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Research Article

Free vibration and buckling analysis of functionally graded sandwich beams resting on a two-parameter elastic foundation using a quasi-3D theory

Ibrahim MOHAMED[1], Sebahat ŞİMŞEK[1], Volkan KAHYA[1,*]

[1]Department of Civil Engineering, Faculty of Engineering, Karadeniz Technical University, Trabzon, 61080, Türkiye

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ABSTRACT

This study examines the free vibration and buckling behavior of functionally graded (FG) sandwich beams supported by a Winkler-Pasternak elastic foundation, utilizing a quasi-3D deformation theory. The material properties of the FG sandwich beams are modeled to vary continuously through the thickness according to a power-law distribution. Using Hamilton's principle, the governing equations of motion are derived. Analytical solutions are obtained for simply supported FG sandwich beams with homogeneous cores by employing Navier's method. The accuracy of the proposed model is demonstrated by comparing the current results with the higher-order deformation theories-based solutions available in literature. A comprehensive parametric study is also carried out to explore the effect of the skin-core-skin thickness ratio, the power-law index, beam span-to-depth ratio, normal strain, core material, and elastic foundation on fundamental natural frequencies and critical buckling loads.

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INTRODUCTION

The necessity to discover or invent new materials has significantly increased with the advancement of knowledge and technology, along with the evolution of materials from monolithic to the emergence of advanced composite materials. A composite material is a type of advanced material consisting of two or more different materials with significantly distinct properties that benefit each part's superior characteristics [1].

Functionally Graded Materials (FGMs) represent a class of advanced composite materials distinguished by their gradual variation in properties across a specific direction. Unlike traditional composites, FGMs eliminate distinct boundaries between constituent regions, replacing them with a smooth gradient transition [2]. This unique feature provides FGMs with a combination of the desirable properties of their components, such as thermal resistance, wear resistance, and corrosion resistance of ceramics, along with the toughness and mechanical strength of metals. Commonly composed of ceramic and metallic phases,

***Corresponding author.**
*E-mail address: volkan@ktu.edu.tr

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FGMs are particularly suitable for high-temperature conditions and precision-demanding applications [3]. FGMs are employed in various structural forms, including beams, plates, and shells, across diverse engineering sectors such as aerospace, automotive, and civil engineering, offering enhanced durability and performance.

Over the past few decades, FG sandwich beams have been extensively studied to emphasize their behaviors, according to the recent reviews by Sayyad and Ghugal [4] and Aman et al. [5]. Birman and Kardomatea [6] presented an extensive review of the theoretical frameworks used for analyzing sandwich structures, which are primarily categorized into classical beam theory, first-order shear deformation theory, and higher-order shear deformation theories. The classical beam theory (CBT), also known as the Euler-Bernoulli beam theory, represents the most straightforward approach to beam analysis [7]. This theory has been widely adopted by researchers to investigate the free vibration, buckling, and bending behavior of FG beams, as documented in numerous studies [8–15]. Despite its simplicity, CBT does not account for transverse shear deformation, making it applicable only to slender beams.

The first-order shear deformation theory (FSDT), introduced by Timoshenko in 1921, addresses the limitations of classical beam theory by incorporating the influence of shear deformation. This enhancement enables FSDT to deliver more accurate predictions for thick beams, where the assumptions of classical beam theory are insufficient. Many investigations have employed the FSDT to explore the dynamic, buckling, and static behaviors of FG beams, as documented in various research works [16–22]. Kahya and Turan [23] developed a finite element (FE) model for the buckling and vibration analysis of FG beams using FSDT. Turan et al. [24] employed the Ritz method, finite element analysis (FE), and artificial neural networks (ANNs) based on the first-order shear deformation theory (FSDT) to study the free vibration and buckling behavior of FG porous beams under different boundary conditions. Additionally, Turan and Kahya analyzed the free vibration and buckling characteristics of FG sandwich beams, including those with homogeneous ceramic cores and FG cores, using the Navier method in conjunction with FSDT [25]. It is worth noting that FSDT requires appropriate shear correction factors to accurately capture the effects of transverse shear deformation.

Higher-order shear deformation theories (HSDTs) have been developed to eliminate the need for shear correction factors while accurately accounting for transverse shear deformation. These theories utilize polynomial or non-polynomial shape functions to describe the displacement field [26–34] Reddy [35] introduced a third-order polynomial shear deformation theory for analyzing isotropic and anisotropic composite structures. Sayyad and Ghugal [36] proposed a modified exponential shear deformation theory for studying the free vibration, buckling, and bending behaviors of exponential FG beams under various boundary conditions. Avcar et al. [37] applied HSDT to examine the natural frequencies of sigmoid FG sandwich beams. Ramteke et al. [38] utilized finite element (FE) solutions based on HSDT for the static analysis of FG structures with variable grading patterns and porosity effects. Derikvand et al. [39] investigated the buckling behavior of FG sandwich beams with porous ceramic cores using third-order shear deformation theory. Ramteke and Panda [40] explored the free vibration frequencies of multi-directional FG structures, considering the effects of variable grading and porosity distributions with HSDT. Nguyen et al. [41] introduced a hyperbolic HSDT for evaluating the buckling and free vibration characteristics of isotropic and FG sandwich beams under various boundary conditions. Vo et al. [42] proposed an FE model based on a refined parabolic shear deformation theory for analyzing the vibration and buckling properties of FG sandwich beams.

Quasi-3D theories have been introduced as an extension of HSDTs to better capture the behavior of FG sandwich beams, particularly by incorporating the effects of transverse normal stress. These theories account for thickness-stretching effects in the transverse displacement through higher-order shear shape functions, enabling more precise predictions. Sayyad and Ghumare [43] developed analytical solutions for bending and buckling analysis of FG beams using a fifth-order shear and normal deformation theory. Bennai et al. [44] proposed a novel higher-order shear and normal deformation theory for studying the free vibration and buckling of FG sandwich beams under various boundary conditions. Sayyad and Shinde [45] applied a quasi-3D polynomial shear and normal deformation theory to analyze the bending behavior of laminated composite and FG sandwich beams. Nguyen et al. [46] introduced a Ritz-based quasi-3D solution for the free vibration and buckling analysis of FG sandwich beams under diverse boundary conditions. Karamanli and Aydogdu [47] utilized the quasi-3D theory in conjunction with the Ritz method to investigate the free vibration and buckling characteristics of laminated composite and sandwich microbeams with arbitrary boundary conditions. Karamanli Karamanli [48] examined the free vibration and buckling behaviors of two-directional FG beams using the Ritz method and quasi-3D theory. Vo et al. [49,50] applied a quasi-3D theory for the buckling and free vibration analysis of FG sandwich beams using both finite element (FEM) and Navier methods. Osofero et al. [51] developed Navier-based solutions for bending, buckling, and free vibration analyses of FG sandwich beams employing non-polynomial quasi-3D theories.

Various models have been developed to describe the interaction between beams and elastic foundations, with the Winkler and Pasternak models being among the most commonly used. The Winkler model simplifies the foundation as a series of independent vertical springs [52], assuming that the foundation behaves elastically, and the beam's deflection is proportional to the applied load. The Pasternak model enhances this by introducing additional shear springs to account for the shear interaction between adjacent supports [53], enabling the foundation to exhibit non-linear behavior and coupling effects between vertical and horizontal deformations. Thi [54] conducted bending, buckling, and free vibration analyses of FG sandwich curved beams on Pasternak foundations using an analytical method and FSDT. Zenkour et al. [55] performed the buckling analysis of size-dependent FG nanobeams on a two-parameter elastic foundation via third-order shear deformation theory. Mohammed et al. [56] investigated the bending and buckling behaviors of FG Euler-Bernoulli beams resting on Winkler-Pasternak foundations.

Songsuwan et al. [57] studied the free vibration and dynamic response of FG sandwich Timoshenko beams subjected to a moving harmonic load on an elastic foundation. Hung and Truong [58] analyzed the free vibration of sandwich beams with FG porous cores supported by a Winkler foundation, using different shear deformation theories. Fahsi et al. [59] proposed a refined quasi-3D theory for free vibration, bending, and buckling analyses of FG porous beams on elastic foundations. Atmane et al. [60] extended quasi-3D theory to evaluate the effects of porosity on the vibration, bending, and buckling behavior of FG beams resting on a two-parameter elastic foundation.

The review of existing literature highlights that most studies focus on single-layered FG beams and shear deformation theories. To the best of the authors' knowledge, no research has specifically addressed the effects of elastic foundations on the free vibration and buckling behaviors of FG sandwich beams while accounting for both shear and normal deformations. Moreover, there is a notable gap in studies exploring the free vibration and buckling characteristics of FG sandwich beams with soft cores using higher-order shear and normal deformation theories. To address this gap, the primary objective of this paper is to analyze the free vibration and buckling behaviors of symmetric FG sandwich beams with homogeneous cores (both hardcore and softcore) resting on a two-parameter Winkler-Pasternak elastic foundation, employing a quasi-3D theory. This work also aims to provide benchmark results for the fundamental natural frequencies and critical buckling loads of FG sandwich beams with soft cores. The material properties of the beams are assumed to vary continuously through the thickness following a power-law distribution. Analytical solutions for simply supported FG sandwich beams are derived using Navier's method. Extensive numerical studies have been conducted, and the nondimensional results are validated by comparison with other higher-order theories reported in the literature to confirm the accuracy and convergence of the proposed model. Furthermore, a comprehensive parametric analysis is performed to examine the influence of factors such as the skin-core-skin thickness ratio, power-law index, span-to-depth ratio, normal strain, and elastic foundation parameters on the fundamental natural frequencies and critical buckling loads of FG sandwich beams.

PROBLEM

Geometrical Configuration

Consider a three-layered FG sandwich beam, where the face layers are made of a mixture of ceramic and metal, and the core is an isotropic homogeneous material, as depicted in Figure 1. The beam has a length L and the overall thickness h and width b, with the width normalized to unity. The top and bottom face layers are positioned at $z = \pm h/2$ The beam is assumed to be supported by a two-parameter elastic foundation, which includes Winkler and Pasternak's shear layer springs with constants k_w and k_p, respectively. As shown in Figure 1, the homogeneous core can either be ceramic (hardcore) or metal (softcore). The face layers of the first type are graded from metal to ceramic, while in the second type, they are graded from ceramic to metal.

Material Properties

The material properties of FG sandwich beams are distributed progressively and smoothly across the thickness direction, following a power-law variation:

$$\begin{aligned} E(z) &= E_m + (E_c - E_m)V_c(z) \\ \rho(z) &= \rho_m + (\rho_c - \rho_m)V_c(z) \end{aligned} \tag{1}$$

for homogeneous hardcore, and

$$\begin{aligned} E(z) &= E_c + (E_m - E_c)V_c(z) \\ \rho(z) &= \rho_c + (\rho_m - \rho_c)V_c(z) \end{aligned} \tag{2}$$

for homogeneous softcore. In Eqs. (1) and (2), $E(z)$ is the modulus of elasticity and $\rho(z)$ is the density of the material. Here, the subscripts m and c represent the metallic and ceramic components, respectively. The volume fraction of the FG sandwich beam is described by a power-law function along the thickness direction, defined as:

$$\begin{aligned} V_c(z) &= \left(\frac{z-h_0}{h_1-h_0}\right)^p \text{ for } z \in [h_0, h_1] \\ V_c(z) &= 1 \text{ for } z \in [h_1, h_2] \\ V_c(z) &= \left(\frac{z-h_3}{h_2-h_3}\right)^p \text{ for } z \in [h_2, h_3] \end{aligned} \tag{3}$$

where $V_c(z)$ is the volume fraction p is the power-law index.

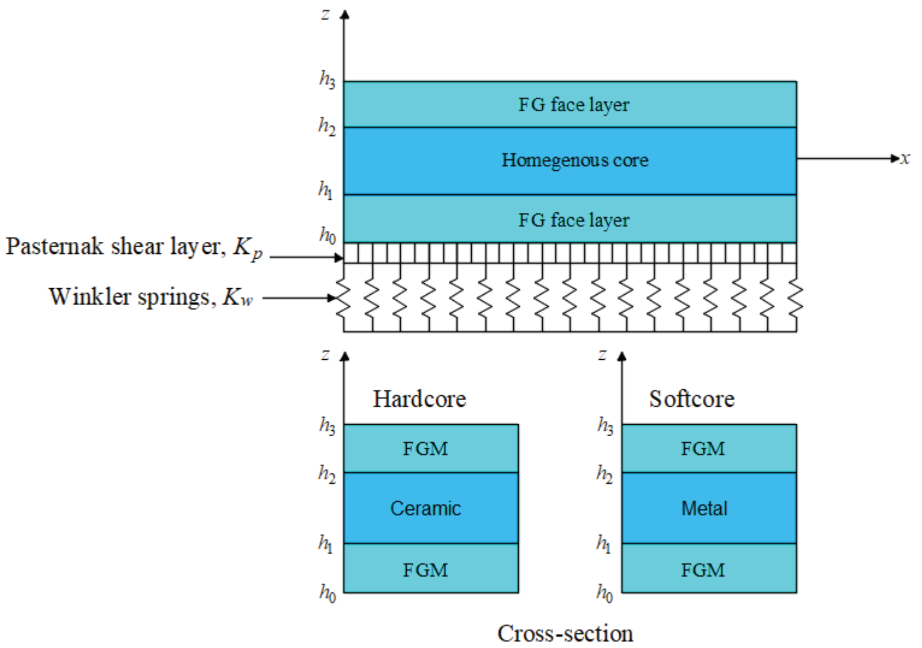

Figure 1. Geometry and dimensions of FG sandwich beam resting on two-parameter elastic foundation

THEORETICAL FORMULATION

Kinematics

The displacement field of the present quasi-3D theory is given as follows [46]:

$$\begin{aligned} u(x,z,t) &= u_0(x) - z\frac{\partial w_o}{\partial x} + f(z)\psi_x(x) \\ w(x,z,t) &= w_0(x) + g(z)\psi_z(x) \end{aligned} \tag{4}$$

where u and w denote the displacements of a generic point within the FG sandwich beam along the x- and z--axes, respectively. The variables u_0 and w_0 represent the displacements at the beam's mid-line, while ψ_x and ψ_z correspond to the shear slopes associated with transverse shear and normal deformations. Here, $g(z) = f'(z)$, and the shear shape function $f(z)$ is chosen as follows [35]:

$$f = z - \frac{4z^3}{3h^2} \tag{5}$$

The strain field is derived using the strain-displacement relationships from elasticity theory and can be written as:

$$\varepsilon_x = \varepsilon_x^0 - zk_x + f\varepsilon_x^1,\ \ \varepsilon_z = g'(z)\psi_z(x),\ \ \gamma_{xz} = g(z)\gamma_{xz}^0 \tag{6}$$

where

$$\varepsilon_x^0 = \frac{\partial u_0}{\partial x},\ k_x = \frac{\partial^2 w_0}{\partial x^2},\ \varepsilon_x^1 = \frac{\partial \psi_x}{\partial x},\ \gamma_{xz}^0 = \psi_x + \frac{\partial \psi_z}{\partial x} \tag{7}$$

The stress-strain relationship of the FG sandwich beam is given as the following:

$$\begin{Bmatrix} \sigma_x \\ \sigma_z \\ \tau_{xz} \end{Bmatrix} = \begin{bmatrix} Q_{11}(z) & Q_{13}(z) & 0 \\ Q_{13}(z) & Q_{33}(z) & 0 \\ 0 & 0 & Q_{55}(z) \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_z \\ \gamma_{xz} \end{Bmatrix} \tag{8}$$

where

$$Q_{11}(z) = Q_{33}(z) = \frac{E(z)}{1-\upsilon^2}, Q_{13}(z) = \frac{\upsilon E(z)}{1-\upsilon^2}, Q_{55}(z) = \frac{E(z)}{2(1+\upsilon)} \tag{9}$$

where v is Poisson's ratio.

Equation of Motion

The governing differential equations of the proposed theory are derived by applying Hamilton's principle, which can be expressed as follows:

$$\int_0^T (\delta U + \delta U_F + \delta V - \delta K)dt = 0 \tag{10}$$

where the symbol δ denotes the variational operator and U, U_F, V, and K represent the strain energy, additional strain energy induced by the elastic foundations, potential energy, and kinetic energy, respectively. The variation of the strain energy of the beam can be expressed as follows:

$$\delta U=\int_0^L\int_{-h/2}^{h/2}(\sigma_x\delta\varepsilon_x+\sigma_z\delta\varepsilon_z+\tau_{xz}\delta\gamma_{xz})dzdx$$
$$=\int_0^L\left(N_x\frac{\partial\delta u_0}{\partial x}-M_x^b\frac{\partial^2\delta w_0}{\partial x^2}+M_x^s\frac{\partial\delta\psi_x}{\partial x}+Q_z^s\delta\psi_z\right.\quad(11)$$
$$\left.+Q_{xz}\delta\psi_x+Q_{xz}\frac{\partial\delta\psi_z}{\partial x}\right)dx$$

where $N_x, M_x^b, M_x^s, Q_{xz}$ and Q_z^s are force and moment resultants defined by

$$[N_x, M_x^b, M_x^s]=\int_{-h/2}^{h/2}\sigma_x[1,z,f(z)]dz$$
$$Q_{xz}=\int_{-h/2}^{h/2}\tau_{xz}g(z)dz,\quad Q_z^s=\int_{-h/2}^{h/2}\sigma_z g'(z)dz \tag{12}$$

Substituting Eqs. (4), (5), (7), and (8) into Eq. (12) yields

$$N_x=A_{11}\frac{\partial u_0}{\partial x}-B_{11}\frac{\partial^2 w_0}{\partial x^2}+C_{11}\frac{\partial\psi_x}{\partial x}+D_{13}\psi_z$$
$$M_x^b=B_{11}\frac{\partial u_0}{\partial x}-A_{S11}\frac{\partial^2 w_0}{\partial x^2}+B_{S11}\frac{\partial\psi_x}{\partial x}+C_{S13}\psi_z$$
$$M_x^s=C_{11}\frac{\partial u_0}{\partial x}-B_{S11}\frac{\partial^2 w_0}{\partial x^2}+D_{S11}\frac{\partial\psi_x}{\partial x}+E_{13}\psi_z \tag{13}$$
$$Q_z^s=D_{13}\frac{\partial u_0}{\partial x}-C_{S13}\frac{\partial^2 w_0}{\partial x^2}+E_{13}\frac{\partial\psi_x}{\partial x}+F_{33}\psi_z$$
$$Q_{xz}=E_{55}\left(\psi_x+\frac{\partial\psi_z}{\partial x}\right)$$

where

$$[A_{11},B_{11},C_{11},A_{S11},B_{S11},D_{S11},F_{33}]=$$
$$\int_{-h/2}^{h/2}Q_{11}(z)[1,z,f(z),z^2,zf(z),f^2(z),[g'(z)]^2]dz$$
$$[D_{13,}E_{13},C_{S13}]=\int_{-h/2}^{h/2}Q_{13}(z)g'(z)[1,f(z),z]dz \tag{14}$$
$$E_{S55}=\int_{-h/2}^{h/2}Q_{55}(z)[g(z)]^2dz$$

The variation of the kinetic energy can be defined as

$$\delta K=\int_0^L\int_{-h/2}^{h/2}\rho(z)\left(\frac{\partial u}{\partial \mathrm{t}}\frac{\partial\delta u}{\partial \mathrm{t}}+\frac{\partial w}{\partial \mathrm{t}}\frac{\partial\delta w}{\partial \mathrm{t}}\right)dzdx$$
$$=\int_0^L\left(I_1\frac{\partial^2 u_0}{\partial t^2}-I_2\frac{\partial^3 w_0}{\partial x\,\partial t^2}+I_4\frac{\partial^2\psi_x}{\partial t^2}\right)\delta u_0 dx$$
$$+\int_0^L\left(-I_2\frac{\partial^2 u_0}{\partial t^2}-I_3\frac{\partial^3 w_0}{\partial x\,\partial t^2}-I_5\frac{\partial^2\psi_x}{\partial t^2}\right)\frac{\partial\delta w_o}{\partial x}dx$$
$$+\int_0^L\left(I_4\frac{\partial^2 u_0}{\partial t^2}-I_5\frac{\partial^3 w_0}{\partial x\,\partial t^2}+I_6\frac{\partial^2\psi_x}{\partial t^2}\right)\delta\psi_z dx \tag{15}$$
$$+\int_0^L\left(I_1\frac{\partial^2 w_0}{\partial t^2}+I_7\frac{\partial^2\psi_z}{\partial t^2}\right)\delta w_0 dx$$
$$+\int_0^L\left(I_7\frac{\partial^2 w_0}{\partial t^2}+I_8\frac{\partial^2\psi_z}{\partial t^2}\right)\delta\psi_z dx$$

where

$$[I_1,I_2,I_3,I_4,I_5,I_6,I_7,I_8]=$$
$$\int_{-h/2}^{h/2}\rho(z)[1,z,z^2,f(z),zf(z),f^2(z),g(z),g^2(z)]dz \tag{16}$$

The variation of potential energy due to external axial force can be written as

$$\delta V=\int_0^L\left[\left(N_0\frac{\partial^2 w_0}{\partial x^2}\right)\delta w_0\right]dx \tag{17}$$

where N_0 is the axial force.

The variation of strain energy induced by the elastic foundation can be expressed as

$$\delta U_F=-\int_0^L\left[\left(k_{\mathrm{w}}w-k_p\frac{\partial^2 w}{\partial x^2}\right)\delta w\right]dx \tag{18}$$

where k_w and k_p are the constants of Winkler and shear layer springs.

Substituting Eqs. (11), (15), (17), and (18) into Eq. (10), performing integration by parts, collecting the coefficients of the unknown displacement variables (δu_0, δw_0, $\delta\psi_x$, $\delta\psi_z$), and setting them equal to zero, the following equations of motion can be obtained:

$$\delta u_0:\ \frac{\partial N_x}{\partial x}=I_1\frac{\partial^2 u_0}{\partial t^2}-I_2\frac{\partial^3 w_0}{\partial t^2\partial x}+I_4\frac{\partial^2\psi_x}{\partial t^2}$$
$$\delta w_0:\ \frac{\partial^2 M_x^b}{\partial x^2}+N_0\frac{\partial^2 w_0}{\partial x^2}-k_w(w_0+g\psi_z)+k_p\frac{\partial^2(w_0+g\psi_z)}{\partial x^2}$$
$$=I_2\frac{\partial^3 u_0}{\partial t^2\partial x}-I_3\frac{\partial^4 w_0}{\partial t^2\partial x^2}+I_5\frac{\partial^3\psi_x}{\partial t^2\partial x}+I_1\frac{\partial^2 w_0}{\partial t^2}+I_7\frac{\partial^2\psi_z}{\partial t^2}$$
$$\delta\psi_x:\ \frac{\partial M_x^s}{\partial x}-Q_{xz}=I_4\frac{\partial^2 u_0}{\partial t^2}-I_5\frac{\partial^3 w_0}{\partial t^2\partial x}+I_6\frac{\partial^2\psi_x}{\partial t^2} \tag{19}$$
$$\delta\psi_z:\ \frac{\partial Q_{xz}}{\partial x}+g\left(-k_w(w_0+g\psi_z)+k_p\frac{d^2(w_0+g\psi_z)}{dx^2}\right)-Q_z^s$$
$$=I_7\frac{\partial^2 w_0}{\partial t^2}+I_8\frac{\partial^2\psi_z}{\partial t^2}$$

Analytical Solution

Analytical solutions for free vibration and buckling analysis of simply supported FG beams on an elastic foundation are derived using Navier's method. In this approach, the unknown displacement variables are expressed as [45]:

$$u_0(x,t)=\sum_{m=1}^{\infty}U_m\cos(\alpha x)e^{i\omega t}$$
$$w_0(x,t)=\sum_{m=1}^{\infty}W_m\sin(\alpha x)\,e^{i\omega t}$$
$$\psi_x(x,t)=\sum_{m=1}^{\infty}\psi_{xm}\cos(\alpha x)\,e^{i\omega t} \tag{20}$$
$$\psi_z(x,t)=\sum_{m=1}^{\infty}\psi_{zm}\sin(\alpha x)e^{i\omega t}$$

where U_m, W_m, ψ_{xm} and ψ_{zm} are unknown coefficients, ω refers to the natural frequency of the beam, $\alpha = m\pi/L$ is a nondimensional parameter, m is a positive integer, which is taken as $m = 1$, and $\sqrt{i} = -1$ represents the imaginary unit. Substituting Eqs. (20) into Eqs. (19), the following matrix equations are obtained:

$$[\mathbf{K} - \omega^2\mathbf{M}]\boldsymbol{\Delta} = \mathbf{0} \quad (21)$$

for free vibration and

$$[\mathbf{K} - N_0\mathbf{G}]\boldsymbol{\Delta} = \mathbf{0} \quad (22)$$

For buckling. Here, **K** denotes the stiffness matrix, **M** represents the mass matrix, **G** is the geometric matrix, and **Δ** is the vector of unknown coefficients. Detailed expressions for the components of these matrices are provided in the Appendix.

NUMERICAL RESULTS AND DISCUSSION

This section provides numerical examples and discusses their results to validate the accuracy of the proposed study. It also examines the influence of the elastic foundation on the fundamental natural frequencies and critical buckling loads of FG sandwich beams. The FG layers of the beams are assumed to consist of a mixture of Alumina (Al_2O_3) and Aluminum (Al),), while the core layer is modeled as a homogeneous material, considering both hardcore and softcore configurations. A comprehensive parametric study is conducted to analyze the effects of the power-law index, span-to-depth ratio, skin-to-core thickness ratio, and elastic foundation parameters on the free vibration and buckling behaviors. The material properties utilized in this study are as follows: E_c = 380 GPa, ρ_c = 3960 kg/m^3, ν = 0.3 for ceramic material and E_m = 70 GPa, ρ_m = 2702 kg/m^3, ν = 0.3 for metal material. For simplicity, the fundamental natural frequency, critical buckling load, and elastic foundation parameters are, respectively, defined in the following non-dimensional forms:

$$\bar{\omega} = \frac{\omega L^2}{h}\sqrt{\frac{\rho_m}{E_m}},\ \bar{N}_{cr} = \frac{12N_{cr}L^2}{E_m h^3},\ \xi_w = \frac{k_w\, L^2}{E_m h},\ \xi_p = \frac{k_p}{E_m h} \quad (23)$$

Effect of the Power-Law Index and Skin-Core-Skin Thickness Ratio

To validate the accuracy of the proposed quasi-3D theory, free vibration and buckling analyses of various types of simply supported FG sandwich beams without elastic foundations are conducted. Tables 1–4 compare the non-dimensional fundamental natural frequencies and critical buckling loads of FG sandwich beams with homogeneous hardcore and softcore configurations. The analysis considers four symmetric FG sandwich beam configurations with different skin-to-core thickness ratios: 1-0-1, 2-1-2, 1-1-1, and 1-2-1, for span-to-depth ratios of L/h = 5 and 20. The results are benchmarked against existing studies using HSDT [42] and quasi-3D theories [46,49]. The findings indicate excellent agreement between the current theory and previous studies that account for transverse normal deformation effects. Notably, while the displacement field in the present theory aligns with the quasi-3D approach in [46], the results more closely match those in [49], which employ polynomial shape functions. This highlights the improved accuracy of polynomial shape functions in capturing transverse shear and normal deformation effects. Existing quasi-3D studies have primarily focused on FG sandwich beams with ceramic cores (homogeneous hardcore), examining only their free vibration and buckling characteristics. However, no studies in the literature provide results for FG sandwich beams with metal cores (homogeneous softcore) using quasi-3D theory. Thus, the present results for softcore configurations are validated against HSDT and also offer benchmark data. The slight deviations from HSDT can be attributed to the neglect of transverse normal strain in HSDT, which, when considered, leads to higher predictions for natural frequencies and buckling loads. This underscores the importance of accounting for transverse normal strain effects in FG sandwich beams. The data also reveal that as the power-law index p increases, the fundamental natural frequencies and critical buckling loads decrease in FG sandwich beams with homogeneous hardcore but increase in those with homogeneous softcore. This behavior is explained by the material composition: a higher p value indicates a greater metal fraction, making hardcore beams more flexible. In contrast, for softcore configurations, a higher p value corresponds to a larger ceramic fraction, resulting in increased stiffness and rigidity. These observations emphasize the critical role of the power-law index in determining the mechanical performance of FG sandwich beams. The choice of p directly influences the free vibration and buckling responses, highlighting its importance in tailoring FG sandwich beams for specific engineering applications.

Figures 2 and 3 demonstrate the influence of the power-law index and the skin-core-skin thickness ratio on the fundamental natural frequencies and critical buckling loads for a span-to-depth ratio of L/h = 5. The results reveal that for FG sandwich beams with homogeneous hardcore, the 1-0-1 configuration yields the highest values for natural frequencies and buckling loads, while the 1-2-1 configuration exhibits the lowest. Conversely, for beams with homogeneous softcore, the trends are reversed, with the 1-2-1 configuration achieving the highest values and the 1-0-1 configuration the lowest. As the power-law index increases and the core thickness decreases, the fundamental natural frequencies and critical buckling loads decline for FG sandwich beams with homogeneous hardcore, whereas they increase for beams with homogeneous softcore. This behavior is attributed to the material-dependent

Table 1. Nondimensional fundamental natural frequencies of simply supported FG sandwich beams with homogeneous hardcore

p	Theory	*L/h* = 5				*L/h* = 20			
		1-0-1	2-1-2	1-1-1	1-2-1	1-0-1	2-1-2	1-1-1	1-2-1
0	Present	5.1616	5.1616	5.1616	5.1616	5.4610	5.4610	5.4610	5.4610
	HSDT [42]	5.1528	5.1528	5.1528	5.1528	5.4603	5.4603	5.4603	5.4603
	Quasi-3D [46]	5.1620	5.1620	5.1620	5.1620	5.4611	5.4611	5.4611	5.4611
	Quasi-3D [49]	5.1618	5.1618	5.1618	5.1618	5.4610	5.4610	5.4610	5.4610
0.5	Present	4.1342	4.2427	4.3381	4.4879	4.3153	4.4295	4.5330	4.6985
	HSDT [42]	4.1268	4.2351	4.3303	4.4798	4.3148	4.4290	4.5324	4.6979
	Quasi-3D [46]	4.1329	4.2417	4.3373	4.4874	4.3137	4.4284	4.5321	4.6979
	Quasi-3D [49]	4.1344	4.2429	4.3383	4.4881	4.3153	4.4296	4.5330	4.6985
1	Present	3.5801	3.7367	3.8828	4.1183	3.7151	3.8773	4.0333	4.2895
	HSDT [42]	3.5735	3.7298	3.8755	4.1105	3.7147	3.8768	4.0328	4.2889
	Quasi-3D [46]	3.5804	3.7369	3.8830	4.1185	3.7153	3.8774	4.0334	4.2896
	Quasi-3D [49]	3.5803	3.7369	3.883	4.1185	3.7152	3.8773	4.0333	4.2895
2	Present	3.0736	3.2425	3.4255	3.7408	3.1768	3.3469	3.5394	3.8774
	HSDT [42]	3.0680	3.2365	3.4190	3.7334	3.1764	3.3465	3.5389	3.8769
	Quasi-3D [46]	3.0739	3.2428	3.4258	3.7410	3.1769	3.3471	3.5395	3.8775
	Quasi-3D [49]	3.0737	3.2427	3.4257	3.7410	3.1785	3.3488	3.5413	3.8793
5	Present	2.7492	2.8487	3.0236	3.3839	2.8443	2.9314	3.1115	3.4926
	HSDT [42]	2.7446	2.8439	3.0181	3.3771	2.8439	2.9310	3.1111	3.4921
	Quasi-3D [46]	2.7497	2.8491	3.0239	3.3840	2.8444	2.9315	3.1116	3.4927
	Quasi-3D [49]	2.7493	2.8489	3.0238	3.3840	2.8443	2.9314	3.1115	3.4926
10	Present	2.6977	2.7398	2.8858	3.2421	2.8045	2.8191	2.9665	3.3411
	HSDT [42]	2.6932	2.7355	2.8808	3.2356	2.8041	2.8188	2.9662	3.3406
	Quasi-3D [46]	2.6982	2.7402	2.8862	3.2423	2.8046	2.8192	2.9666	3.3412
	Quasi-3D [49]	2.6978	2.7400	2.8860	3.2422	2.8045	2.8191	2.9665	3.3411

Table 2. Nondimensional fundamental natural frequencies of simply supported FG sandwich beams with homogeneous softcore

p	Theory	*L/h* = 5				*L/h* = 20			
		1-0-1	2-1-2	1-1-1	1-2-1	1-0-1	2-1-2	1-1-1	1-2-1
0	Present	2.6819	2.6819	2.6819	2.6819	2.8375	2.8375	2.8375	2.8375
	HSDT [42]	2.6773	2.6773	2.6773	2.6773	2.8371	2.8371	2.8371	2.8371
0.5	Present	4.4501	4.3115	4.1904	3.9979	4.8585	4.7466	4.6300	4.4165
	HSDT [42]	4.4427	4.3046	4.1839	3.9921	4.8579	4.7460	4.6294	4.4160
1	Present	4.8612	4.7260	4.5935	4.3730	5.2997	5.2224	5.1167	4.8944
	HSDT [42]	4.8525	4.7178	4.5858	4.3663	5.2990	5.2217	5.1160	4.8938
2	Present	5.1040	5.0063	4.8828	4.6536	5.5247	5.5120	5.4418	5.2452
	HSDT [42]	5.0945	4.9970	4.8740	4.6459	5.5239	5.5113	5.4410	5.2445
5	Present	5.1976	5.1703	5.0799	4.8650	5.5653	5.6391	5.6250	5.4851
	HSDT [42]	5.1880	5.1603	5.0703	4.8564	5.5645	5.6382	5.6242	5.4843
10	Present	5.1942	5.2066	5.1400	4.9416	5.5309	5.6460	5.6629	5.5583
	HSDT [42]	5.1848	5.1966	5.1301	4.9326	5.5302	5.6452	5.6621	5.5575

Table 3. Nondimensional critical buckling loads of simply supported FG sandwich beams with homogeneous hardcore

p	Theory	*L/h* = 5				*L/h* = 20			
		1-0-1	2-1-2	1-1-1	1-2-1	1-0-1	2-1-2	1-1-1	1-2-1
0	Present	49.5906	49.5906	49.5906	49.5906	53.3145	53.3145	53.3145	53.3145
	HSDT [42]	48.5959	48.5959	48.5959	48.5959	53.2364	53.2364	53.2364	53.2364
	Quasi-3D [46]	49.5970	49.5970	49.5970	49.5970	53.3175	53.3175	53.3175	53.3175
	Quasi-3D [49]	49.5906	49.5906	49.5906	49.5906	49.5906	53.3145	53.3145	53.3145
0.5	Present	28.4623	30.6824	32.5698	35.5155	29.7625	32.1021	34.1379	37.3724
	HSDT [42]	27.8574	30.0301	31.8784	34.7653	29.7175	32.2629	34.0862	37.3159
	Quasi-3D [46]	28.4407	30.6650	32.5547	35.5032	29.7410	32.0853	34.1242	37.3626
	Quasi-3D [49]	28.4624	30.6825	32.5699	35.5156	29.7626	32.1022	34.1380	41.8227
1	Present	20.0894	22.7065	25.1075	29.0755	20.7530	23.4572	25.9989	30.2774
	HSDT [42]	19.6525	22.2108	24.5596	28.4447	20.7212	23.4211	25.9588	30.2307
	Quasi-3D [46]	20.0899	22.7061	25.1060	29.0723	20.7541	23.4584	26.0001	30.2785
	Quasi-3D [49]	20.7425	22.7065	25.1075	29.0755	20.7530	23.4572	25.9989	30.2774
2	Present	13.8838	16.2761	18.7772	23.3042	14.2190	16.6307	19.2299	24.0276
	HSDT [42]	13.5801	15.9152	18.3587	22.7863	14.1973	16.6050	19.3116	23.9900
	Quasi-3D [46]	13.8852	16.2761	18.7756	23.3002	14.2199	16.6317	19.2309	24.0284
	Quasi-3D [49]	13.8839	16.2761	18.7772	23.3042	14.2190	16.6307	19.2299	24.0276
5	Present	10.3673	11.9301	14.0352	18.5092	10.6330	12.1068	14.2505	18.9172
	HSDT [42]	10.1460	11.6676	13.7212	18.0914	10.6171	12.0883	14.2284	18.8874
	Quasi-3D [46]	10.3708	11.9320	14.0352	18.5058	10.6341	12.1078	14.2515	18.9180
	Quasi-3D [49]	10.3673	11.9301	14.0353	18.5092	10.6330	12.1068	14.2505	18.9172
10	Present	9.6535	10.7689	12.5393	16.7574	9.9994	10.9239	12.7014	17.0712
	HSDT [42]	9.4515	10.5348	12.2605	16.3783	9.9847	10.9075	12.6819	17.0443
	Quasi-3D [46]	9.6573	10.7715	12.5402	16.7550	10.0003	10.9246	12.7023	17.0723
	Quasi-3D [49]	9.6535	10.7689	12.5393	16.7574	9.9995	10.9239	12.7014	17.0712

Table 4. Nondimensional critical buckling loads of simply supported FG sandwich beams with homogeneous softcore

p	Theory	*L/h* = 5				*L/h* = 20			
		1-0-1	2-1-2	1-1-1	1-2-1	1-0-1	2-1-2	1-1-1	1-2-1
0	Present	9.1351	9.1351	9.1351	9.1351	9.8211	9.8211	9.8211	9.8211
	HSDT [42]	8.9519	8.9519	8.9519	8.9519	9.8067	9.8067	9.8067	9.8067
0.5	Present	28.9557	26.4203	24.4839	21.7615	33.2659	30.8982	28.8571	25.6412
	HSDT [42]	28.4280	25.9503	24.054	21.3821	33.2187	30.8546	28.8167	25.6086
1	Present	36.8893	33.4917	30.7778	26.9366	42.2413	39.4682	36.8962	32.6255
	HSDT [42]	36.2103	32.8974	30.2449	26.4801	42.1810	39.4124	36.8445	32.5803
2	Present	43.2667	39.5732	36.3395	31.5445	48.7919	46.2696	43.6024	38.7731
	HSDT [42]	42.4501	38.8589	35.7058	31.0152	48.7215	46.2035	43.5408	38.7192
5	Present	47.5776	44.3595	41.0572	35.6386	52.4419	50.8343	48.5858	43.8252
	HSDT [42]	46.6504	43.5338	40.3235	35.0357	52.3655	50.7608	48.5163	43.7637
10	Present	48.7461	45.9842	42.8470	37.3250	53.1108	52.0560	50.1623	45.6685
	HSDT [42]	47.7825	45.1141	42.0693	36.6874	53.0331	51.9804	50.0902	45.604

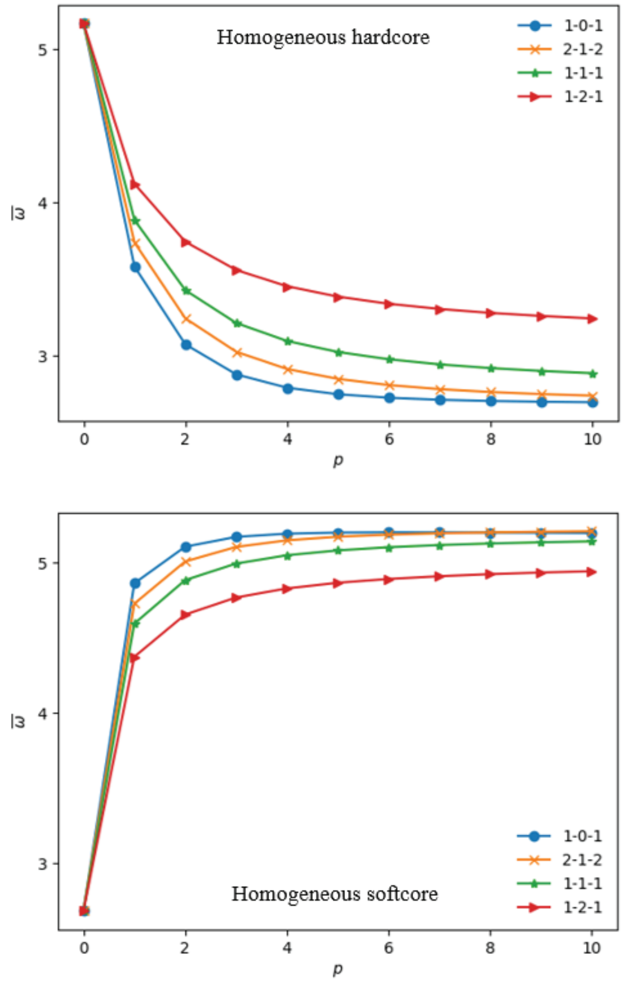

Figure 2. Nondimensional fundamental natural frequencies of simply supported FG sandwich beams for various skin-core-skin thickness ratios ($L/h = 5$)

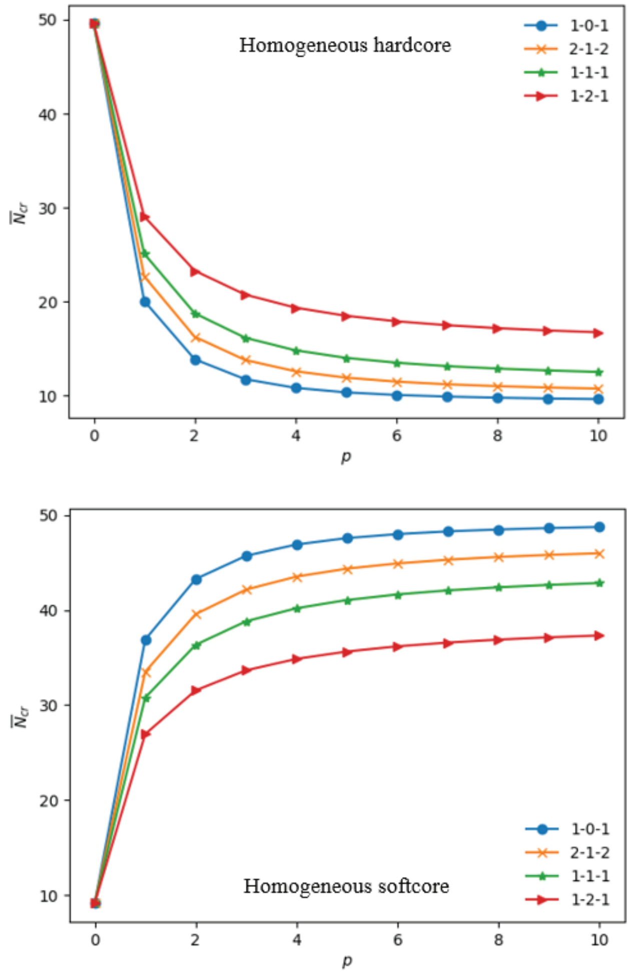

Figure 3. Nondimensional critical buckling loads of simply supported FG sandwich beams for various skin-core-skin thickness ratios ($L/h = 5$)

bending stiffness of the beam: for homogeneous hardcore, an increase in the power-law index or a reduction in core thickness enhances bending flexibility, reducing stiffness. In contrast, for homogeneous softcore, these changes lead to an increase in the ceramic fraction, which enhances the beam's bending stiffness.

Effect of Elastic Foundation

The influence of the elastic foundation on the free vibration and buckling behavior of simply supported FG sandwich beams is analyzed. For this purpose, free vibration and buckling analyses are performed on various FG sandwich beam configurations resting on a two-parameter elastic foundation. Figures 4 and 5 depict the variations in nondimensional fundamental natural frequencies and critical buckling loads for simply supported FG sandwich beams with a 2-1-2 configuration, different power-law indices, and $L/h = 10$, under three foundation conditions: no elastic foundation, a Winkler foundation, and a Pasternak foundation. The results show that the inclusion of elastic foundation models enhances the nondimensional fundamental natural frequencies and critical buckling loads for all cases. The addition of the Winkler parameter (k_w) provides a modest increase in these values due to the added stiffness and support provided by the Winkler foundation. In contrast, the Pasternak parameter (k_p) significantly amplifies the fundamental natural frequencies and critical buckling loads by increasing the shear stiffness of the foundation, resulting in notable improvements in the beam's free vibration and buckling performance. These findings highlight that the Pasternak parameter has a far more substantial effect on the natural frequencies and critical buckling loads compared to the Winkler parameter. This suggests that incorporating the Pasternak foundation model is particularly beneficial for

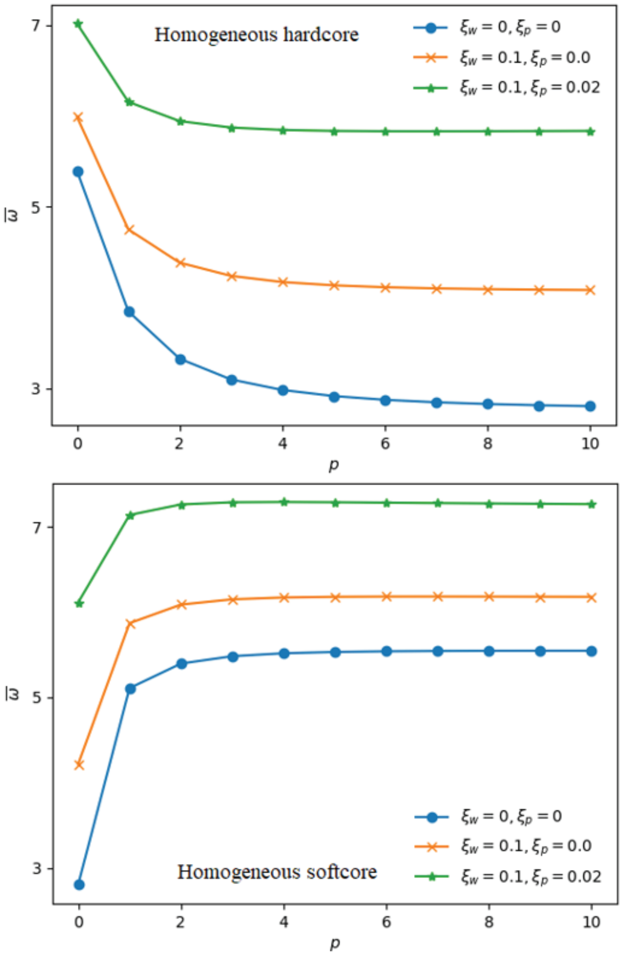

Figure 4. Effect of foundation parameters on nondimensional fundamental natural frequencies (2-1-2, L/h = 10)

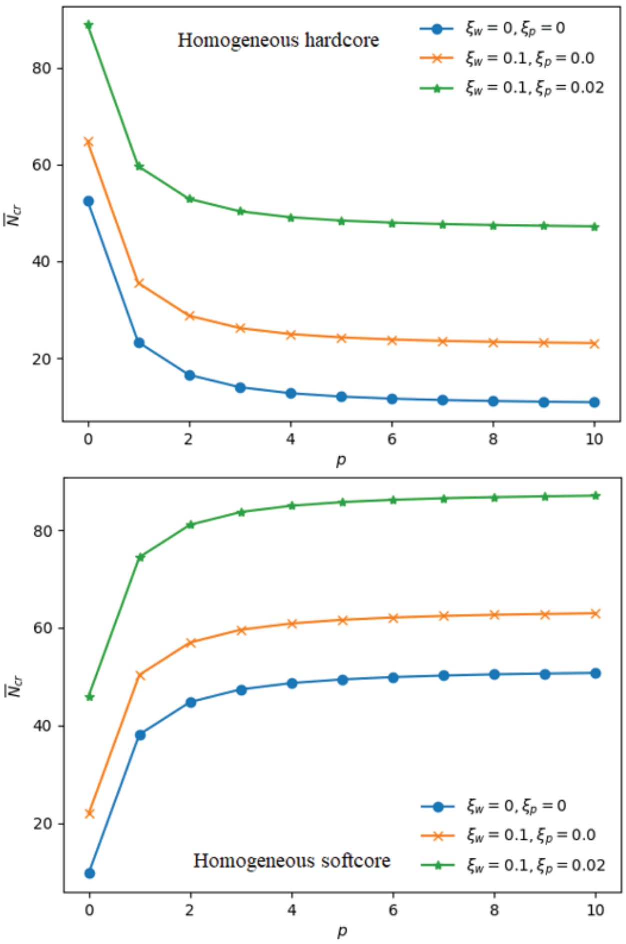

Figure 5. Effect foundation parameters on nondimensional critical buckling loads (2-1-2, L/h = 10)

enhancing the stability and vibrational characteristics of FG sandwich beams.

Figures 6 and 7 illustrate the effect of the span-to-depth ratio L/h on the fundamental natural frequencies and critical buckling loads of FG sandwich beams with homogeneous hardcore and softcore under three different scenarios: Case 1 (ξ_w = 0, ξ_p = 0), Case 2 (ξ_w = 0.01, ξ_p = 0), Case 3 (ξ_w = 0.01, ξ_p = 0.01). A beam with a 1-2-1 configuration and a power-law index of p = 2 is analyzed. For Case 1, the results show a slight increase in the nondimensional fundamental natural frequencies and critical buckling loads with increasing L/h. This behavior is attributed to the enhanced bending and buckling resistance associated with the higher span-to-depth ratio, as a longer beam exhibits greater overall stiffness. In Case 2, the introduction of spring constants from the elastic foundation leads to a notable increase in both natural frequencies and buckling loads, demonstrating the influence of the Winkler parameter in enhancing beam stability and vibrational performance. In Case 3, the inclusion of the Pasternak parameter results in a dramatic rise in the nondimensional fundamental natural frequencies and critical buckling loads. This significant improvement is due to the additional shear stiffness provided by the Pasternak foundation, which greatly enhances the overall stiffness of the FG sandwich beams. The observed trends indicate that as L/h. increases, the impact of the Pasternak parameter becomes increasingly pronounced. This suggests that the interaction between the beam and the elastic foundation intensifies with beam length, further emphasizing the importance of considering both foundation parameters, especially for longer beams.

Figures 8 and 9 display the variations in nondimensional fundamental natural frequencies and critical buckling loads of FG sandwich beams with a 1-1-1 configuration, homogeneous hardcore, and softcore, as functions of the Winkler spring constant (ξ_w) with p = 2 and L/h = 5 with varying values of the Pasternak shear layer parameter (ξ_p). These figures aim to separately assess the effects of the Winkler and Pasternak

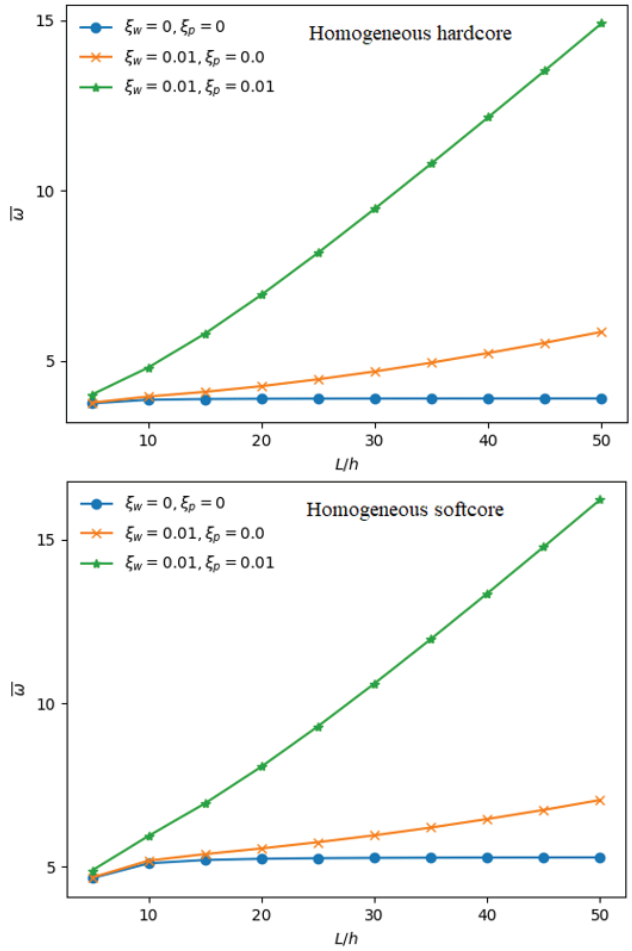

Figure 6. Variation of nondimensional fundamental natural frequencies of FG sandwich beams with the spring constants and beam slenderness (1-2-1, $p = 2$)

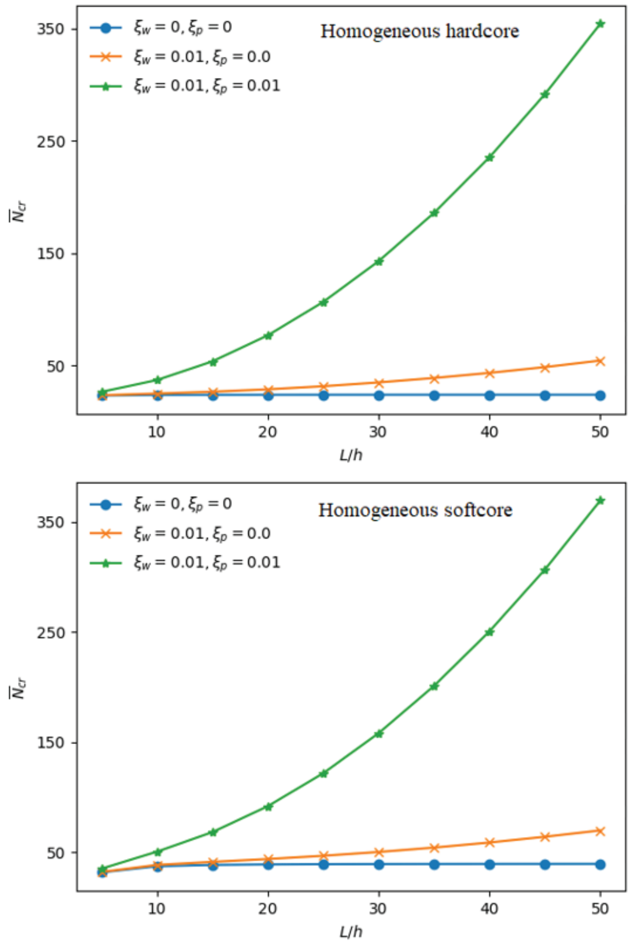

Figure 7. Variation of nondimensional critical buckling loads of FG sandwich beams with the spring constants and beam slenderness (2-1-2, $p = 2$)

elastic foundation parameters. The results reveal that both the nondimensional fundamental natural frequencies and critical buckling loads follow a linear relationship with the foundation parameters. Additionally, it is evident that as the Pasternak parameter increases, there is a significant rise in the fundamental natural frequencies and critical buckling loads. This enhancement can be attributed to the stronger shear interaction between the shear layer and the beam, provided by a (ξ_p) value, which in turn improves the overall stability and stiffness of the beam. This effect is linked to the additional lateral support offered by the Pasternak foundation. Therefore, it can be concluded that the increase in (ξ_p) has a more pronounced effect on the fundamental natural frequencies and critical buckling loads than the increase in ξ_w. This highlights the significant role of shear interaction in influencing the dynamic and stability behaviors of FG sandwich beams.

CONCLUSION

In this paper, Navier-type analytical solutions for the free vibration and buckling analysis of FG sandwich beams with homogeneous hardcore and softcore, resting on a Winkler-Pasternak elastic foundation, are presented. The proposed model is based on a quasi-3D deformation theory, and the governing differential equations of motion are derived using Hamilton's principle. To validate the model, several numerical examples are considered, and the results are compared with those available in the literature. A comprehensive parametric study is conducted to investigate the effects of various parameters, such as the skin-core-skin thickness ratio, power-law index, span-to-depth ratio, normal strain, and elastic foundation parameters, on the fundamental natural frequencies and critical buckling loads of FG sandwich beams with homogeneous hardcore and softcore. The main findings of the study can be summarized as follows:

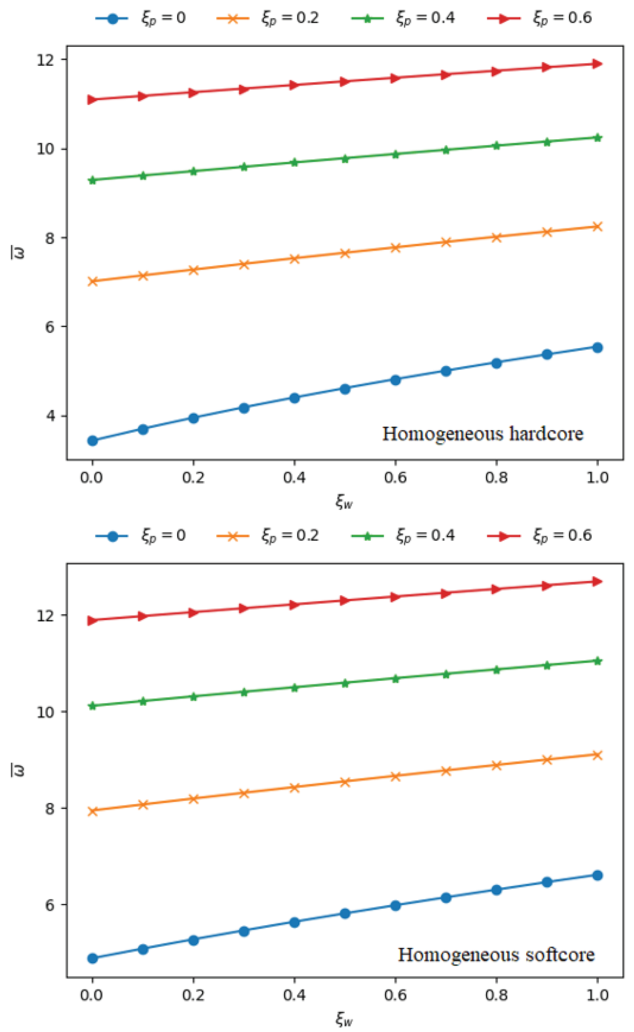

Figure 8. Variation of nondimensional fundamental natural frequencies of FG sandwich beams (1-1-1) with the foundation parameters ($L/h = 5$, $p = 2$)

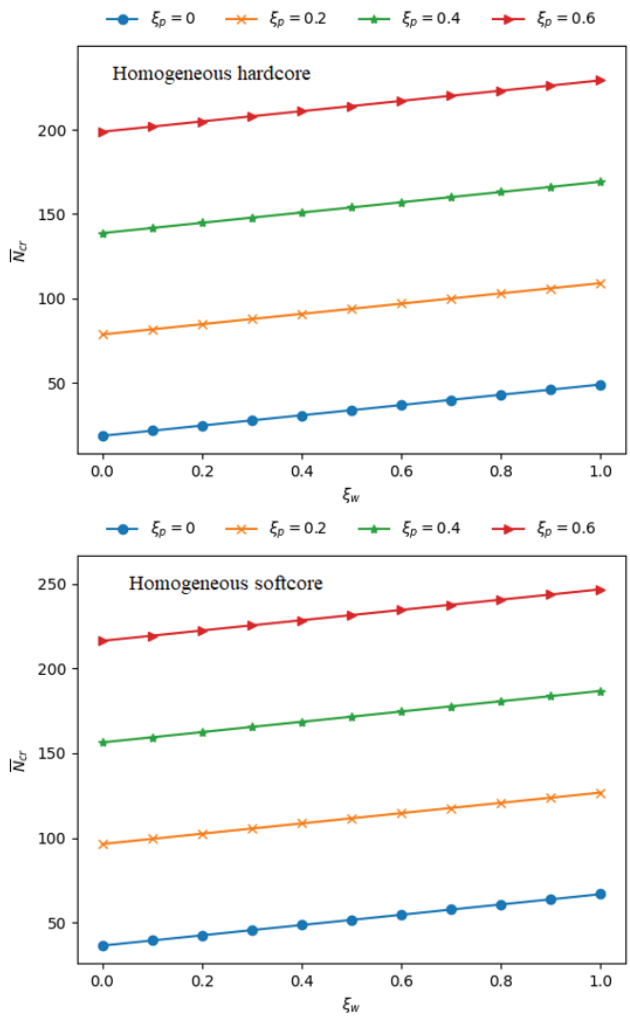

Figure 9. Variation of nondimensional critical buckling loads of FG sandwich beams (1-1-1) with the foundation parameters ($L/h = 5$, $p = 2$)

- The proposed model provides more accurate and efficient predictions for the free vibration and buckling responses of FG sandwich beams with homogeneous hardcore and softcore on a two-parameter Winkler-Pasternak elastic foundation.
- For FG sandwich beams with homogeneous hardcore, the fundamental natural frequencies and critical buckling loads decrease as the power-law index increases. Conversely, these values increase for FG sandwich beams with homogeneous softcore as the power-law index rises.
- As the span-to-depth ratio increases, the fundamental natural frequencies and critical buckling loads of FG sandwich beams with homogeneous cores also increase.
- The fundamental natural frequencies and critical buckling loads of the FG sandwich beams increase significantly with higher spring and shear constants of the elastic foundation, especially when the shear layer constant increases.

APPENDIX

$$\mathbf{K} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{12} & K_{22} & K_{23} & K_{24} \\ K_{13} & K_{23} & K_{33} & K_{34} \\ K_{14} & K_{24} & K_{34} & K_{44} \end{bmatrix}, \quad \boldsymbol{G} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \alpha^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{M} = \begin{bmatrix} I_1 & -I_2\alpha & I_4 & 0 \\ -I_2\alpha & I_3\alpha^2 + I_1 & -I_5\alpha & I_7 \\ I_4 & -I_5\alpha & I_6 & 0 \\ 0 & I_7 & 0 & I_8 \end{bmatrix}, \quad \boldsymbol{\Delta} = \begin{Bmatrix} U_m \\ W_m \\ \psi_{xm} \\ \psi_{zm} \end{Bmatrix} \tag{A1}$$

$$\begin{aligned} &K_{11} = A_{11}\alpha^2, \quad K_{12} = -B_{11}\alpha^3, \quad K_{13} = C_{11}\alpha^2, \\ &K_{14} = -D_{13}\alpha, \; K_{22} = A_{S11}\alpha^4 + A_{S11}\alpha^4 + k_w + k_p\alpha^2, \\ &K_{23} = -\,B_{S11}\alpha^3, \quad K_{24} = C_{S13}\alpha^2 + g(k_w + k_p\alpha^2), \\ &K_{33} = (\,D_{S11}\alpha^2 + E_{S55}), \;\; K_{34} = (E_{S55} - E_{13})\alpha, \\ &K_{44} = (\,E_{S55}\alpha^2 + F_{33}) + g^2(k_w + k_p\alpha^2) \end{aligned} \tag{A2}$$

where $\alpha = m\pi/L$, $m = 1$.

AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

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