Sigma J Eng Nat Sci, Vol. 43, No. 1, pp. 1–14, February, 2025

Sigma Journal of Engineering and Natural Sciences
Web page info: https://sigma.yildiz.edu.tr
DOI: 10.14744/sigma.2025.00002

Research Article

A complete study on and-product of soft sets

Aslıhan SEZGİN[1,*], Akın Osman ATAGÜN[2], Naim ÇAĞMAN[3]

[1]*Department of Mathematics and Science Education, Faculty of Education, Amasya University, Amasya, 20815, Türkiye*
[2]*Department of Mathematics, Faculty of Arts and Science, Kırşehir Ahi Evran University, Kırşehir, 40100, Türkiye*
[3]*Department of Mathematics, Faculty of Arts and Science, Tokat Gaziosmanpaşa University, Tokat, 60000, Türkiye*

ARTICLE INFO

Article history
Received: 15 August 2023
Revised: 08 October 2023
Accepted: 22 October 2023

Keywords:
AND-Product; Hemiring; Soft Equal Relations; Soft Sets

ABSTRACT

Soft set theory is a general mathematical framework for dealing with uncertainty. In this regard, soft set operations can be regarded as crucial concepts in soft set theory, since they offer new perspectives for dealing with issues containing parametric information. In this paper, we give a theoretical study on AND-product (∧-product), which is an essential concept in decision making problems, by investigating its whole algebraic properties in detail regarding soft F-subsets and soft M- equality, the strictest type of soft equality. Moreover, in order to complete some incomplete results concerning AND-product in the literature, we compare our properties by the formerly obtained properties regarding soft L-equality and soft J-equality. Furthermore, we handle the whole relations between AND-product and OR-product, the other keystone in decision making. Besides, by establishing some new results on distributive properties of AND-product over restricted, extended, and soft binary piecewise soft set operations, we prove that the set of all the soft sets over U together with restricted/extended union and AND-product is a commutative hemiring with identity as the set of all the soft sets over U together with restricted/extended symmetric difference and AND-product forms a commutative hemiring with identity in the sense of soft L-equality. As analyzing the algebraic structure of soft sets from the standpoint of operations gives profound insight into the potential uses of soft sets in classical and nonclassical logic and since theoretical foundations of soft computing approaches are derived from purely mathematical principles, this research will pave the way for a wide range of applications, including new decision-making approaches and innovative cryptography techniques based on soft sets.

Cite this article as: Sezgin A, Atagün AO, Çağman N. A complete study on and-product of soft sets. Sigma J Eng Nat Sci 2025;43(1):1–14.

INTRODUCTION

Mathematical models have been widely used in practical problems related to engineering and computer science, economics, social, natural and medical sceinces, etc. Fuzzy set theory, initiated by Zadeh [1], has been developed in mathematics as an important tool for solving mathematical problems of uncertainty and ambiguity. However, there are some limitations and inadequacies related to parameterization in fuzzy set theory.

***Corresponding author.**
*E-mail address: aslihan.sezgin@amasya.edu.tr

This paper was recommended for publication in revised form by Editor-in-Chief Ahmet Selim Dalkilic

Published by Yıldız Technical University Press, İstanbul, Turkey

Therefore, Molodtsov [2] in 1999 initiated research on "Soft Set Theory" to overcome defects and inadequacies regarding fuzzy set theory. Since then, soft set theory has been used in problem solving for different areas, especially in decision making, analysis, forecasting, information science, mathematics, etc. Since Maji et al. [3] described the application of soft set theory to a decision making problem, many researchers have developed new decision making methods with the help of soft sets [4-10].

Çağman and Enginoğlu [11] initiated a salient soft set based decision making, called uni-int decision making and it achieved the optimal decision from the experiment. Further, Çağman and Enginoğlu [12] defined soft matrices and they established min-max decision making methods for OR, AND, AND-NOT and OR-NOT product of the soft matrices and applied them to the real life problems that contain uncertainties. Since then, soft set theory has been widely and successfully applied to decision making problems [13-24].

First contributions to soft set operations were made in [25,26]. A thorough theoretical analysis of the soft set theory was conducted by Maji et al. [25]. In particular, they introduced the concepts of soft subset, soft intersection and soft union while discussing soft sets properties. These operations allowed for the creation of new soft sets from existing soft set datasets, but certain fundamental characteristics that are not universal was highlighted and refined in [27,28]. Moreover, Ali et al. [28] defined some restricted operations on soft sets, such as restricted intersection, restricted union, and restricted difference and improved the notion of the complement of a soft set. After then, in [29-45], the authors pointed out several gaps as regards basic operations in soft set theory in the existing literature and introduced new and improved operations and discussed the algebraic structure of the set of all soft sets defined on a fixed universe and fixed set of parameters. In recent years, studies on soft sets have been progressing rapidly. A new type of difference operation of soft sets was proposed by Eren and Çalışıcı [46]. Sezgin et al. [47] introduced the concept of "extended difference" of soft sets and Stojanovic [48] defined the notion of "extended symmetric difference" of soft sets and their basic properties were studied in [47,48], respectively. Also, Yavuz [49] introduced a new type of intersection, union and symmetric difference of soft sets, and studied their basic properties.

In the context of soft set theory, soft subsets together with soft equal relations are essential concepts. Maji et al. [25] were the pioneers in using a very strict definition of soft subsets. Without mathematical validation, they presented some results on the soft distributive laws of such operations (see Proposition 2.6 in [25]). Soft distributive laws of Maji are not applicable to the soft equal relation mentioned in Remark 2.8 in [28] as pointed out by Ali et al. [29] and refuted by others. The concept of soft subsets were expanded by Pei and Miao [26] and Feng et al. [29] which can be considered as a generalization of Majis' previous methods. Congruence relations on soft sets were introduced by Qin and Hong [50], who also discussed two new types of soft equal relations. During their research on fuzzy soft sets with interval values, Jun and Yang [51] examined a broadening of soft subsets and attempted to modify Maji's soft distributive laws using generalized soft equal relations (which we call J-soft equal relation for the sake of designation throughout the study). They presented a recent result that is known as generalized soft distributive laws of soft product operations. In a concise research note, Liu, Feng, and Jun [52] were motivated by the new ideas of Jun and Yang to define soft L-subsets and soft L-equal relations. A significant finding is that distributive laws do not apply to all forms of soft equal relations existing in literature.

Consequently, Feng et al. [53] also dealt with soft subsets and the operation of soft products defined in [24] and continued the research started in [52]. Compared to notes [52], Feng et al. [53] mainly focused on different types of soft subsets and the algebraic properties of soft product operations. In addition to the distributional laws that have been widely studied by many researchers, they also considered commutative laws, association rules, normal laws, and other fundamental properties. They also provided theoretical research related to the operation of soft products, namely the AND-product and the OR-product by soft L-subset and some related concepts. They consummated some incomplete results concerning soft product operations existing in the literature and studied in detail the algebraic properties of the soft product operations for the J-equality and L-equality. It was shown that soft L-equal relations are congruent relations on free soft algebras and corresponding quotient structures that form commutative semigroups. (For more about soft equal relations, see [54-58]). Also, Singh and Onyeozili [39] established some results related to left distributive properties of AND-product and OR-product over the restricted intersection, restricted union, restricted difference, extended intersection, and extended union. But they do not investigate the right distributive laws of AND-product and OR-product over the aforementioned operations and they ignored the case that the intersection of the parameter sets can be an empty set.

In 1934, the concept of semiring was first introduced by Vandiver [59]. Semirings have been the subject of several hypotheses and observations from various academics, including [60, 61].

Research on semirings with additive inverse has also been done by certain scholars [62-65]. Especially in terms of applications, semirings have been the subject of much research recently (see [66]). Despite their importance in geometry, semirings are also important in pure mathematics and play a major role in many applications of practical mathematics and the information sciences [67–75]. A hemiring is a unique semiring with a zero and commutative addition. Hemirings are also important in the field of theoretical computer science. Hemirings arises naturally in a number of applications to automata, computer sciences, and formal language theory [74–75].

This paper is a theoretical study on AND-product of soft sets, which has been the cornerstone and has been used by decision makers for many years. In the literature, AND-product and its properties have been investigated by different authors as regards different kinds of soft equalities such as soft L-equality and soft J-equality. However, in this paper, we investigate the whole algebraic properties of AND-product such as commutative laws, associative laws, idempotent laws, and other all basic properties in detail as regards soft F-subsets and soft M-equality and compare them with the formerly obtained properties as regards soft L-equality and soft J-equality. Also, we give the relation between AND-product and OR-product. Moreover, by examining ֆ the distributive properties of AND- product over restricted, extended, and soft binary piecewise operations, we obtain that the set of all the soft sets over U together with restricted/extended union and AND-product form a commutative hemiring with identity as restricted/extended symmetric difference and AND-product in the sense of soft L-equality. We, by this study, complete the results concerning the AND-product in the literature totally.

The following is the format of this document. In Section 2, we review some basic ideas related to semirings and soft set theory. In Section 3, we completely handle the AND-product and its whole algebraic properties as regards soft F-subsets and soft M-equality and compare our results with the formerly obtained. Section 4 is focuses on the distribution rules of AND-product over other types of soft set operations in order to reveal the algebraic structures of soft sets with AND-product and other soft set operations. A short conclusion is stated in the Conclusion Section.

PRELIMINARIES

Definition 2.1. Let U be the universal set, E be the parameter, P(U) be the power set of U and ֆ⊆E . A pair ŧ(ŧ,ֆ) is called a soft set over U where ŧ is a set-valued function such that ŧ:ֆ → P(U). [1]

It should be noted that Çağman and Enginoğlu [11] modified Molodstov's notion of soft sets; nonetheless, we adhere to the original definition in this study and apply it. The sets of all the soft sets defined over U are referred as $S_E(U)$ throughout this work. The collection of all soft sets over U with the fixed parameter set A is denoted by $S_A(U)$, where A is a fixed subset of E. That is, while in the set $S_A(U)$, there are only soft sets whose parameter sets are A; in the set $S_E(U)$, there are soft sets whose parameter sets may be any set. Clearly $S_A(U)$ is a subset of $S_E(U)$. Henceforth, for convenience's sake, soft set(s) shall be identified by SS(s) and parameter set(s) by PS(s).

Definition 2.2. Let (ŧ,ֆ) be a SS over U. (ŧ,ֆ) is called a relative null SS (with respect to the PS ֆ), denoted by $\emptyset_{ֆ}$, if ŧ(t)=∅ for all t∈ֆ and (ŧ,ֆ) is called a relative whole SS (with respect to the PS ֆ), denoted by $U_{ֆ}$ if ŧ(t)=U for all t∈ֆ. The relative whole SS U_E with respect to the universe set of parameters E is called the absolute SS over U [28].

The unique SS over U with an empty PS is denoted by $\emptyset_\emptyset$ and is referred to as the empty SS over U. Keep in mind that the SSs over U by $\emptyset_\emptyset$ and by $\emptyset_{ֆ}$ are distinct [31]. Unless otherwise indicated, we always take into account SSs with non-empty PSs in the universe U in the following.

Maji et al. [25] was the first to define the notion of soft subset, which we here call soft M-subset for avoiding confusion, in a very strict manner as follows:

Definition 2.3. Let (O,W) and (L,B) be two SSs over U. (O,W) is called a soft M-subset of (L,B), denoted by $(O,W)\tilde{\subseteq}_M(L,B)$, if W⊆B and O(τ)=L(τ) for all τ∈W. Two SSs (O,W) and (L,B) are said to be soft M-equal, denoted by $(O,W)=_M(L,B)$, if $(O,W)\tilde{\subseteq}_M(L,B)$ and $(L,B)\tilde{\subseteq}_M(O,W)$ [25].

Definition 2.4. Let (O,W) and (L,B) be two SSs over U. (O,W) is called a soft F-subset of (L,B), denoted by $(O,W)\tilde{\subseteq}_F(L,B)$, if W⊆B and O(τ)⊆L(τ) for all τ∈W. Two SSs (O,W) and (L, B) are said to be soft F-equal, denoted $(O,W)=_F(L,B)$, if $(O,W)\tilde{\subseteq}_F(L,B)$ and $(L,B)\tilde{\subseteq}_F(O,W)$ [26].

Note that the definitions of soft F-subset and soft F-equal were actually first given by Pei and Miao in [26], however in some of the SS papers, it was stated that these definitions were first given by Feng et al. in [29], and for this reason, this relation is designated by the letter "F".

In [52], it was shown that soft equal relations $=_M$ and $=_F$ coincide with each other. That is; $(O,W)=_M(L,B)$ ⇔$(O,W)=_F(L,B)$. If two SSs on U satisfy such soft equivalence, they are actually identical, because they have the same set of parameter and approximate function [52].

Jun and Yang [51], by relaxing the conditions on PSs, generalized the concepts of F-soft subsets and soft F-equal relations. Although in [51], it is called generalized soft subset and generalized soft equal relation, hereinafter, we call soft J-subsets and soft J-equal relations, the first letter of Jun.

Definition 2.5. Let (O,W) and (L,B) be two SSs over U. (O,W) is called a soft J-subset of (L,B), denoted by $(O,W)\tilde{\subseteq}_J(L,B)$, if for all τ∈W, there exist ω∈B such that O(τ)⊆L(ω). Two SSs (O,W) and (L,B) are said to be soft J-equal, denoted $(O,W)=_J(L,B)$, if $(O,W)\tilde{\subseteq}_J(L,B)$ and $(L,B)\tilde{\subseteq}_J(O,W)$ [51].

In [52] and [53], it was shown that $(O,W)\tilde{\subseteq}_M(L, B)$⇒$(O,W)\tilde{\subseteq}_F(L, B)$⇒$(O,W)\tilde{\subseteq}_J(L, B)$; but the converse may not be true.

Motivated by the notions of soft J-subset [51] and ontology-based soft subsets [30], Liu, Feng and Jun [52] also introduced the following new kind of soft subsets (henceforth referred to as soft L- subsets and soft L-equality) that generalize both soft M-subsets and ontology-based soft subsets:

Definition 2.6. Let (O,W) and (L,B) be two SSs over U. (O,W) is called a soft L-subset of (L,B), denoted by $(O,W)\tilde{\subseteq}_L(L,B)$, if for all τ∈W, there exist ω∈B such that O(τ)=L(ω). Two SSs (O,W) and (L,B) are said to be soft

L-equal, denoted $(O,W)=_L(L,B)$, if $(O,W)\tilde{\subseteq}_L(L,B)$ and $(L,B)\tilde{\subseteq}_L(O,W)$ [52].

In [52], it was proved that, as regard soft subset relation, we have: $(O,W)\tilde{\subseteq}_M(L, B)\Rightarrow(O,W)\tilde{\subseteq}_L(L, B)\Rightarrow(O,W)\tilde{\subseteq}_J(L,B)$. And also as regards soft relations, we have: $(O,W)=_M(L,B)\Rightarrow(O,W)=_L(L,B)\Rightarrow(O,W)=_J(L, B)$. But the converses may be true. Here recall that $(O,W)=_M(L,B)$ if and only if $(O,W)=_F(L,B)$.

Thus, we can conclude that soft J-equality =J is the weakest soft equal relation, while soft M-equality (hence soft F-equality) is the strictest sense. Soft L-equal relation =L is a concept midway between them [52].

Example 2.7. Let $E=\{e_1,e_2,e_3,e_4,e_5,e_6\}$ be the PS, $A=\{e_2,e_5\}$ and $B=\{e_2,e_5,e_6\}$ be the subsets of E and $U=\{h_1,h_2,h_3,h_4,h_5\}$ be the initial universe set. Let $(\Gamma,A)=\{(e_2, \{h_2,h_4\}), (e_5, \{h_3,h_4,h_6\})\}$, $(6,B)=\{(e_2, \{h_2,h_4\}), (e_5, \{h_3,h_4\}), (e_6, \{h_2,h_3,h_4,h_6\})\}$. $(W,B)=\{(e_2, \{h_3,h_4,h_6\}), (e_5, \{h_2,h_4\}), (e_6, \{h_2,h_3,h_4,h_6\})\}$.

Since $\Gamma(e_2)\subseteq 6(e_2)$ (and also $\Gamma(e_2)\subseteq 6(e_6)$) and $\Gamma(e_5)\subseteq 6(e_6)$, it is obvious that $(\Gamma,A)\tilde{\subseteq}_J(6,B)$. However, since $\Gamma(e_5)\neq 6(e_2)$, $\Gamma(e_5)\neq 6(e_5)$, and $\Gamma(e_5)\neq 6(e_6)$, we can deduce that (Γ,A) is not a soft L-subset of $(6,B)$. Moreover, as $\Gamma(e_5)\neq 6(e_5)$, (Γ,A) is not a soft M-subset of $(6,B)$.

Now, since, $\Gamma(e_2)=W(e_5)$ and $\Gamma(e_5)=W(e_2)$, it is obvious that $(\Gamma,A)\tilde{\subseteq}_L(W,B)$. However, as $\Gamma(e_2)\neq W(e_2)$, $\Gamma(e_5)\neq W(e_5)$, (Γ,A) is not again a soft M-subset of (W,B).

Example 2.8. Let $E=\{e_1,e_2,e_3,e_4,e_5\}$ be the PS, $A=\{e_2,e_5\}$ and $B=\{e_2,e_5,e_6\}$ be the subsets of E and $U=\{h_1,h_2,h_3,h_4,h_5\}$ be the initial universe set. Let $(\Gamma,A)=\{(e_2, \{h_2,h_4\}), (e_5, \{h_2,h_3,h_4,h_6\})\}$, $(6,B)=\{(e_2, \{h_2,h_3,h_4\}), (e_5, \{h_2,h_3,h_4,h_6\}), (e_6,\{h_2\})\}$.

Since $\Gamma(e_2)\neq 6(e_2)$, $\Gamma(e_2)\neq 6(e_5)$ and $\Gamma(e_2)\neq 6(e_6)$, it is obvious that $(\Gamma,A)\neq_L(6,B)$. However, since $\Gamma(e_2)\subseteq 6(e_2)$ (moreover $\Gamma(e_2)\subseteq 6(e_5)$) and $\Gamma(e_5)\subseteq 6(e_5)$, we can deduce that $(\Gamma,A)\tilde{\subseteq}_J(6,B)$.

Moreover, since $6(e_2)\subseteq\Gamma(e_5)$ and $6(e_5)\subseteq\Gamma(e_5)$, and $6(e_6)\subseteq\Gamma(e_2)$, we can deduce that $(6, B)\tilde{\subseteq}_J (\Gamma,A)$. Therefore, $(\Gamma,A)=_J(6, B)$. As $\Gamma(e_2)\neq 6(e_2)$ and $\Gamma(e_5)\neq 6(e_5)$, it is obvious that (Γ,A) is not a soft M- subset of $(6,B)$.

For more on soft F-equality, soft M-equality, soft J-equality, soft L-equality, and some other existing definitions of soft subsets and soft equal relations in the literature, we refer to [50-58].

Definition 2.9. Let (ŧ,ȹ) be a SS over U. The relative complement of a SS (ŧ,ȹ), denoted by $(ŧ,ȹ)^r$, is defined by $(ŧ,ȹ)^r = (ŧ^r,ȹ)$, where $ŧ^r$: ȹ → P(U) is a mapping given by $(ŧ,ȹ)^r = U\setminus ŧ(t)$ for all t ∈ ȹ [28]. From now on, $U\setminus ŧ(t)=[ŧ(t)]'$ will be designated by ŧ'(t) for the sake of designation.

Definition 2.10. Let (O,W) and (L,B) be two SSs over U. The AND-product (∧-product) of the SSs (O,W) and (L,B) is a SS defined by $(O,W)\wedge(L,B)=(H,WxB)$, where $H(x,y)=O(x)\cap L(y)$ for all $(x, y) \in WxB$ [25].

Definition 2.11 Let (O,W) and (L,B) be two SSs over U. The OR-product (∨-product) of the SSs (O,W) and (L,B) is a SS defined by $(O,W)\vee(L,B)=(H,WxB)$, where $H(x,y)=O(x)\cup L(y)$ for all $(x,y)\in WxB$ [25].

Here note that we prefer to use AND-product instead of ∧-product and OR-product instead of ∨-product throughout the paper.

Example 2.12. Let $E=\{e_1,e_2,e_3,e_4\}$ be the PS, $A=\{e_1,e_3\}$ and $B=\{e_2,e_4\}$ be the subsets of E and $U=\{h_1,h_2,h_3,h_4,h_5\}$ be the initial universe set. Let $(\Gamma,A)=\{(e_1, \{h_2\}), (e_3, \{h_2,h_5\})\}$, $(6,B)=\{(e_2, \{h_2,h_4\}), (e_4, \{h_1,h_5\})\}$. Since, $AxB=\{(e_1,e_2), (e_1,e_4), (e_3,e_2), (e_3,e_4)\}$, then $(\Gamma,A)\wedge(6,B)=\{((e_1,e_2), \{h_2\}), ((e_1,e_4),\emptyset), ((e_3,e_2), \{h_2\}), ((e_3,e_4), \{h_5\})\}$, $(\Gamma,A)\vee(6,B)=\{((e_1,e_2), \{h_2,h_4\}), ((e_1,e_4), \{h_1,h_2,h_5\}), ((e_3,e_2), \{h_2,h_4,h_5\}), ((e_3,e_4), \{h_1,h_2,h_5\})\}$.

Let "∇" be used to represent the set operations such as ∩, ∪,\, Δ. Then, restricted SS operations, extended SS operations, and soft binary piecewise operations are defined as follows:

Definition 2.13. Let (ŧ,ȹ) and (L,B) be SSs over U. The restricted ∇ operation of (ŧ,ȹ) and (L,B) is the SS (Y,S), denoted by, $(ŧ,ȹ)\nabla_R(L,B)=(Y,S)$, where S=ȹ∩B≠∅ and ∀τ∈S, Y(τ)=ŧ(τ)∇L(τ) [28,32]. Here note that if ȹ∩B=∅, then $(ŧ,ȹ)\nabla_R(L,B)=\emptyset_\emptyset$ [31].

Definition 2.14. Let (ŧ,ȹ) and (L,B) be SSs over U. The extended ∇ operation of (ŧ,ȹ) and (L, B) is the SS (Y,S), denoted by, $(ŧ,ȹ)\nabla_\varepsilon(L,B)=(Y,S)$, where S=ȹ∪B and ∀τ∈S,

$$Y(\tau)=\begin{cases} ŧ(\tau), & \tau \in ȹ\setminus B,\\ L(\tau), & \tau \in B\setminus ȹ,\\ ŧ(\tau)\,\nabla L(\tau), & \tau \in ȹ \cap B.\end{cases}$$

[25,28,47,48].

Definition 2.15. Let (ŧ,ȹ) and (L,B) be SSs over U. The soft binary piecewise ∇ operation of (ŧ,ȹ) and (L,B) is the SS (Y,P), denoted by, $(ŧ,ȹ)\nabla(L,B)=(Y, P)$, where ∀τ∈ȹ

$$Y(\tau)=\begin{cases} ŧ(\tau), & \tau\in ȹ\setminus B\\ ŧ(\tau)\,\nabla L(\tau), & \tau\in ȹ\cap B\end{cases}$$

[46,49].

Compared to rings, semirings are more general. Typically, addition and multiplication are two binary operations that combine to form a semiring (R,+, ·), which is an algebraic structure made up of a non-empty set R and two semigroups (R,+) and (R,.), where multiplication is distributive over addition from both sides. A semiring is called a semiring with identity if it has identity with multiplication, and a commutative semiring if it has commutative multiplication. A zero is said to exist in R if there is an element 0 ∈ R such that 0. a = a. 0 = 0 and 0 + a = a + 0 = a for every a ∈ R. A hemiring is a semiring with a zero element and commutative addition. See [59-75] for further information on semirings and hemirings and [76] for further possible applications of soft sets and graph applications.

PROPERTIES OF AND-PRODUCT

In this section, a whole investigation on AND-product as regards its algebraic properties will be handled such as commutative laws, associative laws, idempotent laws and other all basic properties concerning soft F-subsets and soft M-equality and we will compare them with the formerly obtained properties as regards soft L-equality and soft J-equality.

The set $S_E(U)$ is closed under AND-product. That is, when (Γ,A) and (6,B) are two SSs over U, then so is (Γ,A) ∧(6,B). In fact, AND-product is a binary operation in $S_E(U)$, namely,

$\wedge : S_E(U)xS_E(U)\rightarrow S_E(U)$

$((\Gamma,A), (6,B)) \rightarrow (\Gamma,A)\wedge(6,B)=(H,AxB)$

as the set $S_E(U)$ contains all the SS over U. Here note that, the set $S_A(U)$ is not closed under AND-product, since if (Γ,A) and (6,A) are the elements of $S_A(U)$, (Γ,A)∧(6,A) is an element of $S_{AxA}(U)$ not $S_A(U)$.

Maji et al. [25] stated without any proof that associative law holds for AND-product. But in [28], it was shown that $(\Gamma,A)\wedge((6,B)\wedge(Z,C))\neq_M ((\Gamma,A)\wedge(6,B))\wedge(Z,C)$. Since the SS (Γ,A)∧((6,B)∧(Z,C)) has the PS Ax(BxC) and the SS ((Γ,A) ∧ (6,B))∧(Z,C) has the PS (AxB)xC and strictly speaking, the cartesian product is not associative (unless one of the involved sets is empty), that is Ax(BxC) and (AxB)xC are not exactly identical from a set theoretic point of view, they cannot be soft equal in the sense of soft M-equality (and so soft F-equality). However in [52], it was shown that the associative laws of AND-product operation (which they call "Generalized Soft Associative Laws") only hold in the sense of soft L-equality instead of soft M-equality (and so soft F-equality).

Proposition 3.1. $(\Gamma,A)\wedge((6,B)\wedge(Z,C))=_L((\Gamma,A)\wedge(6,B))\wedge(Z,C)$ [52].

It was highlighted in [52] that although the SSs on the two sides are soft equal with regard to soft L-equal relations, they are not the same since they have distinct PSs. Put otherwise, we may state that rather than holding true for Maji's soft M-equality, the associative rules of soft product operations only hold in the context of soft L-equality. For algebraic structures like semigroups, this differs from the standard associative principles of binary operations.

Thus, we can deduce that ($S_E(U)$, ∧) is a semigroup only in the sense of soft L-equality, not in the sense of soft M-equality. Also, since AND-product is not closed in the set $S_A(U)$, ($S_A(U)$, ∧) can not be a semigroup even in the sence of soft L-equality.

In classical set theory; the Cartesian product of sets is not commutative, that is AxB≠BxA. In [53], as the name of "generalized soft commutative laws" (see Proposition 3.2), it was shown that commutative law holds for AND-product with respect to soft L-equal relations.

Proposition 3.2. Let (Γ,A) and (6,B) be two SSs over U. Then, $(\Gamma,A)\wedge(6,B)=_L(6,B)\wedge(\Gamma,A)$ [53].

We now have the following comparison:

Proposition 3.3. Let (Γ,A), (6,A) and (6,B) be SSs over U. Then, $(\Gamma,A)\wedge(6,B)\neq_M(6,B)\wedge(\Gamma,A)$, moreover $(\Gamma,A)\wedge(6,A)\neq_M(6,A)\wedge(\Gamma,A)$.

Proof: Let first handle the case of (Γ,A)∧(6,B). Since the PS of the left hand side is AxB, and the PS of the right hand side is BxA, and AxB≠BxA, $(\Gamma,A)\wedge(6,B)\neq_M(6,B)\wedge(\Gamma,A)$ is obvious.

Now let handle the case of (Γ,A) ∧(6,A). Since the PS of the left hand side is AxA, and the PS of the right hand side is AxA, the initial requirement for the soft M-equality has been met. Moreover,

i) Let $(\tau,\omega)\in AxA$ such that $\tau=\omega$. Then, $(\Gamma,A)\wedge(6,A)=(H,AxA)$, where $H(\tau,\tau)=\Gamma(\tau)\cap6(\tau)$ for all $(\tau,\tau)\in AxA$. Let $(6,A)\wedge(\Gamma,A)=(K,AxA)$, where $K(\tau,\tau)=6(\tau)\cap\Gamma(\tau)$ for all $(\tau,\tau)\in AxA$. Since $\Gamma(\tau)\cap6(\tau)=6(\tau)\cap\Gamma(\tau)$, then $(\Gamma,A)\wedge(6,A)=(6,A)\wedge(\Gamma,A)$ for the elements of AxA such that $\tau = \omega$. However,
ii) Now suppose that $(\tau,\omega)\in AxA$ such that $\tau\neq\omega$ and let $(\Gamma,A)\wedge(6,A)=(H,AxA)$, where $H(\tau,\omega)=\Gamma(\tau)\cap6(\omega)$ for all $(\tau,\omega)\in AxA$ and $(6,A)\wedge(\Gamma,A)=(K,AxA)$, where $K(\tau,\omega)=6(\tau)\cap\Gamma(\omega)$ for all $(\tau,\omega)\in AxA$. Since $\Gamma(\tau)\cap6(\omega)$ needs not be equal to $6(\tau)\cap\Gamma(\omega)$, $(\Gamma,A)\wedge(6,A)\neq_M(6,A)\wedge(\Gamma,A)$.

Proposition 3.3. shows that for satisfying $(\Gamma,A)\wedge(6,B)=_M(6,B)\wedge(\Gamma,A)$, first of all the PSs of the SSs of both sides should be equal; that is A sould be equal to B; but this is not a sufficient condition for the commutativity; as (Γ,A)∧(6,A) is not soft M-equal to (6,A)∧(Γ,A) although both sides have the same PSs.

In classical set theory, if A=B, then AxB=BxA. We now have the following comparison:

Proposition 3.4. Let (Γ,A) and (6,B) be two SSs over U. If $(\Gamma,A)=_M(6,B)$, then $(\Gamma,A)\wedge(6,B)=_M(6,B)\wedge(\Gamma,A)$.

Proof: Since $(\Gamma,A)=_M(6,B)$, then, A=B. Hence, AxB=BxA. Moreover, $(\Gamma,A)\wedge(6,B)=_M(\Gamma,A)\wedge(\Gamma,A)$ and $(6,B)\wedge(\Gamma,A)=_M(\Gamma,A)\wedge(\Gamma,A)$. Thus, the proof is completed.

In classical theory, AxB=BxA if and only if A=B, or A=∅ or B=∅; however the following example (Example 3.5) illustrates that Proposition 3.4 cannot be reversed in general, that is, $(\Gamma,A)\wedge(6,B)=_M(6,B)\wedge(\Gamma,A)$ does not imply that $(\Gamma,A)=_M(6,B)$.

Example 3.5. Let $E=\{e_1,e_2,e_3,e_4\}$ be the PS, $A=B=\{e_1,e_3\}$ be the subsets of E and $U=\{h_1,h_2,h_3,h_4,h_5\}$ be the initial universe set. Let (Γ,A) and (6,B) be SSs over U defined as following:

$(\Gamma,A)=\{(e_1, \{h_2,h_5\}), (e_3, \{h_5\})\}$

$(6,B)=\{(e_1, \{h_1,h_4\}), (e_3, \{h_3,h_4\})\}$

Then, $(\Gamma,A)\wedge(6,B)=_M\{((e_1,e_1),\emptyset), ((e_1,e_3),\emptyset), ((e_3,e_1),\emptyset), ((e_3,e_3),\emptyset)\}$ and $(6,B)\wedge(\Gamma,A)=_M\{((e_1, e_1),\emptyset), ((e_1,e_3), \emptyset), ((e_3,e_1),\emptyset), ((e_3, e_3),\emptyset)\}$. It is observed that $(\Gamma,A)\wedge(6,B)=_M(6,B)\wedge(\Gamma,A)$; but $(\Gamma,A)\neq_M(6,B)$.

Note 3.6: In classical set theory, if AxB=BxA, then A=B or [A=∅ or B=∅]. Example 3.5 also shows us that, as a nonanalogy to the classical set theory, in SS theory $(\Gamma,A)\wedge(6,B)=_M(6,B)\wedge(\Gamma,A)$ neither implies that $(\Gamma,A)=_M(6,B)$ nor $(\Gamma,A)=_M \emptyset_A$ or $(6,B)=_M \emptyset_B$.

In classical set theory, if AxB=∅, then A=∅ or B=∅. Example 3.5 also illustrates that this case is not valid for SS theory as regards AND-product. For satisfying (Γ,A)

$\wedge(6,B)=_M \emptyset_{AxB}$, it is compulsory that $\Gamma(\tau)\cap 6(\omega)=\emptyset$ for all $(\tau,\omega)\in AxB$ as in Example 3.5.

In classical set theory, if $A=\emptyset$ or $B=\emptyset$, then $AxB=BxA$ and also $Ax\emptyset=\emptyset xA=\emptyset$. That is, the empty set commutes with any set under the cartesian product and $\emptyset$ is the absorbing element under cartesian product in the set of sets. In [53], it was shown that $\emptyset_A$ commutes with any SS whose PS is A under AND-product and also $\emptyset_A$ is the absorbing element for AND-product in $S_A(U)$ as regards L- equality. That is:

$(\Gamma,A)\wedge\emptyset_A=_L \emptyset_A\wedge(\Gamma, A)=_L \emptyset_A$.

However, as regards soft M-equality, we have Proposition 3.7. and Proposition 3.8.:

Proposition 3.7. Let (Γ,A) and $(6,B)$ be two SSs over U. If $(\Gamma,A)=\emptyset_A$ or $(6,B)=\emptyset_B$, then $(\Gamma,A)\wedge(6,B)$ needs not be soft M-equal to $(6,B)\wedge(\Gamma,A)$.

Proof: Without loss of generality, let $(6,B)=_M\emptyset_B$. Then, $(\Gamma,A)\wedge(6,B)=_M(\Gamma,A)\wedge\emptyset_B=_M\emptyset_{AxB}$ and $(6,B)\wedge(\Gamma,A)=\emptyset_B\wedge(\Gamma,A)=_M\emptyset_{BxA}$ and since $\emptyset_{AxB}\neq_M\emptyset_{BxA}$, $(\Gamma, A)\wedge(6, B)\neq_M(6,B)\wedge(\Gamma,A)$.

We now have the following comparison:

Proposition 3.8. Let (Γ,A) be a SS over U. Then, $(\Gamma,A)\wedge\emptyset_A=_M \emptyset_A\wedge(\Gamma,A)=_M \emptyset_{AxA}$.

Proof: Let $\emptyset_A=(S,A)$, where $S(\tau)=\emptyset$ for all $\tau\in A$. Then, $(\Gamma,A)\wedge\emptyset_A=_M(\Gamma,A)\wedge(S,A)=_M(H,AxA)$, where $H(\tau,\omega)=\Gamma(\tau)\cap S(\omega)=\Gamma(\tau)\cap\emptyset=\emptyset$ for all $(\tau,\omega)\in AxA$. Hence, $(H,AxA)=_M \emptyset_{AxA}$. Now, let $\emptyset_A\wedge(\Gamma,A)=_M(S,A)\wedge(\Gamma, A)=_M(K,AxA)$, where $K(\tau,\omega)=S(\tau)\cap F(\omega)=\emptyset\cap F(\omega)=\emptyset$ for all $(\tau,\omega)\in AxA$. Hence, $(K,AxA)=_M \emptyset_{AxA}$.

Here note that while $\emptyset_A$ is absorbing element for AND-product in $S_A(U)$ as regards L-equality, it is not the absorbing element for AND-product in $S_A(U)$ as regards M-equality.

Corollary 3.9. Let (Γ,A) and $(6,B)$ be SSs over U. If $(\Gamma,A)=\emptyset_A$ or $(6,B)=\emptyset_A$, then $(\Gamma,A)\wedge(6,B)=_M(6,B)\wedge(\Gamma,A)$.

Now by being inspired by Proposition 3.8. and Corollary 3.9., we have the following:

Proposition 3.10. Let (Γ,A) and $(6,B)$ be SSs over U. $(\Gamma,A)\wedge(6,B)=_M(6,B)\wedge(\Gamma,A)=_M \emptyset_\emptyset$ if and only if $(\Gamma,A)=_M \emptyset_\emptyset$ or $(6,B)=_M \emptyset_\emptyset$.

Proof: Necessity: Let $(\Gamma,A)\wedge(6,B)=_M(6,B)\wedge(\Gamma,A)=_M \emptyset_\emptyset$. Since $AxB=BxA=\emptyset$ if and only if $A=\emptyset$ or $B=\emptyset$, and since $\emptyset_\emptyset$ is the unique SS over U with an empty PS, $(\Gamma,A)=\emptyset_\emptyset$ or $(6,B)=\emptyset_\emptyset$.

Sufficiency: Let $(\Gamma,A)=\emptyset_\emptyset$ or $(6,B)=\emptyset_\emptyset$. Without loss of generality, let $(6,B)=\emptyset_\emptyset$. Then, $B=\emptyset$ and since $Ax\emptyset=\emptyset xA=\emptyset$, and since $\emptyset_\emptyset$ is the unique SS over U with an empty PS, the rest of the proof is obvious.

By Proposition 3.10, we have $\emptyset_\emptyset\wedge(\Gamma,A)=_M(\Gamma,A)\wedge\emptyset_\emptyset=_M \emptyset_\emptyset$ (1). This shows us that $\emptyset_\emptyset$ commutes with any SS under AND-product and $\emptyset_\emptyset$ is the absorbing element for AND-product in $S_E(U)$ as regards soft M-equality (and thus, soft L-equality and soft J-equality, as soft M-equality requires soft L-equality and soft J-equality.)

Moreover, one can easily show that $(\Gamma,A)\wedge\emptyset_E=_L \emptyset_E\wedge(\Gamma,A)=_L \emptyset_E$ (2). This means that $\emptyset_E$ commutes with any SS under AND-product and also $\emptyset_E$ is the absorbing element for AND-product in SE(U) as regards L-equality, too. Thus, $\emptyset_\emptyset$ and $\emptyset_E$ are all the absorbing element for AND-product in SE(U) as regards L-equality. It is well-known a magma can have at most one absorbing element.

Here, we want to draw attention to one crucial point: Although the SSs on either side of Equations (1) and (2) differ in their PSs, they are soft equal in reference to soft L-equal relations. In other words, we can say that there are two different absorbing elements for AND-product operation in $S_E(U)$ (they are $\emptyset_\emptyset$ and $\emptyset_E$), only in the sense of soft L-equality. Having two distinct absorbing elements in the sense of soft M-equality is ofcourse impossible. In fact, only $\emptyset_\emptyset$ is the only absorbing element of AND-product in the set SE(U) in the sense of soft M-equality.

In [53], it was shown that $(\Gamma,A)\wedge U_A=_L U_A\wedge(\Gamma,A)=_L(\Gamma,A)$. That is, U_A commutes with any SS whose PS is A under AND-product and also U_A is the identity element for AND-product in $S_A(U)$ as regards L-equality. Also, similarly one can show that $(\Gamma,A)\wedge U_E=_L U_E\wedge(\Gamma,A)=_L(\Gamma,A)$. That is, U_E commutes with any SS under AND-product and U_E is the identity element for AND-product in SE(U) as regard L-equality. Now, we have the following for soft M-equality:

Proposition 3.11. Let (Γ,A) and $(6,A)$ be SSs over U. If $(\Gamma,A)=_M U_A$ or $(6,A)=_M U_A$, then $(\Gamma,A)\wedge(6,A)$ needs not to be soft M-equal to $(6,A)\wedge(\Gamma,A)$.

Example 3.12. Let $E=\{e_1,e_2,e_3,e_4\}$ be the PS, $A=B=\{e_1,e_3\}$ be the subset of E and $U=\{h_1,h_2,h_3,h_4,h_5\}$ be the initial universe set. Let (Γ,A) and $(6,A)$ be SSs over U defined as following:

$(\Gamma,A)=\{(e_1,U), (e_3,U)\}=U_A$ and $(6,B)=\{(e_1, \{h_1,h_4\}), (e_3, \{h_3,h_4\})\}$. Then, $(\Gamma,A)\wedge(6,A)=\{((e_1,e_1), \{h_1,h_4\}), ((e_1,e_3), \{h_3,h_4\}), ((e_3,e_1), \{h_1,h_4\}), ((e_3,e_3), \{h_3,h_4\})\}$ and $(6,A)\wedge(\Gamma,A)=\{((e_1,e_1), \{h_1,h_4\}), ((e_1,e_3), \{h_1,h_4\}), ((e_3,e_1), \{h_3,h_4\}), ((e_3,e_3), \{h_3,h_4\})\}$. It is seen that $(\Gamma,A)\wedge(6,A)\neq_M (6,A)\wedge(\Gamma,A)$.

We now have the following comparison:

Proposition 3.13. Let (Γ,A) and $(6,A)$ be SSs over U. If one of the SSs is the whole SS with respect to A, then $(\Gamma,A)\wedge(6,A)=_M (6,A)\wedge(\Gamma,A)$ if and only if the other SS is a constant function.

Proof: Necessity: Let (Γ,A) and $(6,A)$ be SSs over U such that $(\Gamma,A)=_M U_A$ and $(\Gamma,A)\wedge(6,A)=_M (6,A)\wedge(\Gamma,A)$, that is $U_A\wedge(6,A)=_M (6,A)\wedge U_A$.

Let $(\tau,\omega)\in AxA$ such that if $\tau=\omega$. Then, $U_A\wedge(6,A)=_M(6,A)\wedge U_A$ implies that $U\cap 6(\tau)=6(\tau)\cap U$. That is $6(\tau)=6(\tau)$ is already satisfied for all $(\tau,\omega)\in AxA$ such that $\tau=\omega$.

Therefore let $(\tau,\omega)\in AxA$ such that $\tau\neq\omega$. Then, $U_A\wedge(6,A)=_M (6,A)\wedge U_A$ implies that $U\cap 6(\omega)=6(\tau)\cap U$. That is, $6(\tau)=6(\omega)$ for all $(\tau,\omega)\in AxA$ such that $\tau\neq\omega$. This implies that $(6,A)$ is a constant function.

Sufficiency: Let (Γ,A) and $(6,A)$ be SSs over U such that $(\Gamma,A)=_M U_A$ and $(6,A)$ is a constant function. Let $(\Gamma,A)\wedge(6,A)=_M(H,AxA)$, where $H(\tau,\omega)=\Gamma(\tau)\cap 6(\omega)=U\cap$

Ϭ(ω)=Ϭ(ω) for all (τ,ω)∈AxA. Let (Ϭ,A)∧(Γ,A)=(W,AxA), where W(τ,ω)=Ϭ(τ)∩Γ(ω)=Ϭ(τ)∩U=Ϭ(τ) for all (τ,ω)∈AxA. Since Ϭ is a constant function, then Ϭ(τ)=Ϭ(ω) for all τ, ω∈A. Hence, H(τ,ω)=W(τ,ω) for all (τ,ω)∈AxA, implying that (H,AxA)$=_M$ (W,AxA), and so (Γ,A)∧(Ϭ,A)$=_M$ (Ϭ,A)∧(Γ,A).

With a generalization; we have the following proposition for the commutative property of the SSs under AND-product whose PSs are the same.

Corollary 3.14. Let (Γ,A) and (Ϭ,B) be SSs over U. (Γ,A)∧(Ϭ,B)$=_M$(Ϭ,B)∧(Γ,A) if and only if A=B and Γ(τ)∩Ϭ(ω)=Ϭ(τ)∩Γ(ω) for all (τ,ω)∈ AxA such that τ≠ ω.

Example 3.15. Let E={e_1,e_2,e_3,e_4} be the PS, A=B={e_1,e_3} be the subset of E and U= {h_1,h_2,h_3,h_4,h_5} be the initial universe set. Let (Γ,A) and (Ϭ,A) be the SSs over U defined as following:

(Γ,A)={(e_1, {h_1,h_3}), (e_3, {h_1,h_3,h_4,h_5})}

(Ϭ,A)={(e_1, {h_1, h_2, h_3 }), (e_3, {h_1,h_3,h_4})}.

Then, (Γ,A)∧(Ϭ,A)={((e_1,e_1), {h_1,h_3}), ((e_1,e_3), {h_1,h_3}), ((e_3,e_1), {h_1,h_3}), ((e_3,e_3) {h_1,h_3,h_4})} and (Ϭ,A)∧(Γ,A)={((e_1,e_1), {h_1,h_3}), ((e_1,e_3), {h_1,h_3}), ((e_3,e_1), {h_1,h_3}), ((e_3,e_3), {h_1,h_3,h_4})}.

Since for all τ≠ ω, Γ(τ)∩Ϭ(ω)=Ϭ(τ)∩Γ(ω), then it t is observed that (Γ,A)∧(Ϭ,A)$=_M$ (Ϭ,A)∧(Γ,A).

Proposition 3.16. Let (Γ,A) and (Ϭ,B) be SSs over U. (Γ,A)∧(Ϭ,B)$=_M$ U_{AxB} if and only if (Γ,A)$=_M$ U_A and (G,B)$=_M$ U_B .

Proof: Necessity: Let (Γ,A)∧(Ϭ,B)$=_M$(Z,AxB), where Z(τ,ω)=Γ(τ)∩Ϭ(ω) for all (τ,ω)∈AxB. Let $U_{AxB}$$=_M$(L,AxB), where L(τ,ω)=U for all (τ,ω)∈AxB. Since, Z(τ,ω)=L(τ,ω)= Γ(τ)∩Ϭ(ω)=U, then Γ(τ)=U for all τ∈A and Ϭ(ω)=U for all ω∈B. Hence, (Γ,A)$=_M$ U_A and (Ϭ,B)$=_M$ U_B.

Sufficiency: It is obvious, hence omitted.

In classical set theory; A⊆B ⇒ AxC⊆BxC. We now have the following comparison:

Proposition 3.17. Let (Γ,A), (Ϭ,B) and (Z,C) be SSs over U. If (Γ,A)$\tilde{\subseteq}_F$(Ϭ,B), then (Γ,A)∧(Z,C)$\tilde{\subseteq}_F$(Ϭ,B)∧(Z,C).

Proof: Let (Γ,A)$\tilde{\subseteq}_F$(Ϭ,B). Then, A⊆B and hence, AxC⊆BxC. Moreover since (Γ,A)$\tilde{\subseteq}_F$(Ϭ,B), then Γ(τ)⊆Ϭ(τ) for all τ∈A. Thus, Γ(τ)∩Z(ω)⊆Ϭ(τ)∩Z(ω) for all (τ,ω) ∈AxC. So, (Γ,A)∧(Z,C)$\tilde{\subseteq}_F$(Ϭ,B)∧(Z,C).

In classical set theory; B⊆C ⇒ AxB⊆AxC. We now have the following comparison:

Proposition 3.18. Let (Γ,A), (Ϭ,B) and (Z,C) be SSs over U. (Ϭ,B)$\tilde{\subseteq}_F$(Z,C)⇒(Γ,A)∧(Ϭ,B)$\tilde{\subseteq}_F$(Γ,A)∧(Z,C).

The proof is similar to the proof of Proposition 3.17, hence omitted.

In classical set theory, if AxC⊆BxC, then A needs not be a soft subset of B; but if AxC⊆BxC where C≠∅, then A⊆B. We now have the following comparison:

Proposition 3.19. Let (Γ,A), (Ϭ,B) and (Z,C) be SSs over U. If (Γ,A)∧(Z,C)$\tilde{\subseteq}_F$(Ϭ,B)∧(Z,C) and (Z,C)$\neq_F$ $\emptyset_\emptyset$ and (Z,C)$\neq_F$ $\emptyset_C$, then (Γ,A)$\tilde{\subseteq}_F$(Ϭ,B).

Proof: Let (Γ,A)∧(Z,C)$\tilde{\subseteq}_F$(Ϭ,B)∧(Z,C). Then, AxC⊆BxC. Since (Z,C)$\neq_F$ $\emptyset_\emptyset$, then C≠∅. (As $\emptyset_\emptyset$ is the unique SS with an empty PS); hence AxC⊆BxC implies that A⊆B. Hence, the first condition for being the soft F-subset is satisfied.

Now, let (Γ,A)∧(Z,C)=(T,AxC), where T(τ,c)=Γ(τ)∩Z(c) for all (τ,c)∈AxC. Since A⊆ B, τ∈A implies that τ∈B. Also let (Ϭ,B)∧(Z,C)=F(P,BxC), where P(τ,c)=Ϭ(τ)∩Z(c). By assumption, since (T,AxC)$\tilde{\subseteq}_F$(P,BxC), then T(τ,c)⊆P(τ,c) for all (τ,c)∈AxC. Thus, T(τ,c)=Γ(τ)∩Z(c)⊆ P(τ,c)=Ϭ(τ)∩Z(c) and so Γ(τ)⊆Ϭ(τ) for all τ∈A (Since (Z,C)≠$\emptyset_C$, Z(c)≠∅ for all c∈C). Thus, (Γ,A)$\tilde{\subseteq}_F$(Ϭ,B).

In classical set theory; A⊆B and C⊆D⇒AxC⊆BxD. We now have the following comparison:

Proposition 3.20. Let (Γ,A), (Ϭ,B), (Z,C) and (T,D) be SSs over U. (Γ,A)$\tilde{\subseteq}_F$(Ϭ,B) and (Z,C)$\tilde{\subseteq}_F$(T,D)⇒(Γ,A)∧(Z,C) $\tilde{\subseteq}_F$(Ϭ,B)∧(T,D).

Proof: Let (Γ,A)$\tilde{\subseteq}_F$(Ϭ,B) and (Z,C)$\tilde{\subseteq}_F$(T,D), then A⊆B and C⊆D. Hence, AxC⊆BxD. Moreover, by assumption, Γ(τ)⊆Ϭ(τ) for all τ∈A and Z(ω)⊆T(ω) for all ω∈C. Thus, Γ(τ)∩Z(ω)⊆Ϭ(τ)∩Z(ω) for all (τ,ω)∈AxC. So, (Γ,A)∧(Z,C)$\tilde{\subseteq}_F$(Ϭ,B)∧(T,D).

In classical set theory, (AxB)∩(CxD)=(A∩C)x(B∩D). We now have the following comparison:

Proposition 3.21. Let (Γ,A), (Ϭ,B), (Z,C) and (T,D) be SSs over U. Then, [(Γ,A)∧(Ϭ,B)]$\cap_R$[(Z,C)∧(T,D)]$=_M$ [(Γ,A)$\cap_R$(Z,C)]∧[(Ϭ,B)$\cap_R$(T,D)]

Proof: The PS of the SS of the left hand side is (AxB)∩(CxD); and the PS of the SS of the right hand side is (A∩C)x(B∩D). Since, (AxB)∩(CxD)=(A∩C) x(B∩D), the parameter condition for the soft M-equality is satisfied.

Let τ∈A∩C and ω∈B∩D. Hence, τ∈A and τ∈C, and ω∈B and ω∈D. Thus, (τ,ω) ∈AxB and (τ,ω)∈CxD. Now, let (Γ,A)∧(Ϭ,B)$=_M$(N,AxB), where N(τ,ω)=Γ(τ)∩Ϭ(ω) for all (τ, ω) ∈AxB and let (Z,C)∧(T,D)$=_M$(K,CxD) where K(τ,ω)=Z(τ) ∩T(ω) for all (τ,ω)∈CxD. Now, suppose that (N,AxB)$\cap_R$(K,CxD)$=_M$(M,(AxB)∩(CxD)), where M(τ,ω)=N(τ,ω)∩K(τ,ω) for all (τ,ω)∈AxB and (τ,ω) ∈CxD. Hence M(τ,ω)=N(τ,ω)∩K(τ,ω)=[Γ(τ)∩Ϭ(ω)]∩ [Z(τ)∩T(ω)].

Suppose that (Γ,A)$\cap_R$(Z,C)$=_M$(R,A∩C), where R(τ)=Γ(τ)∩Z(τ) for all τ∈A∩C and (Ϭ,B)$\cap_R$(T,D)$=_M$(S,B∩D), for all ω∈B∩D where S(ω)=Ϭ(ω)∩T(ω). Let (R,A∩C)∧(S,B∩D)$=_M$(Y,(A∩C) x(B∩D)), where Y(τ,ω)=R(τ)∩S(ω) for all (τ, ω)∈(A∩C) x(B∩D). Hence; Y(τ,ω)=R(τ)∩S(ω)=[Γ(τ) ∩Z(τ)]∩[Ϭ(ω) ∩T(ω)]. When we consider the properties of operations on set theory, we obtain that M and Y are the same set-valued mappings, so the proof is completed.

In classical set theory, for all A, ∅⊆A. We now have the following comparison:

Proposition 3.22. Let (Γ,A) and (Ϭ,B) be SSs over U. Then, $\emptyset_{AxB}$$\tilde{\subseteq}_F$(Γ,A)∧(Ϭ,B), $\emptyset_{BxA}$$\tilde{\subseteq}_F$(Ϭ,B)∧(Γ,A).

In classical set theory, for all A, A⊆U. We now have the following comparison:

Proposition 3.23. Let (Γ,A) and (Ϭ,B) be SSs over U. Then, (Γ,A)∧(Ϭ,B)$\tilde{\subseteq}_F$$U_{AxB}$ and (Ϭ,B)∧(Γ, A)$\tilde{\subseteq}_F$$U_{BxA}$.

In [53], it was proved that the AND-product is idempotent in the sense of soft J-equality, that is $(\Gamma,A)\wedge(\Gamma,A)=_J(\Gamma,A)$; however in the sense of L-equality, the AND-product is not idempotent, that is, $(\Gamma,A)\wedge(\Gamma,A)\neq_L(\Gamma,A)$. Also, AND-product is not idempotent in the sense of soft M-equality, that is, $(\Gamma,A)\wedge(\Gamma,A)\neq_M(\Gamma,A)$ due to the inequality of the PSs of the SSs of both sides. In [53], it was also proved that for certain types of SSs (sublattice SS), soft product operations may satisfy idempotent laws with respect to soft L-equal relations.

In what follows, we give the interrelations of AND-product with OR-product.

Proposition 3.24. Let (Γ,A) and (σ,B) be SSs over U. Then, $((\Gamma,A)\wedge(\sigma,B))^r=_M(\Gamma, A)^r\vee(\sigma,B)^r$ [32].

In Maji [25], it was stated without any proof that AND-product distributes over OR-product, and OR-product distributes over AND-product as regards soft M-equality. But in [28], it was proved that AND-product does not distribute over OR-product, and OR-product does not distribute over AND-product as regards soft M-equality due to the inequality of the PSs of the SSs of both sides. In [51], it was proved that AND-product distributes over OR-product, and OR-product distributes over AND-product as regards soft J-equality; but in [52,53], it was shown with a counterexample that AND-product does not distribute over OR-product, and OR-product does not distribute over AND-product as regards soft J-equality. Finally, with the help of soft L-subsets, in [52,53] the right answer to the above question concerning soft distributive laws was given follows:

Proposition 3.25. Let (Γ,A), (σ,B) and (Z,C) be SSs over U. Then,

i) $(\Gamma,A)\wedge((\sigma,B)\vee(Z,C))\tilde{\subseteq}_L((\Gamma,A)\wedge(\sigma,B))\vee((\Gamma,A)\wedge(Z,C))$ [52].

ii) $(\Gamma,A)\vee((\sigma,B)\wedge(Z,C))\tilde{\subseteq}_L((\Gamma,A)\vee(\sigma,B))\wedge((\Gamma,A)\vee(Z,C))$ [53].

iii) $((\Gamma,A)\wedge(\sigma,B))\vee(Z,C)\tilde{\subseteq}_L((\Gamma,A)\vee(Z,C))\wedge((\sigma,B)\vee(Z,C))$ [52].

iv) $((\Gamma,A)\vee(\sigma,B))\wedge(Z,C)\tilde{\subseteq}_L((\Gamma,A)\wedge(Z,C))\vee((\sigma,B)\wedge(Z,C))$ [53].

In classical set theory, $A\cap B\subseteq A\cup B$. We now have the following comparison:

Proposition 3.26. Let (Γ,A) and (σ,B) be SSs over U. Then, $(\Gamma,A)\wedge(\sigma,B)\tilde{\subseteq}_F(\Gamma,A)\vee(\sigma,B)$.

Proof: Let $(\Gamma,A)\wedge(\sigma,B)=(H,AxB)$, where $Z(\tau,\omega)=\Gamma(\tau)\cap\sigma(\omega)$ for all $(\tau,\omega)\in AxB$. Let $(\Gamma,A)\vee(\sigma,B)=(J,AxB)$, where $J(\tau,\omega)=\Gamma(\tau)\cup\sigma(\omega)$ for all $(\tau,\omega)\in AxB$. Since, $\Gamma(\tau)\cap\sigma(\omega)\subseteq\Gamma(\tau)\cup\sigma(\omega)$ for all $(\tau,\omega)\in AxB$, the rest of the proof is obvious.

In classical set theory, $A\cap B=A\cup B\Leftrightarrow A=B$. However in the following example, we show that $(\Gamma,A)\wedge(\sigma,B)=_M(\Gamma,A)\vee(\sigma,B)$ does not imply that $(\Gamma,A)=_M(\sigma,B)$.

Example 3.27. Let $E=\{e_1,e_2,e_3,e_4\}$ be the PS, $A=\{e_1,e_2\}$ and $B=\{e_1,e_3\}$ be the subsets of E and $U=\{h_1,h_2,h_3,h_4,h_5\}$ be the initial universe. Consider (Γ,A) and (σ,B) as the SSs over U defined as following:

$(\Gamma,A)=\{(e_1,\{h_2,h_5\}), (e_2,\{h_2,h_5\})\}$

$(\sigma,B)=\{(e_1,\{h_2,h_5\}), (e_3,\{h_2,h_5\})\}$.

Then,

$(\Gamma,A)\wedge(\sigma,B)=\{((e_1,e_1), \{h_2, h_5\}), ((e_1,e_3), \{h_2,h_5\}), ((e_2,e_1), \{h_2,h_5\}), ((e_2,e_3), \{h_2,h_5\})\}$ and $(\Gamma,A)\vee(\sigma,B)=\{((e_1,e_1), \{h_2,h_5\}), ((e_1,e_3), \{h_2,h_5\}), ((e_2,e_1), \{h_2,h_5\}), ((e_2,e_3), \{h_2,h_5\})\}$.

It is seen that $(\Gamma,A)\wedge(\sigma,B)=_M(\sigma,B)\wedge(\Gamma,A)$; but $(\Gamma,A)\neq_M(\sigma,B)$ (since $A\neq B$).

Proposition 3.28. Let (Γ,A) and (σ,B) be SSs over U. $(\Gamma,A)\wedge(\sigma,B)=_M(\Gamma,A)\vee(\sigma,B)$ if and only if Γ and σ are the constant functions such that $\Gamma(\tau)=\sigma(\omega)$ for all $\tau\in A$ and for all $\omega\in B$.

Proof: Necessity: Let $(\Gamma,A)\wedge(\sigma,B)=_M(\Gamma,A)\vee(\sigma,B)$ and $(\Gamma,A)\wedge(\sigma,B)=M(Z,AxB)$, where $Z(\tau,\omega)=\Gamma(\tau)\cap\sigma(\omega)$ for all $(\tau,\omega)\in AxB$ and $(\Gamma,A)\vee(\sigma,B)=M(J,AxB)$, where $J(\tau,\omega)=\Gamma(\tau)\cup\sigma(\omega)$ for all $(\tau,\omega)\in AxB$. Since, $(Z,AxB)=M(J,AxB)$, then $\Gamma(\tau)\cap\sigma(\omega)=\Gamma(\tau)\cup\sigma(\omega)$ for all $(\tau,\omega)\in AxB$. This implies that Γ and σ are the constant functions satisfying $\Gamma(\tau)=\sigma(\omega)$ for all $\tau\in A$ and for all $\omega\in B$.

Sufficiency: Let (Γ,A) and (σ,B) be SSs over U satisfying that Γ and σ are constant functions such that $\Gamma(\tau)=\sigma(\omega)$ for all $\tau\in A$ and for all $\omega\in B$. Let $(\Gamma,A)\wedge(\sigma,B)=_M(Z,AxB)$, where $Z(\tau,\omega)=\Gamma(\tau)\cap\sigma(\omega)$ for all $(\tau,\omega)\in AxB$ and $(\Gamma,A)\vee(\sigma,B)=_M(J,AxB)$, where $J(\tau,\omega)=\Gamma(\tau)\cup\sigma(\omega)$ for all $(\tau,\omega)\in AxB$. Since $\Gamma(\tau)\cap\sigma(\omega)=\Gamma(\tau)\cup\sigma(\omega)$ for all $(\tau,\omega)\in AxB$, this implies that $(Z,AxB)=_M(J,AxB)$. Thus, $(\Gamma,A)\wedge(\sigma,B)=_M(\Gamma,A)\vee(\sigma,B)$.

Corollary 3.29: Let (Γ,A) be a SS over U. $(\Gamma,A)\wedge(\Gamma,A)=_M(\Gamma,A)\vee(\Gamma,A)$ if and only if Γ is a constant function.

DISTRIBUTIONS

In this section, we explore more about the distributions of AND-product over restricted SS operations, extended SS operations and soft binary piecewise operations, respectively.

Distributions of AND-Product Over Restricted SS Operations

In this subsection, we examine the distributions of AND-product over restricted SS operations.

When considering the necessary condition for the soft M-equality, that is the equality of PSs of the SSs of both sides, we can deduce that the distributions of restricted SS operations over AND-product can not be examined, since the intersection operation does not distribute over cartesian product. Thus, the distribution of restricted SS operations over AND-product is never satisfied.

Here it should be noted that in [39], the distributions of AND-product over restricted intersection, restricted union and restricted difference are obtained but as regards restricted intersection and restricted union, only left distributions are examined. Moreover, in [39], the condition where the intersection of the PSs is empty was ignored.

Hence in this section, we handle the distributions of AND-product over restricted SS operations completely.

Left-distributions of AND-product over restricted SS operations

1) $(\Gamma,A)\wedge[(6,B)\cup_R(Z,C)]=_M[(\Gamma,A)\wedge(6,B)]\cup_R[(\Gamma,A)\wedge(Z,C)]$

Proof: Let's first check the equality of PSs of the SSs of both sides. The PS of the left hand side is Ax(B∩C), and the PS of the right hand side is (AxB)∩(AxC). Since Ax(B∩C)=(AxB)∩(AxC), the first condition for the soft M-equality is satisfied.

Let's first consider the left side (LS). Assume that $(6,B)\cup_R(Z,C)=_M(K,B\cap C)$, where $K(\omega)=6(\omega)\cup Z(\omega)$ for all $\omega\in B\cap C$. Let $(\Gamma,A)\wedge(K,B\cap C)=_M(L,Ax(B\cap C))$, where $L(\tau,\omega)=\Gamma(\tau)\cap K(\omega)$ for all $(\tau,\omega)\in Ax(B\cap C)$. Hence, $L(\tau,\omega)=\Gamma(\tau)\cap K(\omega)=\Gamma(\tau)\cap(6(\omega)\cup Z(\omega))=[\Gamma(\tau)\cap 6(\omega)]\cup[\Gamma(\tau)\cap Z(\omega)]$.

Now, let $(\Gamma,A)\wedge(6,B)=_M(M,AxB)$, where $M(\tau,\omega)=\Gamma(\tau)\cap 6(\omega)$ for all $(\tau,\omega)\in AxB$ and let $(\Gamma,A)\wedge(Z,C)=_M(N,AxC)$, where $N(\tau,\omega)=\Gamma(\tau)\cap Z(\omega)$ for all $(\tau,\omega)\in AxC$. Let $(M,AxB)\cup_R(N,AxC)=_M(T,(AxB)\cap(AxC))$, where $T(\tau,\omega)=M(\tau,\omega)\cup N(\tau,\omega)$ for all $(\tau,\omega)\in(AxB)\cap(AxC)$. Thus, $T(\tau,\omega)=_M(\tau,\omega)\cup N(\tau,\omega)=[\Gamma(\tau)\cap 6(\omega)]\cup[\Gamma(\tau)\cap Z(\omega)]$ for all $(\tau,\omega)\in(AxB)\cap(AxC)$. When we consider the properties of operations on set theory, we have that L and T are the same set-valued mappings, so the proof is completed.

Here note that if B∩C=∅, then the PS of both sides is ∅; that is, Ax(B∩C)=(AxB)∩(AxC)=∅ and since $\emptyset_\emptyset$ the unique SS over U with an empty PS, both side will be $\emptyset_\emptyset$. Thus, the soft M-equality is again satisfied.

2) $(\Gamma,A)\wedge[(6,B)\cap_R(Z,C)]=_M[(\Gamma,A)\wedge(6,B)]\cap_R[(\Gamma,A)\wedge(Z,C)]$

3) $(\Gamma,A)\wedge[(6,B)\setminus_R(Z,C)]=_M[(\Gamma,A)\wedge(6,B)]\setminus_R[\Gamma,A)\wedge(Z,C)]$

4) $(\Gamma,A)\wedge[(6,B)\Delta_R(Z,C)]=_M[(\Gamma,A)\wedge(6,B)]\Delta_R[(\Gamma,A)\wedge(Z,C)]$

Right-distributions of AND-product over restricted SS operations

1) $[(\Gamma,A)\cap_R(6,B)]\wedge(Z,C)=_M[(\Gamma,A)\wedge(Z,C)]\cap_R[(6,B)\wedge(Z,C)]$

Proof: Let's first check the equality of PSs of the SSs of both sides. The PS of the left hand side is (A∩B)xC, and the PS of the right hand side is (AxC)∩(BxC). Since (A∩B)xC=(AxC)∩(BxC), the first condition of the soft M-equality is satisfied.

Let's first consider the LS. Assume that $(\Gamma,A)\cap_R(6,B)=_M(K,A\cap B)$, where $K(\tau)=\Gamma(\tau)\cap 6(\tau)$ for all $\tau\in A\cap B$. Let $(K,A\cap B)\wedge(Z,C)=_M(L,(A\cap B)xC)$, where $L(\tau,c)=K(\tau)\cap Z(c)$ for all $(\tau,c)\in(A\cap B)xC$. Hence, $L(\tau,c)=[\Gamma(\tau)\cap 6(\tau)]\cap Z(c)=[\Gamma(\tau)\cap Z(c)]\cap[6(\tau)\cap Z(c)]$.

Now, let $(\Gamma,A)\wedge(Z,C)=_M(M,AxC)$, where $M(\tau,c)=\Gamma(\tau)\cap Z(c)$ for all $(\tau,c)\in AxC$ and let $(6,B)\wedge(Z,C)=(N,BxC)$, where $N(\tau,c)=6(\tau)\cap Z(c)$ for all $(\tau,c)\in BxC$. Let $(M,AxC)\cap_R(N,BxC)=_M(T,(AxC)\cap(BxC))$, where $T(\tau,c)=_M(\tau,c)\cap N(\tau,c)$ for all $(\tau,c)\in(AxC)\cap(BxC)$. Thus, $T(\tau,c)=_M(\tau,c)\cap N(\tau,c)=[\Gamma(\tau)\cap Z(c)]\cap[6(\tau)\cap Z(c)]$ for all $(\tau,c)\in(AxC)\cap(BxC)$. When we consider the properties of operations on set theory, we have that L and T are the same set-valued mappings, so the proof is completed.

Here note that if A∩B=∅, then the PS of both sides is ∅; that is, (A∩B)xC=(AxC)∩(BxC)=∅ and since $\emptyset_\emptyset$ the unique SS over U with an empty PS, both sides will be $\emptyset_\emptyset$. Thus, the soft M-equality is again satisfied.

2) $[(\Gamma,A)\cup_R(6,B)]\wedge(Z,C)=_M[(\Gamma,A)\wedge(Z,C)]\cup_R[(6,B)\wedge(Z,C)]$

3) $[(\Gamma,A)\setminus_R(6,B)]\wedge(Z,C)=_M[(\Gamma,A)\wedge(Z,C)]\setminus_R[(6,B)\wedge(Z,C)]$

4) $[(\Gamma,A)\Delta_R(6,B)]\wedge(Z,C)=_M[(\Gamma,A)\wedge(Z,C)]\Delta_R[(6,B)\wedge(Z,C)]$

Theorem 4.1. $(S_E(U),\cup_R,\wedge)$ is a commutative hemiring with identity element as regards soft L-equality (and hence J-equality).

Proof: In [31,32], it was shown that $(S_E(U),\cup_R)$ is a commutative monoid with identiy $\emptyset_E$. Hence, we can deduce that $(S_E(U),\cup_R)$ is a semigroup. $(S_E(U),\wedge)$ is a semgiroup in the sense of soft L-equality (hence J-equality). Furthermore, ∧ distributes over $\cup_R$ from both sides. Therefore, $(S_E(U),\cup_R,\wedge)$ is a semiring. Further, $(F,A)\cup_R(G,B)=_M(G,B)\cup_R(F,A)$. That is to say, $\cup_R$ is commutative in $S_E(U)$ and $(F,A)\cup_R\emptyset_E=_M\emptyset_E\cup_R(F,A)=(F,A)$ and $(F,A)\wedge\emptyset_E=_L\emptyset_E\wedge(F,A)=_L\emptyset_E$. That is to say, $\emptyset_E$ is the zero element of $(S_E(U),\cup_R,\wedge)$. Therefore, $(S_A(U),\cup_R,\wedge)$ is a hemiring as regards soft L-equality (and hence J-equality). Besides, since $(F,A)\wedge(G,B)=_L(G,B)\wedge(F,A)$ and $(F,A)\wedge U_E=_L U_E\wedge(F,A)=_L(F,A)$, $(S_A(U),\cup_R,\wedge)$ is a commutative hemiring with identity U_E as regards soft L-equality (and hence J- equality).

Theorem 4.2. $(S_E(U),\Delta_R,\wedge)$ is a commutative hemiring with identity as regards soft L-equality (and hence J-equality).

Proof: In [32,49], it was shown that $(S_E(U),\Delta_R)$ is a commutative monoid with identiy $\emptyset_E$. Hence, we can deduce that $(S_E(U),\Delta_R)$ is a semigroup. $(S_E(U),\wedge)$ is a semgiroup in the sense of soft L-equality (hence J-equality). Furthermore, ∧ distributes over Δ_R from both sides. Therefore, $(S_E(U),\Delta,\wedge)$ is a semiring. Further, $(F,A)\Delta_R(G,B)=_M(G,B)\Delta_R(F,A)$. That is to say, Δ_R is commutative in $S_E(U)$ and $(F,A)\Delta_R\emptyset_E=_M\emptyset_E\Delta_R(F,A)=_M(F,A)$ and $(F,A)\wedge\emptyset_E=_L\emptyset_E\wedge(F,A)=_L\emptyset_E$. That is to say, $\emptyset_E$ is the zero element of $(S_E(U),\Delta_R,\wedge)$. Therefore, $(S_E(U),\Delta_R,\wedge)$ is a hemiring as regards soft L-equality (and hence J-equality). Besides, since $(F,A)\wedge(G,B)=_L(G,B)\wedge(F,A)$ and $(F,A)\wedge U_E=_L U_E\wedge(F,A)=_L(F,A)$, $(S_E(U),\Delta_R,\wedge)$ is a commutative hemiring with identity U_E as regards soft L-equality (and hence J-equality).

Distributions of AND-Product Over Extended SS Operations

In this subsection, we examine the distributions of AND-product over extended SS operations.

When considering the necessary condition for the soft M-equality of SSs, that is the equality of PSs, we can deduce that the distributions of extended SS operations over AND-product can not be examined, since the union operation does not distribute over cartesian product. Thus, the distribution of extended SS operations over AND-product is never satisfied.

Here it should be noted that in [39], the distributions of AND-product over extended intersection and extended union are obtained but not over extended difference and symmetric difference operations; moreover in [39], only left distributions of AND-product over extended intersection and extended union are obtained. Hence in this section, we handle the distributions of AND-product over extended SS operations completely.

Left-distributions of AND-product over extended SS operations

1) $(\Gamma,A)\wedge[(6,B)\cap_\varepsilon(Z,C)]=_M[(\Gamma,A)\wedge(6,B)]\cap_\varepsilon[(\Gamma,A)\wedge(Z,C)]$

Proof: Let's first check the equality of PSs of the SSs of both sides. The PS of the left hand side is Ax(B∪C), and the PS of the right hand side is (AxB)∪(AxC). Since Ax(B∪C)=(AxB)∪(AxC), the first condition of the soft M-equality is satisfied. Moreover, since B≠∅ and C≠∅ by the assumption, then B∪C≠∅. Thus, the PS of the SSs of both sides can never be ∅.

Now let's first cinsider the LS. Let $(6,B)\cap_\varepsilon(Z,C)=_M(M,B\cup C)$, where

$$M(\omega)=\begin{cases}Ϭ(\omega), & \omega\in B\setminus C\\ Z(\omega), & \omega\in C\setminus B\\ Ϭ(\omega)\cap Z(\omega), & \omega\in B\cap C\end{cases}$$

for ∀ω∈B∪C. Now let $(\Gamma,A)\wedge(M,B\cup C)=_M(N,Ax(B\cup C))$, where $N(\tau,\omega)=\Gamma(\tau)\cap M(\omega)$ for all $(\tau,\omega)\in Ax(B\cup C)$. (Here note that τ∈A and ω∈B∪C). Hence

$$N(\tau)=\begin{cases}\Gamma(\tau)\cap Ϭ(\omega), & (\tau,\omega)\in Ax(B\setminus C)\\ \Gamma(\tau)\cap Z(\omega), & (\tau,\omega)\in Ax(C\setminus B)\\ \Gamma(\tau)\cap[Ϭ(\omega)\cap Z(\omega)], & (\tau,\omega)\in Ax(B\cap C)\end{cases}$$

Now, let's consider the right side (RS). Let $(\Gamma,A)\wedge(6,B)=_M(K,AxB)$, where $K(\tau,\omega)=\Gamma(\tau)\cap 6(\omega)$ for all $(\tau,\omega)\in AxB$. And let $(\Gamma,A)\wedge(Z,C)=_M(T,AxC)$, where $T(\tau,\omega)=\Gamma(\tau)\cap Z(\omega)$ for all $(\tau,\omega)\in AxC$. Now let $(K,AxB)\cap_\varepsilon(T,AxC)=_M(L,(AxB)\cup(AxC))$, where

$$L(\tau,\omega)=\begin{cases}K(\tau,\omega), & (\tau,\omega)\in(AxB)\setminus(AxC)\\ T(\tau,\omega), & (\tau,\omega)\in(AxC)\setminus(AxB)\\ K(\tau,\omega)\cap T(\tau,\omega), & (\tau,\omega)\in(AxB)\cap(AxC)\end{cases}$$

for all $(\tau,\omega)\in(AxB)\cup(AxC)$. Hence,

$$L(\tau)=\begin{cases}\Gamma(\tau)\cap Ϭ(\omega), & (\tau,\omega)\in(AxB)\setminus(AxC)=Ax(B\setminus C)\\ \Gamma(\tau)\cap Z(\omega), & (\tau,\omega)\in(AxC)\setminus(AxB)=Ax(C\setminus B)\\ [\Gamma(\tau)\cap Ϭ(\omega)]\cap[\Gamma(\tau)\cap Z(\omega)], & (\tau,\omega)\in(AxB)\cap(AxC)=Ax(B\cap C)\end{cases}$$

Here note that, if ω∈B∪C, then ω∈B or ω∈C. Let check all the possible cases:

Case 1. Let ω∈B and ω∉C. Then $N(\tau,\omega)=\Gamma(\tau)\cap 6(\omega)$ and $L(\tau,\omega)=K(\tau,\omega)=\Gamma(\tau)\cap 6(\omega)$,

Case 2. Let ω∉B and ω∈C. Then $N(\tau,\omega)=\Gamma(\tau)\cap Z(\omega)$ and $L(\tau,\omega)=T(\tau,\omega)=\Gamma(\tau)\cap Z(\omega)$,

Case 3. Let ω∈B and ω∈C. Then $N(\tau,\omega)=\Gamma(\tau)\cap[6(\omega)\cap Z(\omega)]$ and $L(\tau,\omega)=K(\tau,\omega)\cap T(\tau,\omega)=[\Gamma(\tau)\cap 6(\omega)\cap[\Gamma(\tau)\cap Z(\omega)]=\Gamma(\tau)\cap[6(\omega)\cap Z(\omega)]$.

In all circumstances, N and L are the same set-valued mappings, so the proof is completed.

2) $(\Gamma,A)\wedge[(6,B)\cup_\varepsilon(Z,C)]=_M[(\Gamma,A)\wedge(6,B)]\cup_\varepsilon[(\Gamma,A)\wedge(Z,C)]$

3) $(\Gamma,A)\wedge[(6,B)\setminus_\varepsilon(Z,C)]=_M[(\Gamma,A)\wedge(6,B)]\setminus_\varepsilon[(\Gamma,A)\wedge(Z,C)]$

4) $(\Gamma,A)\wedge[(6,B)\Delta_\varepsilon(Z,C)]=_M[(\Gamma,A)\wedge(6,B)]\Delta_\varepsilon[(\Gamma,A)\wedge(Z,C)]$

Right-distributions of AND-product over extended SS operations

1) $[(\Gamma,A)\cup_\varepsilon(6,B)]\wedge(Z,C)=_M[(\Gamma,A)\wedge(Z,C)]\cup_\varepsilon[(6,B)\wedge(Z,C)]$

Proof: Let's first check the equality of PSs of the SSs of both sides. The PS of the left hand side is (A∪B)xC, and the PS of the right hand side is (AxC)∪(BxC). Since (A∪B)xC=(AxC)∪(BxC), the first condition of the soft M-equality is satisfied. Moreover, since A≠∅ and B≠∅ by assumption, then A∪B≠∅. Thus, the PS of the SSs of both sides can never be ∅.

Now let's first consider the LS. Let $(\Gamma,A)\cup_\varepsilon(6,B)=_M(M,A\cup B)$. Then, ∀τ∈A∪B;

$$M(\tau)=\begin{cases}\Gamma(\tau), & \tau\in A\setminus B\\ Ϭ(\tau), & \tau\in B\setminus A\\ \Gamma(\tau)\cup Ϭ(\tau), & \tau\in A\cap B\end{cases}$$

Now let $(M,A\cup B)\wedge(Z,C)=_M(N,(A\cup B)xC)$, where $N(\tau,\omega)=_M(\tau)\cap Z(\omega)$ for all $(\tau,\omega)\in(A\cup B)xC$. Here τ∈A∪B and ω∈C. Hence,

$$N(\tau,\omega)=\begin{cases}\Gamma(\tau)\cap Z(\omega), & (\tau,\omega)\in(A\setminus B)xC\\ Ϭ(\tau)\cap Z(\omega), & (\tau,\omega)\in(B\setminus A)xC\\ [\Gamma(\tau)\cup Ϭ(\tau)]\cap Z(\omega), & (\tau,\omega)\in(A\cap B)xC\end{cases}$$

Now let's consider the RS. Let $(\Gamma,A)\wedge(Z,C)=_M(K,AxC)$, where $K(\tau,\omega)=\Gamma(\tau)\cap Z(\omega)$ for all $(\tau,\omega)\in(AxC)$ and let $(6,B)\wedge(Z,C)=_M(T,BxC)$, where $T(\tau,\omega)=6(\tau)\cap Z(\omega)$. Hence, let $(K,AxC)\cup_\varepsilon(T,BxC)=_M(L,(AxC)\cup(BxC))$, where

$$L(\tau,\omega)=\begin{cases}K(\tau,\omega), & (\tau,\omega)\in(AxC)\setminus(BxC)\\ T(\tau,\omega), & (\tau,\omega)\in(BxC)\setminus(AxC)\\ K(\tau,\omega)\cup T(\tau,\omega), & (\tau,\omega)\in(AxC)\cap(BxC)\end{cases}$$

Thus,

$$L(\tau,\omega)=\begin{cases}\Gamma(\tau)\cap Z(\omega), & (\tau,\omega)\in(AxC)\setminus(BxC)=(A\setminus B)xC\\ Ϭ(\tau)\cap Z(\omega), & (\tau,\omega)\in(BxC)\setminus(AxC)=(B\setminus A)xC\\ [\Gamma(\tau)\cap Z(\omega)]\cup[Ϭ(\tau)\cap Z(\omega)], & (\tau,\omega)\in(AxC)\cap(BxC)=(A\cap B)xC\end{cases}$$

Here note that, if τ∈A∪B, then τ∈A or τ∈B. Let check all the possible cases:

Case 1. Let τ∈A and x∉B. Then $N(\tau,\omega)=\Gamma(\tau)\cap Z(\omega)$ and $L(\tau,\omega)=K(\tau,\omega)=\Gamma(\tau)\cap Z(\omega)$,

Case 2. Let x∉A and τ∈B. Then $N(\tau,\omega)=6(\tau)\cap Z(\omega)$ and $L(\tau,\omega)=T(\tau,\omega)=6(\tau)\cap Z(\omega)$,

Case 3. Let τ∈A and τ∈B. Then $N(\tau,\omega)=[\Gamma(\tau)\cup 6(\tau)]\cap Z(\omega)$, and $L(\tau,\omega)=K(\tau,\omega)\cup T(\tau,\omega)[\Gamma(\tau)\cap Z(\omega)]\cup[6(\tau)\cap Z(\omega)]=[\Gamma(\tau)\cup 6(\tau)]\cap Z(\omega)$.

In all circumstances, N and L are the same set-valued mappings, so the proof is completed.

2) $[(\Gamma,A)\cap_\varepsilon(6,B)]\wedge(Z,C)=_M[(\Gamma,A)\wedge(Z,C)]\cap_\varepsilon[(6,B)\wedge(Z,C)]$

3) $[(\Gamma,A)\setminus_\varepsilon(6,B)]\wedge(Z,C)=_M[(\Gamma,A)\wedge(Z,C)]\setminus_\varepsilon[(6,B)\wedge(Z,C)]$

4) $[(\Gamma,A)\Delta_\varepsilon(6,B)]\wedge(Z,C)=_M[(\Gamma,A)\wedge(Z,C)]\Delta_\varepsilon[(6,B)\wedge(Z,C)]$

Theorem 4.2.3. $(S_E(U),\cup_\varepsilon,\wedge)$ is a commutative hemiring with identity as regards soft L-equality (and hence J-equality).

Proof: In [31], it was shown that $(S_E(U),\cup_\varepsilon)$ is a commutative monoid with identiy $\emptyset_\emptyset$. Hence, we can deduce that ($S_E(U)$, $\cup_\varepsilon$) is a semigroup. ($S_E(U)$, ∧) is a semgiroup in the sense of soft L-equality (hence J-equality). Furthermore, ∧ distributes over $\cup_\varepsilon$ from both sides. Therefore, ($S_E(U)$, $\cup_\varepsilon$, ∧) is a semiring.

Further, $(F,A)\cup_\varepsilon(G,B)=_M(G,B)\cup_\varepsilon(F,A)$ by [31]. That is to say, $\cup_\varepsilon$ is commutative in $S_E(U)$ and $(F,A)\cup_\varepsilon\emptyset_\emptyset=_M\emptyset_\emptyset\cup_\varepsilon(F,A)=_M(F,A)$ by [31] and $(F,A)\wedge\emptyset_\emptyset=_M\emptyset_\emptyset\wedge(F,A)=_M\emptyset_\emptyset$. Namely, $\emptyset_\emptyset$ is the zero element of ($S_E(U)$, $\cup_\varepsilon$, ∧). Therefore, ($S_E(U)$, $\cup_\varepsilon$, ∧) is a hemiring as regards soft L-equality (and hence J-equality). Besides, since $(F,A)\wedge(G,B)=_L(G,B)\wedge(F,A)$ and $(F,A)\wedge U_E=_L U_E\wedge(F,A)=_L(F,A)$, ($S_E(U)$,$\cup_\varepsilon$,∧) is a commutative hemiring with identity U_E as regards soft L-equality (and hence J-equality).

Theorem 4.2.4 ($S_E(U)$, Δ_ε, ∧) is a commutative hemiring with identity element as regards soft L-equality (and hence J-equality).

Proof: In [48], it was shown that ($S_E(U)$, Δ_ε) is a commutative monoid with identiy $\emptyset_\emptyset$. Hence, we can deduce that ($S_E(U)$, Δ_ε) is a semigroup. ($S_E(U)$, ∧) is a semgiroup in the sense of soft L-equality (hence J-equality). Furthermore, ∧ distributes over Δ_ε from both sides. Therefore, ($S_E(U)$, Δ_ε, ∧) is a semiring.

Further, $(F,A)\Delta_\varepsilon(G,B)=_M(G,B)\Delta_\varepsilon(F,A)$ by [48]. That is to say, Δ_ε is commutative in $S_E(U)$ and $(F,A)\Delta_\varepsilon\emptyset_\emptyset=_M\emptyset_\emptyset\Delta_\varepsilon(F,A)=_M(F,A)$ by [48] and $(F,A)\wedge\emptyset_\emptyset=_M\emptyset_\emptyset\wedge(F,A)=_M\emptyset_\emptyset$. Namely, $\emptyset_\emptyset$ is the zero element of ($S_E(U)$,Δ_ε,∧). Therefore, ($S_E(U)$,Δ_ε,∧) is a hemiring as regards soft L-equality (and hence J-equality). Besides, since $(F,A)\wedge(G,B)=_L(G,B)\wedge(F,A)$ and $(F,A)\wedge U_E=_L U_E\wedge(F,A)=_L(F,A)$, ($S_E(U)$,$\Delta_\varepsilon$,∧) is a commutative hemiring with identity U_E as regards soft L-equality (and hence J-equality).

Distributions of AND-Product Over Soft Binary Piecewise Operations

In this subsection, we examine the distributions of AND-product over soft binary piecewise operations.

Left-distributions of AND-product over soft binary piecewise operations

1) $(\Gamma,A)\wedge[(б,B)\tilde{\cap}(Z,C)]=_M[(\Gamma,A)\wedge(б,B)]\tilde{\cap}[(\Gamma,A)\wedge(Z,C)]$

Proof: Let's first check the equality of PSs of the SSs of both sides. Here since the PS of the SSs of both sides is AxB, the first condition for the soft M-equality is satisfied. Moreover, by assumption since A≠ ∅ and B≠ ∅, then AxB≠∅. Thus, the PS of the SSs of both sides can never be ∅. Let's first consider the LS. Let $(б,B)\tilde{\cap}(Z,C)=_M(M,B)$, where

$$M(\omega)=\begin{cases} б(\omega), & \omega\in B\setminus C \\ б(\omega)\cap Z(\omega), & \omega\in B\cap C \end{cases}$$

for all ∀ω∈B. Now let $(\Gamma,A)\wedge(M,B)=_M(N,AxB)$, where $N(\tau,\omega)=\Gamma(\tau)\cap M(\omega)$ for all $(\tau,\omega)\in AxB$. Hence,

$$N(\tau)=\begin{cases} \Gamma(\tau)\cap б(\omega) & (\tau,\omega)\in Ax(B\setminus C) \\ \Gamma(\tau)\cap[б(\omega)\cap Z(\omega)] & (\tau,\omega)\in Ax(B\cap C) \end{cases}$$

Now, let's consider the RS. Let $(\Gamma,A)\wedge(б,B)=_M(K,AxB)$, where $K(\tau,\omega)=\Gamma(\tau)\cap б(\omega)$ for all $(\tau,\omega)\in AxB$.

And let $(\Gamma,A)\wedge(Z,C)=_M(T,AxC)$, where $T(\tau,\omega)=\Gamma(\tau)\cap Z(\omega)$ for all $(\tau,\omega)\in AxC$. Now let $(K,AxB)\tilde{\cap}(T,AxC)=(L,(AxB))$, where

$$L(\tau,\omega)=\begin{cases} K(\tau,\omega), & (\tau,\omega)\in(AxB)\setminus(AxC) \\ K(\tau,\omega)\cap T(\tau,\omega), & (\tau,\omega)\in(AxB)\cap(AxC) \end{cases}$$

for all $(\tau,\omega)\in(AxB)$. Hence,

$$L(\tau)=\begin{cases} \Gamma(\tau)\cap б(\omega), & (\tau,\omega)\in(AxB)\setminus(AxC)=Ax(B\setminus C) \\ [\Gamma(\tau)\cap б(\omega)]\cap[\Gamma(\tau)\cap Z(\omega)], & (\tau,\omega)\in(AxB)\cap(AxC)=Ax(B\cap C) \end{cases}$$

Since N and L are the same set-valued mappings, the proof is completed. Here note that when we consider the necessary condition for the M-equality of SSs, that is the eqality of PSs, we can not examine the left distributions of soft binary operations over AND-product, since A≠AxA. Thus, the left distribution of soft binary piecewise operations over AND-product is never satisfied.

2) $(\Gamma,A)\wedge[(б,B)\tilde{\cup}(Z,C)]=_M[(\Gamma,A)\wedge(б,B)]\tilde{\cup}[(\Gamma,A)\wedge(Z,C)]$

3) $(\Gamma,A)\wedge[(б,B)\tilde{\setminus}(Z,C)]=_M[(\Gamma,A)\wedge(б,B)]\tilde{\setminus}[(\Gamma,A)\wedge(Z,C)]$

4) $(\Gamma,A)\wedge[(б,B)\tilde{\Delta}(Z,C)]=_M[(\Gamma,A)\wedge(б,B)]\tilde{\Delta}[(\Gamma,A)\wedge(Z,C)]$

Right-distributions of AND-product over soft binary piecewise operations

1) $[(\Gamma,A)\tilde{\cup}(б,B)]\wedge(Z,C)=_M[(\Gamma,A)\wedge(Z,C)]\tilde{\cup}[(б,B)\wedge(Z,C)]$

Proof: Let's first check the equality of PSs of the SSs of both sides. Here since the PS of the SS of both sides is AxC, the first condition for the soft M-equality is satisfied. Moreover, since A≠∅ and C≠∅ by assumption, then AxC≠∅. Thus, the PS of the SSs of both sides can never be ∅. Let's first consider the LS. Let $(\Gamma,A)\tilde{\cup}(б,B)=_M(M,A)$, where

$$M(\tau)=\begin{cases} \Gamma(\tau), & \tau\in A\setminus B \\ \Gamma(\tau)\cup б(\tau), & \tau\in A\cap B \end{cases}$$

for all ∀τ∈A. Now let $(M,A)\wedge(Z,C)=_M(N,AxC)$, where $N(\tau,\omega)=M(\tau)\cap Z(\omega)$ for all $(\tau,\omega)\in AxC$. Hence,

$$N(\tau,\omega)=\begin{cases} \Gamma(\tau)\cap Z(\omega), & (\tau,\omega)\in(A\setminus B)xC \\ [\Gamma(\tau)\cup б(\tau)\cap Z(\omega), & (\tau,\omega)\in(A\cap B)xC \end{cases}$$

Now let's consider the RS. Let $(\Gamma,A)\wedge(Z,C)=_M(K,AxC)$, where $K(\tau,\omega)=\Gamma(\tau)\cap Z(\omega)$ for all $(\tau,\omega)\in(AxC)$ and let $(б,B)\wedge(Z,C)=_M(T,BxC)$, where $T(\tau,\omega)=б(\tau)\cap Z(\omega)$. Hence, let $(K,AxC)\tilde{\cup}(T,BxC)=_M(L,AxC)$, where

$$L(\tau,\omega)=\begin{cases} K(\tau,\omega), & (\tau,\omega)\in(AxC)\setminus(BxC) \\ K(\tau,\omega)\cup T(\tau,\omega), & (\tau,\omega)\in(AxC)\cap(BxC) \end{cases}$$

Thus,

$$L(\tau,\omega)=\begin{cases}\Gamma(\tau)\cap Z(\omega), & (\tau,\omega)\in(AxC)\setminus(BxC)=(A\setminus B)xC\\ [\Gamma(\tau)\cap Z(\omega)]\cup[\text{Ϭ}(\tau)\cap Z(\omega)], & (\tau,\omega)\in(AxC)\cap(BxC)=(A\cap B)xC\end{cases}$$

Since N and L are the same set-valued mappings, the proof is completed.

2) $[(\Gamma,A)\tilde{\cap}(\text{Ϭ},B)]\wedge(Z,C)=_M[(\Gamma,A)\wedge(Z,C)]\tilde{\cap}[(\text{Ϭ},B)\wedge(Z,C)]$

3) $[(\Gamma,A)\tilde{\setminus}(\text{Ϭ},B)]\wedge(Z,C)=_M[(\Gamma,A)\wedge(Z,C)]\tilde{\setminus}[(\text{Ϭ},B)\wedge(Z,C)]$

4) $[(\Gamma,A)\tilde{\Delta}(\text{Ϭ},B)]\wedge(Z,C)=_M[(\Gamma,A)\wedge(Z,C)]\tilde{\Delta}[(\text{Ϭ},B)\wedge(Z,C)]$

Note that, since soft binary piecewise operations are non-associative operations in the set $S_E(U)$ by [49], they can not form a semigroup in the set $S_E(U)$. Moreover, since AND-product is not closed in the set $S_A(U)$, $(S_A(U), \wedge)$ can not be a semigroup even in the sence of soft L-equality. Hence, we have not investigated the soft algabraic structures as regards soft binary piecewise operations neither in the set $S_E(U)$ nor in the set $S_A(U)$.

CONCLUSION

AND-product, a crucial concept of soft set theory as regards decision making, were investigated by different authors concerning different kinds of soft equalities such as soft L-equality and soft J-equality. In this paper, however, we have investigated the AND-product and its whole algebraic properties such as commutative laws, associative laws, idempotent laws and other all basic properties in detail as regards soft F-subsets and soft M-equality which is the strictest type of soft equality. We have also compared our obtained properties with the formerly obtained properties as regards soft L-equality and soft J-equality. Also, by handling some new results related to distributive properties of AND-product over other type of soft set operations, we complete the results concerning AND-product in the literature totally by proving that the set of all soft sets over U together with restricted/extended union and AND-product is a commutative hemiring with identity as restricted/extended symmteric difference and AND-product in the sense of soft L-equality. Since studying the algebraic structure of soft sets from the perspective of operations provides deep insight into the potential applications of soft sets in classical and nonclassical logic, we are of the opinion that this paper contibutes to the literature of soft set in this regard. Furthermore, as soft sets are a powerful mathematical tool for recognizing uncertain objects, and the theoretical foundations of soft computing approaches are derived from purely mathematical principles, this research will lay the groundwork for a wide range of applications, including new decision-making approaches and innovative cryptography techniques based on soft sets. To continue on this research, studies on OR-product and its essential properties in respect to other types of soft equal connections can be handled in depth not only as regards soft sets but only bipolar soft sets in order to contribute to the literature from the theoretical views and hence application aspects.

AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

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