonstrated in [14] and conclude that Jacobi Identity is satisfied.

According to Magri's theorem [3, 4], an evolutionary system is integrable if it satisfies the following equation:

$$\begin{pmatrix} u^r_t \\ v^r_t \end{pmatrix} = J^r_0 \begin{pmatrix} \delta_u H^r_1 \\ \delta_v H^r_1 \end{pmatrix} = J^r_1 \begin{pmatrix} \delta_u H^r_0 \\ \delta_v H^r_0 \end{pmatrix}. \tag{62}$$

That is, the (2 + 1)-dimensional system forms a bi-Hamiltonian structure if a second Hamiltonian density H^r_0 satisfies (62).

The second Hamiltonian density H_0 of the (3 + 1)-dimensional system is given by (4.1.6) in [2]. Applying the transformations (13), we derive H^r_0 for the (2 + 1)-dimensional system as:

$$H^r_0 = -k\left\{\frac{v^2}{2} + \frac{c_9}{2c_8}\left[2u_1 v + (c_4 - c_8)u_1^2\right]\right\}\Delta, \tag{63}$$

where $k = \frac{c_8}{c_8(c_8 - c_4) + c_9}$. By substituting equations (16), (61) and (63) into the matrix equation (62), we confirm that the equation holds. Hence, we have shown that the (2 + 1)-dimensional system admits a bi-Hamiltonian structure, analogous to the (3 + 1)-dimensional case.

NOETHER'S THEOREM AND INTEGRALS OF MOTION

Using the software package REDUCE 1, point symmetries of the new (2 + 1)-dimensional system (16) are identified as follows:

$$\begin{aligned} &X_1 = \partial_2,\ X_2 = u\partial_u + v\partial_v,\ X_3 = (ab+2c)(-z_1\partial_1 + z_2\partial_2) - a^2 z_2\partial_1 \\ &\qquad + (b^2 + 4c)z_1\partial_2 + [(2a-b)z_1]\partial_t, \\ &X_4 = \partial_1,\ X_5 = z_1\partial_1 + z_2\partial_2 + t\partial_t - v\partial_v,\ X_6 = \partial_t,\ X_7 = z_2\partial_u, \\ &X_8 = t\partial_u + \partial_v \quad X_9 = z_1\partial_u \quad X_{10} = \partial_u. \end{aligned} \tag{64}$$

In the framework of Lie theory, point symmetries act as symmetry generators if they form a Lie algebra. We construct a table illustrating the Lie algebra structure of the point symmetries (64). The intersection of the i^{th} row and the j^{th} column in this table shows the result of the commutator operation $[X_i, X_j]$.

For convenience, the following notation is used in the table:

$$\begin{aligned} &X^1{}_{(7,9)} = (ab + 2c)X_7 + (b^2 + 4c)X_9, \\ &X^2{}_{(7,9)} = aX_7 + (2a - b)X_9,\ X^3{}_{(7,9)} = -a^2X_7 - (ab + 2c)X_9, \\ &X^1{}_{(1,4,6)} = (ab + 2c)X_1 - a^2X_4 + aX_6, \\ &X^2{}_{(1,4,6)} = -(b^2 + 4c)X_1 + (ab + 2c)X_4 + (b - 2a)X_6. \end{aligned} \tag{65}$$

For each symmetry generator X, corresponding symmetry characteristics provide the independent variables that remain untransformed under the symmetry transformation. In [35], symmetry generators are defined in the following general form:

$$X = \xi^i \partial_{x^i} + \eta^\alpha \partial_{u^\alpha}, \tag{66}$$

and the corresponding characteristics are defined in the form:

$$\varphi^\alpha = \eta^\alpha - u^\alpha_i \xi^i. \tag{67}$$

These equations are expressed using the Einstein summation convention. In the case of $(2 + 1)$ $-$dimensional

Table 1. Commutators of point symmetry generators of reduced system

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}
X_1	0	0	$X^1{}_{(1,4,6)}$	0	X_1	0	X_{10}	0	0	0
X_2	0	0	0	0	0	0	$-X_7$	$-X_8$	$-X_9$	$-X_{10}$
X_3	$-X^1{}_{(1,4,6)}$	0	0	$X^2{}_{(1,4,6)}$	0	0	$X^1{}_{(7,9)}$	$X^2{}_{(7,9)}$	$X^3{}_{(7,9)}$	0
X_4	0	0	$-X^2{}_{(1,4,6)}$	0	X_4	0	0	0	X_{10}	0
X_5	$-X_1$	0	0	$-X_4$	0	$-X_6$	X_7	X_8	X_9	0
X_6	0	0	0	0	X_6	0	0	X_{10}	0	0
X_7	$-X_{10}$	X_7	$-X^1{}_{(7,9)}$	0	$-X_7$	0	0	0	0	0
X_8	0	X_8	$-X^2{}_{(7,9)}$	0	$-X_8$	$-X_{10}$	0	0	0	0
X_9	0	X_9	$-X^3{}_{(7,9)}$	$-X_{10}$	$-X_9$	0	0	0	0	0
X_{10}	0	X_{10}	0	0	0	0	0	0	0	0

system, indices i take values: $i = 1,2,3$. Additionally, we have two dependent variables u and v, so indices α take values: $\alpha = 1, 2$. Accordingly, we define:

$$
\begin{gathered}
x^1 = t, \quad x^2 = z_1, \quad x^3 = z_2, \quad u_1^1 = u_t, \; u_2^1 = u_1, \\
u_3^1 = u_2, \; u_1^2 = v_t, \; u_2^2 = v_1, \; u_3^2 = v_2, \; \eta^1 = \eta^u, \; \eta^2 = \eta^v, \\
\varphi^1 = \varphi, \; \varphi^2 = \psi.
\end{gathered} \tag{68}
$$

For every generator X, we obtain two characteristics, namely φ and ψ, which are related to the transformations of u and v respectively. Using (68) in the equations (66), (67) and replacing u_t by v, v_t by q according to (16), we find the characteristics pair (φ_i, ψ_i) of each generator X_i $(i = 1, 2, \ldots, 10)$ as the following:

$$
\begin{gathered}
\varphi_1 = -u_2 \quad \psi_1 = -v_2, \quad \varphi_2 = u, \quad \psi_2 = v, \\
\varphi_3 = -v(2a-b)z_1 + u_1[(ab+2c)z_1 + a^2 z_2] - u_2[(b^2+4c)z_1 + (ab+2c)z_2], \\
\psi_3 = -q(2a-b)z_1 + v_1[(ab+2c)z_1 + a^2 z_2] - v_2[(b^2+4c)z_1 + (ab+2c)z_2], \\
\varphi_4 = -u_1, \; \psi_4 = -v_1, \; \varphi_5 = -vt - u_1 z_1 - u_2 z_2, \; \psi_5 = -v - qt - v_1 z_1 - v_2 z_2, \\
\varphi_6 = -v, \; \psi_6 = -q, \; \varphi_7 = z_2, \; \psi_7 = 0, \; \varphi_8 = t \\
\psi_8 = 1, \; \varphi_9 = z_1, \; \psi_9 = 0 \; \; \varphi_{10} = 1, \; \psi_{10} = 0
\end{gathered} \tag{69}
$$

These symmetry characteristics provide a path to find new integrals of motion conserved by the flow of (16). By substituting the time variable "t" with the group parameter "τ", we can employ the Lie equations provided in (37). Upon substituting these Lie equations into the matrix equation (36), we get:

$$
\begin{pmatrix} \varphi \\ \psi \end{pmatrix} = J_0^r \begin{pmatrix} \delta_u H^r \\ \delta_v H^r \end{pmatrix}. \tag{70}
$$

This represents the Noether theorem in Hamiltonian form, providing the conserved density H^r corresponding to the given symmetry. Remarking that the first Hamiltonian operator (33) is the inverse of the symplectic operator, we arrange the matrix equation (70) into the inverse Noether theorem, taking the following form:

$$
\begin{pmatrix} \delta_u H^r \\ \delta_v H^r \end{pmatrix} = K^r \begin{pmatrix} \varphi \\ \psi \end{pmatrix}. \tag{71}
$$

We write this matrix equation for each characteristics pair (φ_i, ψ_i) that we obtained in (69) as:

$$
\begin{pmatrix} \delta_u H_i^r \\ \delta_v H_i^r \end{pmatrix} = K^r \begin{pmatrix} \varphi_i \\ \psi_i \end{pmatrix}. \tag{72}
$$

Solving this equation, we determine the conserved densities, i.e., first integrals H_i^r corresponding to all variational point symmetry generators X_i with characteristics (φ_i, ψ_i) as follows:

$$
\begin{gathered}
H_1^r = v u_2(u_{12} - u_{22}) - \frac{au}{3}(u_{11}u_{22} - u_{12}^2), \\
H_4^r = v u_1(u_{12} - u_{22}) + \tfrac{1}{2} b u_{22} u_1^2, \\
H_6^r = \frac{v^2}{2}(u_{12} - u_{22}) + \frac{c}{3} u (u_{11}u_{22} - u_{12}^2), \\
H_7^r = z_2 v(u_{22} - u_{12}) + \frac{u}{2}(a u_{11} - b u_{12}), \\
H_8^r = \left(vt - \frac{u}{2}\right)(u_{22} - u_{12}), \\
H_9^r = z_1 v(u_{22} - u_{12}) + \frac{u}{2}(b u_{22} - a u_{12}).
\end{gathered} \tag{73}
$$

We observe that the first integrals H_2^r, H_3^r, H_5^r fail to exist. Therefore, the corresponding generators X_2, X_3, X_5 do not count as variational symmetries. We check the time derivative of every density given in (73) along the flow (16) and obtain all the variational symmetries in total divergence form respectively as follows:

$$
\begin{aligned}
D_t H_1^r = {} & D_1\Big[-\frac{1}{2}\Big(u_{22}v + b u_2 u_{22} + \frac{a}{3} u_2 u_{22} - \frac{a}{3} u_1 u_{22} + a u_2 u_{12}\Big) v \\
& + \frac{a}{3}(u_{12}v_2 - u_{22}v_1)u - c u_2 u_1 u_{22}\Big] + D_2\Big[\Big(u_2 v_1 - u_2 v_2 + \tfrac{1}{2} u_{12} v \\
& + b u_2 u_{12} - \tfrac{a}{3} u_1 u_{12} + \tfrac{a}{3} u_2 u_{11} - a u_{11} u_2\Big) v + \tfrac{a}{3}(u_{12}v_1 - u_{11}v_2)u \\
& + c u_2 u_1 u_{12}\Big] \\
D_t H_4^r = {} & D_1\Big[\Big(-\tfrac{1}{2} u_{12} v + a u_1 u_{12}\Big) v - \tfrac{c}{2} u_1^2 u_{22}\Big] + D_2\Big[\Big(v v_1 - v v_2 - a u_{11} v \\
& + \tfrac{c}{2} u_{12} u_1 + \tfrac{b}{2} v_2 u_1\Big) u_1 + \tfrac{1}{2} u_{12} v_2\Big], \\
D_t H_6^r = {} & D_1\Big[\frac{c}{3}(v_1 u_{22} - v_2 u_{12})u + \frac{c}{3}(u_{12}u_2 - u_1 u_{22})v + \frac{1}{2}(v_2 + a u_{12} - b u_{22})v^2\Big] \\
& + D_2\Big[\tfrac{c}{3}(u_{11}v_2 - u_{12}v_1)u + \tfrac{c}{3}(u_1 u_{12} - u_2 u_{11})v + \tfrac{1}{2}(b u_{12} - a u_{11} - v_2)v^2\Big], \\
D_t H_7^r = {} & D_1\Big[-\Big(v_2 z_2 + a z_2 u_{12} + \frac{a}{2} u_1 - b z_2 u_{22}\Big) v + \Big(\frac{a}{2} v_1 - \frac{b}{2} v_2\Big) u + c z_2 u_1 u_{22}\Big] \\
& + D_2\Big[c\Big(\tfrac{1}{2} u_1 - z_2 u_{12}\Big) u_1 + \Big(v_2 z_2 - \tfrac{1}{2} v + a z_2 u_{11} - b z_2 u_{12} + \tfrac{b}{2} u_1\Big) v\Big] \\
D_t H_8^r = {} & D_1\Big[(-a v u_{12} + b v u_{22} + c u_1 u_{22} - v v_2)t + \frac{1}{2}(u v_2 - v u_2)\Big] \\
& + D_2\Big[v v_2 + a v u_{11} - b v u_{12} - c u_1 u_{12} + \tfrac{1}{2}(v u_2 - u v_2)\Big], \\
D_t H_9^r = {} & D_1\Big[-\frac{a}{2} u v_2 + (b z_1 u_{22} - a z_1 u_{12} - z_1 v_2)v + c u_1 u_{22}\Big] \\
& + D_2\Big[(v v_2 + a v u_{11} - b v u_{12}) z_1 + \Big(\tfrac{a}{2} v - c u_{12}\Big) u_1 + \tfrac{1}{2}(v - b u_2)v + \tfrac{b}{2} v_2 u\Big], \\
D_t H_{10}^r = {} & D_1[(-a u_{12} + b u_{22})v + c u_1 u_{22}] + D_2[(v_2 - v_1 + a u_{11} - b u_{12})v - c u_1 u_{12}].
\end{aligned} \tag{74}
$$

We have successfully expressed the first integrals (73) in total divergence form (74). Thus, we can conclude that these integrals are indeed the constants of motion for the flow governed by the system (16). In essence, total divergences provide an independent check that the corresponding functionals H^r are indeed integrals of motion subject to suitable boundary conditions.

CONCLUSION

We studied a symmetry reduction of the recently discovered (3 + 1)-dimensional equation of the Monge-Ampere type. Our goal was to explore if it is possible to obtain a new (2 + 1)-dimensional bi-Hamiltonian system by applying symmetry reduction to a particular case of the (3 + 1)-dimensional equation. We used point symmetry generators of the system and proceeded by choosing a special combination of the symmetries. We determined the transformation of total derivatives under this particular symmetry, then performed the reduction accordingly. We obtained all the parameters L^r, H_1^r, K^r, J_0^r, R^r, J_1^r and H_0^r of the reduced (2 + 1)-dimensional system. Two component representation made it possible to obtain the Hamiltonian operator through Dirac constraint analysis. Being able to find the second Hamiltonian function H_0^r, we state that the reduced system maintains the bi-Hamiltonian structure of the original system. We confirmed that all parameters and operators could also be obtained by direct reduction from the original system, e.g., $L, H_1, K, J_0, R, J_1, H_0$ with the same symmetry choice. We identified the symmetry generators of the reduced (2 + 1)-dimensional system, along with their corresponding characteristic pairs (φ, ψ). By the Noether theorem, we revealed seven new integrals of motion that define the conserved densities of the system. We also proved that the time derivatives of all variational symmetries are total divergences.

Thus, we presented a new method for obtaining (2 + 1)-dimensional bi-Hamiltonian systems starting from (3 + 1)-dimensional bi-Hamiltonian systems. We have illustrated the involved procedure by an explicit example, producing a new bi-Hamiltonian system. We expect the suggested procedure to be a useful supplement to other techniques for generating (2 + 1)-dimensional bi-Hamiltonian systems.

AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

REFERENCES

[1] Sheftel MB, Yazıcı D. Lax pairs, recursion operators and bi-Hamiltonian representations of (3+1)-dimensional Hirota type equations. J Geometry Phys 2019;136:207–227. [CrossRef]

[2] Sheftel MB, Yazıcı D. Symmetries, integrals and hierarchies of (3+1)-dimensional bi-Hamiltonian Systems of Monge-Ampère type. J Geometry Phys 2019;146:103513. [CrossRef]

[3] Magri F. A simple model of the integrable Hamiltonian equation. J Math Phys 1978;19:1156–1162. [CrossRef]

[4] Magri F. A geometrical approach to the nonlinear solvable equations, In: Boiti, M., Pempinelli, F., Soliani, G (Eds.). Nonlinear Evolution Equations and Dynamical Systems. Lecture Notes in Physics. Berlin, Heidelberg: Springer; 1980. p. 233–263. [CrossRef]

[5] Doubrov B, Ferapontov EV. On the integrability of symplectic Monge-Ampère equations, J Geometry Phys 2010;60:1604–1616. [CrossRef]

[6] Ferapontov EV, Kruglikov B, Novikov V. Integrability of dispersionless hirota-type equations and the symplectic monge-ampère property. Int Math Res Not 2021:14220–14251. [CrossRef]

[7] Plebañski JF. Some solutions of complex Einstein equations. J Math Phys 1975;16:2395–2402. [CrossRef]

[8] Yazıcı D, Sheftel MB. Symmetry reductions of second heavenly equation and 2+1-dimensional hamiltonian ıntegrable systems. J Nonlinear Math Phys 2008;15:417–425. [CrossRef]

[9] Yazıcı D. Symmetry reduction of the first heavenly equation and (2+1)-dimensional bi-Hamiltonian system, Turk J Phys 2018;42:183–190. [CrossRef]

[10] Yazıcı D. (2+1)-dimensional bi-Hamiltonian system obtained from symmetry reduction of (3+1)-dimensional Hirota type equation. AIP Conf Proceed 2019;2178:030049. [CrossRef]

[11] Neyzi F, Nutku Y, Sheftel MB. Multi-Hamiltonian structure of Plebanski's second heavenly equation. J Phys A Math Gen 2005;38:8473. [CrossRef]

[12] Nutku Y, Sheftel MB, Kalaycı J, Yazıcı D. Self dual gravity is completely integrable. J Phys A Math Theor 2008;41:395206. [CrossRef]

[13] Sheftel MB, Yazıcı D. Bi-Hamiltonian representation, symmetries and integrals of mixed heavenly and Husain systems. J Nonlinear Math Phys 2010;17:453–484. [CrossRef]

[14] Sheftel MB, Yazıcı D, Malykh AA. Recursion operators and bi-Hamiltonian structure of the general heavenly equation. J Geometry Phys 2017;116:124–139. [CrossRef]

[15] Sheftel MB, Yazıcı D. Recursion operators and tri-hamiltonian structure of the first heavenly equation of pleban'ski. Symmetry Integr Geom Methods Appl 2016;12:091.

[16] Sheftel MB, Yazıcı D. Evolutionary hirota type (2+1)-dimensional equations: lax pairs, recursion operators and bi-hamiltonian structures. Symmetry Integr Geom Methods Appl 2018;14:017. [CrossRef]

[17] Dirac PAM. Lectures on Quantum Mechanics. New York: Belfer Graduate School of Science, Yeshiva University; 1964.

[18] Ai-Deiakeh R, Abu Arqub O, Al-Smadi M, Momani S. Lie symmetry analysis, explicit solutions, and conservation laws of the time-fractional Fisher equation in two-dimensional space. J Ocean Eng Sci 2022;7:345–352. [CrossRef]
[19] Aal M, Abu Arqub O. Lie analysis and laws of conservation for the two-dimensional model of Newell-Whitehead-Segel regarding the Riemann operator fractional scheme in a time-independent variable. Arab J Basic Appl Sci 2023;30:55–67. [CrossRef]
[20] Abu Arqub O, Hayat T, Alhodaly M. Analysis of lie symmetry, explicit series solutions, and conservation laws for the nonlinear time-fractional phi-four equation in two-dimensional space. Int J Appl Comput Math 2022;8:145. [CrossRef]
[21] Li BK, Ma YL. N-order rogue waves and their novel colliding dynamics for a transient stimulated Raman scattering system arising from nonlinear optics. Nonlinear Dyn 2020;101:2449–2461. [CrossRef]
[22] Chen SJ, Ma WX, Lü X. Bäcklund transformation, exact solutions and interaction behaviour of the (3+1)-dimensional Hirota-Satsuma-Ito-like equation, Commun Nonlinear Sci Numer Simul 2020;83:105135. [CrossRef]
[23] Akbar MA, Akinyemi L, Yao SW, Jhangeer A, Rezazadeh H, Khater MMA, et al. Soliton solutions to the Boussinesq equation through sine-Gordon method and Kudryashov method. Results Phys 2021;25:104228. [CrossRef]
[24] Akinyemi L, Rezazadeh H, Yao SW, Akbar MA, Khater MMA, Jhangeer A, et al. Nonlinear dispersion in parabolic law medium and its optical solitons. Results Phys 2021;26:104411. [CrossRef]
[25] Zabusky NJ, Kruskal MD. Interaction of “solitons” in a collisionless plasma and the recurrence of ınitial States. Phys Rev Lett 1965;15:240. [CrossRef]
[26] Ahmad H, Khan TA, Ahmad I, Stanimirovic PS, Chu YM. A new analyzing technique for nonlinear time fractional Cauchy reaction-diffusion model equations. Results Phys 2020;19:103462. [CrossRef]
[27] Ahmad H, Seadawy AR, Khan TA. Study on numerical solution of dispersive water wave phenomena by using a reliable modification of variational iteration algorithm. Math Comput Simul 2020;177:13–23. [CrossRef]
[28] Nutku Y, Neyzi F. Canonical structures for dispersive waves in shallow water. J Math Phys 1987;28:1499–1504. [CrossRef]
[29] Kalaycı J, Nutku Y. Bi-Hamiltonian structure of a WDVV equation in 2-d topological field theoryç Phys Lett A 1997;227:177–182. [CrossRef]
[30] Yazıcı D, Sert H. Symmetry reduction of asymmetric heavenly equation and 2+1-dimensional bi-hamiltonian system. J Geom Symmetry Phys 2014;34:87–96. [CrossRef]
[31] Camcı U, Can Z, Nutku Y, Sucu Y, Yazıcı D. Dirac constraint analysis and symplectic structure of anti-self-dual Yang-Mills equations. Pramana 2006;67:1043–1053. [CrossRef]
[32] Nutku Y. Hamiltonian structure of real Monge-Ampère equations. J Phys A Math Gener 1996;29:3257. [CrossRef]
[33] Yazıcı D. A new 2+1-dimensional hamiltonian integrable system, Sigma J Eng Nat Sci 2009;27:118–128.
[34] Abu Arqub O, Shawagfeh N. Solving optimal control problems of Fredholm constraint optimality via the reproducing kernel Hilbert space method with error estimates and convergence analysis. Math Methods Appl Sci 2021;44:7915–7932. [CrossRef]
[35] Olver PJ. Applications of Lie Groups to Differential Equations. New York: Springer-Verlag; 2009.
[36] Nutku Y. By Magri’s theorem, self-dual gravity is completely ıntegrable. Symmetry Integr Geom Methods Appl 2007;3:034. [CrossRef]
[37] Nutku Y. Lagrangian Approach to Integrable Systems Yields New Symplectic Structures for KDV, In: Aratyn H, Sorin AS (Eds.). Integrable Hierarchies and Modern Physical Theories. NATO Science Series: Springer, Dordrecht; 2001. p. 203–213. [CrossRef]