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Research Article

On unique solvability of linear complementarity problems, horizontal linear complementarity problems and an n-absolute value equations

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ABSTRACT

The complementarity problems is getting a lot of attention because it is connected to real-world problems in scientific computing and engineering. It shows up in various situations like linear and quadratic programming, two person games, circuit simulation, optimal stopping in Markov chains, contact problems with friction, finding a Nash-equilibrium in bimatrix games. The linear complementarity problems (LCP) and absolute value equations (AVE) have an equivalence relation; that is, the AVE can be transformed into an LCP and vice versa. The relationship between LCP and AVE enables the conversion of one problem into another, offering different perspectives for analysis and solution. This equivalence aids in theoretical understanding and the development of numerical methods applicable to both mathematical formulations. In the present study, we discuss the unique solvability of the LCP and the horizontal linear complementarity problems (HLCP). Some superior unique solvability conditions are obtained for LCP and HLCP. The unique solvability of the n-absolute value equations $A^n x - B^n|x| = b$ is also discussed. Some examples are highlighted for improving the current conditions of unique solutions for absolute value equations.

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INTRODUCTION

A linear complementarity problems (LCP) is presented in the following form:

$$z \geq 0, w \geq 0, w = q + Mz, w^T z = 0, \quad (1)$$

where $z, w, q \in \mathbb{R}^m$ and $M \in \mathbb{R}^{m \times m}$ where q and M are known. Generally, this problem is denoted as LCP(q, M).

This problem has numerous applications, such as, linear programming, quadratic programming, economies with institutional limitations on prices, circuit simulation, and game theory problems [1-3, 4]. Over the years, various numerical methods have been developed and suggested [5, 6]. For an extensive and comprehensive discussion of the complementarity theory, please refer to references [1, 4].

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For given $M_1, M_2 \in \mathbb{R}^{m\times m}$ and $q \in \mathbb{R}^m$, the horizontal linear complementarity problems (HLCP) consists of finding a pair of vectors $(z, w) \in \mathbb{R}^m \times \mathbb{R}^m$ such that

$$M_1 w - M_2 z = q, z \geq 0, w \geq 0, w^T z = 0. \quad (2)$$

Generally, It is denoted by $HLCP(M_1, M_2, q)$. The HLCP is widely acknowledged as the extension of the LCP. Moreover, by taking $M_1 = I$ (Identity matrix), the HLCP converts into the LCP. One may refer [7] for an algorithm to reduce HLCP into LCP. The HLCP has applications in multiple fields, including hydrodynamic lubrication, structural mechanics, mechanical and electrical engineering, and transportation science (see [8, 9]).

The issue of unique solvability in both HLCP and LCP is addressed in several references, including [10-13]. Regarding the unique solution of the LCP(q, M), we have the following results (if any one of them is satisfied, then LCP(q, M) has a unique solution):

(i) $\sigma_{\min}(I - M)^{-1}(I + M) > 1$ [11];
(ii) $||(I + M)^{-1}(I - M)|| < 1$ [11, 14];
(iii) $\sigma_{\max}(I - M) < \sigma_{\min}(I + M)$ [14];
(iv) $\rho((I + M)^{-1}(I - M)) < 1$ [11];
(v) the LCP(q, M) has a unique solution if and only if matrix M is a P- matrix (A square matrix A is called a P-matrix if all its principal minors are positive) [1, 12].

The following conditions are discussed for the unique solvability of the $HLCP(M_1, M_2, q)$ (if any one of them is satisfied, then $HLCP(M_1, M_2, q)$ has a unique solution):

(i) $\sigma_{\max}(M_1 - M_2) < \sigma_{\min}(M_1 + M_2)$ [14];
(ii) $||(M_1 + M_2)^{-1}(M_1 - M_2)|| < 1$ [14];
(iii) $(M_1 + M_2)^T(M_1 + M_2) - ||(M_1 + M_2)||^2 I$ is positive definite matrix [14].

Furthermore, various numerical methods for HLCP are explored, as exemplified by Mezzadri et al. [9, 15-16], who present techniques such as splitting methods [15], modulus-based matrix splitting methods [9], and modulus-based nonsmooth Newton's method [16]. The one-layer neural network approach [17], homotopy approach [18], and interior point method [19] have been studied as well for the solution of the HLCP.

In this article, we further discussed the unique solvability of the LCP(q, M) and $HLCP(M_1, M_2, q)$. Some of our conditions are superior to the conditions mentioned above.

An absolute value equation is a mathematical equation that contains one or more unknown variables enclosed within an absolute value. Further, in various research works, the following type of absolute value equations (AVE) has gained significant attention from numerous researchers due to its extensive applications in various fields of optimization.

$$Ax - B|x| = b, \quad (3)$$

where $A, B \in \mathbb{R}^{m\times m}$ and $b, x \in \mathbb{R}^m$. For a matrix $A \in \mathbb{R}^{m\times m}$ and a vector $x \in \mathbb{R}^m$, $|A|$ and $|x|$ denote the component-wise absolute value of the matrix and the vector, respectively. When $B = I$ in (3), Equ. (3) reduces to the following AVE

$$Ax - |x| = b. \quad (4)$$

The AVE (4) is discussed in detail [20-21]. Further, when $A = I$ in (3), Equ. (3) reduces to the following AVE

$$x - B|x| = b, \quad (5)$$

which has been discussed in [22-24].

Rohn [25] first considered the AVE (3) and provided the alternative theorem for the unique solvability of the AVE (3). Mangasarian, as demonstrated in his work [21], that the task of solving the AVE is a problem within the class of NP-hard problems. Mangasarian et al. [20] showed that AVE (4) is equivalent to a bilinear programming. Mangasarian [26] transformed the LCP (1) into an AVE represented as $(M + I)z + q = |(M - I)z + q|$. A more equivalent relation between AVE and LCP/HLCP can be shown in [10, 11, 20, 27]. Kumar et al. [28], as discussed in their paper, explored various extensions of the AVE and provided characterization for its unique solvability. Several bounds for the solutions of the AVE under various assumptions were presented by Hladík [29]. In the study by Hladík [30], an investigation was carried out into the topological attributes of the solution set of the AVE. This analysis included aspects such as convexity, boundedness, connectedness, the presence of a finite number of solutions, and the non-negativity of solutions of the AVE.

Over the past few years, research on the AVE has predominantly revolved around two key areas: the development of numerical techniques (for example, Picard's method [14], interior point method [10], two-step iterative method [31], generalized Newton method [32], iterative methods [33-34]) for solving the AVE and theoretical analysis (see, e.g., [20-22, 28, 35-40]).

The AVE is significant because it can be applied to various domains of mathematics and applied sciences (see, e.g., [10-11, 14, 41-46]). For instance, the LCP, bimatrix games, mixed-integer programming, system of linear interval matrix, the boundary value problems, convex quadratic optimization, absolute value matrix equations and the hydrodynamic equation can be formulated as AVE.

Mangasarian et al. [20], discussed the following singular value condition

$$1 < \sigma_{\min}(A), \quad (6)$$

for the unique solvability of the AVE (4). Further, Rohn [38] extended the condition (6) for the AVE (3) and provided the following condition for the unique solvability of the AVE (3)

$$\sigma_{\max}(|B|) < \sigma_{\min}(A). \quad (7)$$

In 2019, Wu et al. [47] replaced condition (7) with the following superior condition

$$\sigma_{\max}(B) < \sigma_{\min}(A), \quad (8)$$

which also ensure the unique solvability of the AVE (3). Further, Wu et al. [48], provided the following singular value condition to check the unique solution of the AVE (3)

$$\sigma_{\max}(A^{-1}B) < 1. \quad (9)$$

In the literature, different types of generalizations of the AVE are considered by researchers. For instance, the new generalized absolute value equation (NGAVE) $Ax - |Bx| = b$ is considered by Wu [49] and provides necessary and sufficient conditions for the unique solvability of NGAVE. Zhou et al. [50] first analyzed the new class of AVE $Ax - B|Cx| = b$ and provided unique solvability conditions. Inspired by different generalizations of the AVE, we further considered the following generalization of the AVE

$$A^n x - B^n |x| = b, \quad (10)$$

where $A, B \in \mathbb{R}^{m\times m}$ and $b, x \in \mathbb{R}^m$ and $n \in N$ (set of natural numbers), which we call an n-absolute value equations (n-AVE). Clearly, for n=1, the AVE (10) converts into the AVE (3) and for $B = I$ (or $A = I$), the AVE (10) converts into the AVE (4) (or AVE (5). We are interested in how singular value conditions can be used to determine the unique solvability of the AVE (10).

The further main reason to consider the AVE (10) is that Kumar et al. [28] show that under different transformations, the AVE (4) preserves the unique solvability, but other transformations, such as squaring the matrix may result in a loss of unique solvability [28, Corollary 2 and Example 1]. The AVE (3) not preserve the unique solvability under the matrix transposition, that is $Ax - B|x| = b$ can be uniquely solvable for each $b \in \mathbb{R}^m$, but $A^T x - B^T|x| = b$ need not [28, Example 2]. So here we provide some conditions for the unique solvability of the AVE (3) under the multiplying same matrix n times; that is, we focus on the unique solvability of the AVE (10).

The key contributions of this paper are outlined as follows:

1. Analysed the new type of AVE, and unique solvability conditions are discussed.
2. New unique solvability conditions are suggested for the HLCP and LCP.

A summary of the findings presented in this paper is outlined in Table 1.

Notations

For a matrix $A \in \mathbb{R}^{m\times m}$, we use $\rho(A)$, $\sigma_{\max}(A)$ and $\sigma_{\min}(A)$ to denote the spectral radius, maximum singular value and minimum singular value of the matrix A, respectively. The identity matrix of size $m \times m$ is denoted by I. $Diag(a_i)$, $Diag_{min}(a_i)$, and $Diag_{max}(a_i)$ denotes respectively diagonal elements, minimum diagonal elements and maximum diagonal elements of matrix A. For matrix A, A^T denotes the transpose of A and $\det(A)$ shows the determinant of the matrix A.

The rest of the paper is organized as: some definitions and results are recalled for further uses in Section (2). The Section (3) contains three subsections, where we discussed the unique solvability conditions for the n-AVE, LCP and HLCP. In Section (4), we conclude our discussion with an open problem.

PRELIMINARIES

Here, we revisit and recall some definitions and results.

Definition 2.1. [13] Let $\mathcal{M} = \{M_1, M_2\}$ denote the set of matrices with $M_1, M_2 \in \mathbb{R}^{m\times m}$. A matrix $R \in \mathbb{R}^{m\times m}$ is called a column representative of $\mathcal{M}$, if $R_{.j} \in \{(M_1)_{.j}, (M_2)_{.j}\}$, j=1, 2,..., n, where $R_{.j}$, $(M_1)_{.j}$, and $(M_2)_{.j}$ denote the j^{th}column of R, M_1 and M_2, respectively.

Table 1. Summary of the results presented in the paper

Class	Statement	Description
n-AVE	Lemma 3.1	Equivalent relation between n-AVE and HLCP/LCP
	Theorem 3.3	Unique solvability conditions for n-AVE
	Theorem 3.6	Unique solvability conditions for n-AVE
	Corollary 3.12	Equivalent relation between n-AVE (10) and AVE (3)
	Theorem 3.13	Unique solvability conditions for n-AVE
LCP	Proposition 3.14	Equivalent relation between LCP and AVE
	Theorem 3.15	Unique solvability conditions for LCP
	Theorem 3.16	Unique solvability conditions for LCP
	Theorem 3.17	Unique solvability conditions for LCP
	Theorem 3.19	Unique solvability conditions for LCP
HLCP	Proposition 3.21	Equivalent relation between HLCP and AVE
	Theorem 3.22	Unique solvability conditions for HLCP
	Theorem 3.24	Unique solvability conditions for HLCP
	Theorem 3.25	Unique solvability conditions for HLCP
	Theorem 3.26	Unique solvability conditions for HLCP
	Theorem 3.27	Unique solvability conditions for HLCP

Definition 2.2. [13] The set $\mathcal{M}$ holds the column W-property if the determinants of all column representative matrices of $\mathcal{M}$ are positive.

Theorem 2.1. [13] For $\mathcal{M} = \{M_1, M_2\}$ be a set of matrices in $\mathbb{R}^{m\times m}$, then the following statements are equivalent:-

(i) The HLCP (2) has a unique solution;
(ii) $\mathcal{M}$ has the column W-property;
(iii) M_1 is invertible and $\overline{\mathcal{M}} = \{I, {M_1}^{-1}M_2\}$ has the column W-property;
(iv) $\det(M_1D_1 + M_2D_2) \neq 0$ for arbitrary nonnegative diagonal matrices $D_1, D_2 \in \mathbb{R}^{m\times m}$ with Diag $(D_1 + D_2) > 0$.

Lemma 2.2. [48] If $\sigma_{\min}(A + I) > 2$, then AVE (4) has a unique solution for each $b \in \mathbb{R}^m$.

Lemma 2.3. [48] If $\sigma_{\min}(B^{-1}A) > 1$ or $\sigma_{\max}(A^{-1}B) < 1$, then AVE (3) has a unique solution for each $b \in \mathbb{R}^m$.

Lemma 2.4. [34] If $\det(A) \neq 0$ and $\min\{\sigma_{\max}(A^{-1}), \rho(A^{-1})\} < 1$ then AVE (4) has a unique solution for each $b \in \mathbb{R}^m$.

Lemma 2.5. [48] If $\det(A) \neq 0$ and $\rho(A^{-1}B\overline{D}) < 1$ or $\rho(B\overline{D}A^{-1}) < 1$ for any diagonal matrix $\overline{D} \in [-I, I]$, then AVE (3) has a unique solution for each $b \in \mathbb{R}^m$.

Lemma 2.6. [48] The AVE (3) has a exactly one solution for each $b \in \mathbb{R}^m$ if and only if $A - B\overline{D}$ is nonsingular, for any diagonal matrix $\overline{D} \in [-I, I]$.

RESULTS AND DISCUSSION

In this section, based on the relation among the AVEs, HLCP and LCP, we obtain necessary and sufficient conditions for unique solution of Equ.(10), Equ.(1) and Equ.(2), respectively.

Unique Solvability of the n-AVE

In the following Lemma, n-AVE (10) equivalently can be written into HLCP and LCP forms.

Lemma 3.1. The n-AVE (10) is can be written as the following HLCP form

$$Px^+ - Qx^- = b, x^+ \geq 0, x^- \geq 0, (x^+)^T(x^-) = 0, \quad (11)$$

where $P = A^n - B^n, Q = A^n + B^n$.

If the matrix $A^n - B^n$ is invertible, then n-AVE (10) can be written into the following LCP

$$x^+ = (A^n - B^n)^{-1}(A^n + B^n)\, x^- + (A^n - B^n)^{-1}b, \quad (12)$$

where $x^+ = (|x| + x)/2$ and $x^- = (|x| - x)/2$.

Proof. Let $x^+ = (|x| + x)/2$ and $x^- = (|x| - x)/2$. Then

$$x = x^+ - x^- \text{ and } |x| = x^+ + x^-. \quad (13)$$

Now with the help of Equ. (13), n-AVE (10) can be written as

$$(A^n - B^n)x^+ - (A^n + B^n)x^- = b, x^+ \geq 0, x^- \geq 0, (x^+)^T(x^-) = 0, \quad (14)$$

which is a required HLCP (11).

If the matrix $(A^n - B^n)$ is invertible, then Equ. (14) can be written as

$$x^+ = (A^n - B^n)^{-1}(A^n + B^n)\, x^- + (A^n - B^n)^{-1}b, \quad (15)$$

which is a required LCP (12).

The following result is needed for Theorem 3.3.

Lemma 3.2. If matrices A and B of real entries satisfy the condition $\sigma_{\max}(B^n) < \sigma_{\min}(A^n)$ then matrix $(A^n - B^n)$ is invertible.

Proof. Given $\lambda_{max}((B^n)^TB^n) < \lambda_{min}((A^n)^TA^n)$.

This implies $x^T(B^n)^T(B^n)x < x^T(A^n)^T(A^n)x$ for all non-zero $x \in \mathbb{R}^m$. Let us assume that, $(A^n - B^n)$ is singular.

Then, there exists a non-zero vector x such that $(A^n - B^n)x = 0$.

Now $x^T(A^n + B^n)^T(A^n - B^n)x = x^T(A^n + B^n)^T0$.

This implies $x^T[(A^n)^T(A^n) - (A^n)^T(B^n) + (B^n)^T(A^n) - (B^n)^T(B^n)]x = 0$.

This implies $x^T(A^n)^T(A^n)x - x^T(A^n)^T(A^n)x + x^T(B^n)^T(A^n)x - x^T(B^n)^T(B^n)x = 0$.

This implies $x^T(A^n)^T(A^n)x = x^T(B^n)^T(B^n)x$.

Which is a contradiction. So $(A^n - B^n)$ is nonsingular.

Theorem 3.3. If matrices A and B of real entries satisfy the condition $\sigma_{\max}(B^n) < \sigma_{\min}(A^n)$. Then n-AVE (10) has a unique solution for any $b \in \mathbb{R}^m$.

Proof. From Lemma 3.1, the n-AVE (10) can be written as the following LCP form:

$$x^+ = (A^n - B^n)^{-1}(A^n + B^n)\, x^- + (A^n - B^n)^{-1}b.$$

By Lemma (3.2), $(A^n - B^n)^{-1}$ exists.

For the unique solution of n-AVE (10), we will show that matrix $x^+ = (A^n - B^n)^{-1}(A^n + B^n)$ is a P-matrix. Since $\sigma_{\max}(B^n) < \sigma_{\min}(A^n)$ for non-zero vector $x \in \mathbb{R}^m$.

Then $x^T(A^n)(A^n)^Tx \geq \lambda_{min}(A^n)(A^n)^T > \lambda_{max}(B^n)(B^n)^T \geq x^T(B^n)(B^n)^Tx$.

This implies $x^T(A^n)(A^n)^Tx > x^T(B^n)(B^n)^Tx$.

This implies $x^T[(A^n)(A^n)^T - (B^n)(B^n)^T]x > 0$.

This implies $x^T[(A^n)(A^n)^T - (B^n)(B^n)^T + (B^n)(A^n)^T - (A^n)(B^n)^T]x > 0$.

This implies $x^T[A^n\{(A^n)^T - (B^n)^T\} + B^n\{(A^n)^T - (B^n)^T\}]x > 0$.

This implies $x^T[(A^n + B^n)((A^n)^T - (B^n)^T)]x > 0$.

By choosing $[(A^n)^T - (B^n)^T]x = y$, we get $y^T(A^n - B^n)^{-1}(A^n + B^n)y > 0$.

This implies that matrix $(A^n - B^n)^{-1}(A^n + B^n)$ is a P-matrix.

Remark 3.4. The converse of Theorem 3.3 is not true in general. Let see the following example.

Example 3.1. Consider the matrices A and B

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} -0.9 & 0.2 \\ -0.4 & -0.8 \end{bmatrix}.$$

Here n-AVE (10) for $n = 1$ and 2, have a unique solutions respectively $x = [6.6667, -38.3334]^T$ and $x = [11.1429, 5.8571]^T$ for $b = [5, -5]^T$, but condition $\sigma_{\max}(B^n) < \sigma_{\min}(A^n)$ does not hold for $n = 1$ and 2. As $\sigma_{\min}(A) = 1.0$, $\sigma_{\max}(B) = 1.0132$, $\sigma_{\min}(A^2) = 1.0$ and $\sigma_{\min}(B^2) = 1.01233$.

Remark 3.5. Choosing $n = 1$, Theorem 3.3 will become the main result in [48] and here no need to consider the nonsingularity of $(A - B)$ separately.

The Theorem 3.6 is the particular case of Theorem 3.3.

Theorem 3.6. If A and B are diagonal matrices such that $Diag(b_i) > 0$ for all i and $Diag_{min}(a_i) > Diag_{max}(b_i)$ then n-AVE (10) and AVE (3) have a unique solution for any $b \in \mathbb{R}^m$.

Proof. We know that, in diagonal matrix each diagonal element are singular value of that matrix. Condition $Diag(b_i) > 0$ ensure that matrix B has non-zero singular values and condition $Diag_{min}(a_i) > Diag_{max}(b_i)$ implies to conditions (7), (8) and $\sigma_{\max}(B^n) < \sigma_{\min}(A^n)$.

This completes the proof.

By fixing $n = 1$ and appropriate choices of B in Theorem 3.3, we get the following corollaries.

Corollary 3.7. If matrices A and B of real entries satisfy the condition $\sigma_{\max}(B) < \sigma_{\min}(A)$ then AVE (3) has a unique solution for any $b \in \mathbb{R}^m$.

Corollary 3.8. If matrices A and B of real entries and either $B \geq 0$ or $B \leq 0$ satisfy the condition $\sigma_{\max}(|B|) < \sigma_{\min}(A)$ then AVE (3) has a unique solution for any $b \in \mathbb{R}^m$.

Corollary 3.9. If matrices A of real entries satisfy the condition $1 < \sigma_{\min}(A)$ then AVE (4) has a unique solution for any $b \in \mathbb{R}^m$.

Remark 3.10. Corollary 3.7 is the main result of [47]. Corollary 3.8 is the variation of the main result of Theorem 2 in [38] and Corollary 3.9 is the main result of Proposition 3(i) in [20].

Remark 3.11. If we take $B = 0$ (zero matrix), $n = 1$ then Theorem 3.3 turns into the fundamental theorem of the linear system. i.e., **"**The linear system $Ax = b$ has a unique solution for any $b \in \mathbb{R}^m$, where A is nonsingular [1, 4].**"**

By the simple property of the idempotent matrices, we get the following result.

Corollary 3.12. If matrices A and B are idempotent then n-AVE (10) and AVE (3) are equivalent and conditions (8) and $\sigma_{\max}(B^n) < \sigma_{\min}(A^n)$ are also equivalent.

Based on Theorem 2.1, we have the following result for n-AVE (10).

Theorem 3.13. The following statements are equivalent:-

(i) The n-AVE (10) has a unique solution for any $b \in \mathbb{R}^m$;
(ii) $\{A^n - B^n, A^n + B^n\}$ has the column W-property;
(iii) $(A^n - B^n)$ is invertible and $\{I, (A^n - B^n)^{-1} (A^n + B^n)\}$ has the column W-property;
(iv) $\det((A^n - B^n)\, D_1 + (A^n + B^n)\, D_2\,) \neq 0$ for arbitrary nonnegative diagonal matrices D_1, $D_2 \in \mathbb{R}^{m\times m}$ with $\mathrm{Diag}(D_1 + D_2\,) > 0$.

Proof. By Lemma 3.1, HLCP (11) is equivalent to the n-AVE (10), then by Theorem 2.1 our results holds.

Unique Solvability of the Linear Complementarity Problems

In this subsection, we discuss the unique solvability conditions for the LCP.

Proposition 3.14. The LCP (1) is can be written as the following AVE form

$$(I + M)s - (I - M)|s| = -q \tag{16}$$

and, if the matrix $(I - M)$ is invertible, then (16) converted into the following AVE form

$$(I - M)^{-1}(I + M)s - |s| = -(I - M)^{-1}\, q \tag{17}$$

where z = |s| + s and w = |s| − s.

Theorem 3.15. If $\sigma min((I - M)^{-1}(I + M) + I\,) > 2$, then LCP (1) has a unique solution.

Proof. Since LCP (1) is equivalent to the AVE (17), then our result is directly hold by Lemma 2.2.

Theorem 3.16. If matrix $(I + M)$ is nonsingular and $\min\{\rho(|(I + M)^{-1}(I - M)|), \sigma_{max}((I + M)^{-1}(I - M))\} < 1$ satisfy, then LCP (1) has a unique solution.

Proof. Since LCP (1) is equivalent to the AVE (17), then our result is directly hold by Lemma 2.4.

Theorem 3.17. If det $(I + M) \neq 0$ and $\rho\{(I + M)^{-1}(I - M)\overline{D}\} < 1$ for any diagonal matrix $\overline{D} \in [-I, I]$, then the LCP (1) has a unique solution.

Proof. Since LCP (1) is equivalent to the AVE (16), then by Lemma 2.4, our result is holds.

Remark 3.18. Our condition $\rho\{(I + M)^{-1}(I - M)\overline{D}\} < 1$ is superior than the condition $\rho\{|(I + M)^{-1}(I - M)\,|\} < 1$ of [11, Theorem 3.1]. Because $(I + M)^{-1}(I - M)\overline{D} \leq |(I + M)^{-1}(I - M)|\,|\overline{D}| \leq |(I + M)^{-1}(I - M)|$.

Theorem 3.19. The LCP (1) has a unique solution if and only if matrix $(I + M) - (I - M)\overline{D}$ is nonsingular for any diagonal matrix $\overline{D} \in [-I, I]$.

Proof. Since LCP (1) is equivalent to the AVE (16), then by Lemma 2.6, our result is holds.

Remark 3.20. Based on the fact, "The LCP(q, M) has a unique solution if and only if M is a P-matrix", we can say that, if matrix M fulfills the criteria outlined in Theorems 3.15, 3.16, 3.17, and 3.19, it can be classified as a P-matrix.

Unique Solvability of the Horizontal LCP

In this subsection, we discuss the unique solvability conditions for the HLCP.

Proposition 3.21. The HLCP (2) is can be written as the following AVE form

$$(M_1 + M_2)s - (M_1 - M_2)|s| = -q \tag{18}$$

and, if the matrix $(M_1 - M_2)$ is invertible, then (18) converted into the following AVE form

$$(M_1 - M_2)^{-1}(M_1 + M_2)s + |s| = -\,(M_1 - M_2)^{-1}q \tag{19}$$

where $z = |s| + s$ and w = $|s| - s$.

Theorem 3.22. If $\sigma_{min}((M_1 - M_2)^{-1}(M_1 + M_2)) > 1$ or $\sigma_{max}((M_1 + M_2)^{-1}(M_1 - M_2)) < 1$, then HLCP (2) has a unique solution.

Proof. The HLCP (2) is equivalent to the AVE (18), then we can apply Lemma (2.3). So our result is hold.

Remark 3.23. This condition is slightly superior than the condition $\sigma_{min}(M_1 + M_2) > \sigma_{max}(M_1 + M_2)$ of [14] because

$$\sigma_{min}\big((M_1 - M_2)^{-1}(M_1 + M_2)\big) \geq$$
$$\sigma_{min}((M_1 - M_2)^{-1}\sigma_{min}(M_1 + M_2)) = \frac{\sigma_{min}(M_1+M_2)}{\sigma_{max}(M_1-M_2)}.$$

Theorem 3.24. If $\sigma_{min}((M1 - M2)^{-1}(M1 + M2) + I) > 2$, then HLCP(2) has a unique solution.

Proof. The HLCP (2) is equivalent to the AVE (19), then we can apply Lemma (2.2). So our result is hold.

Theorem 3.25. If matrix $(M_1 + M_2)$ is nonsingular and $\min\{\rho(|(M_1 + M_2)^{-1}(M_1 - M_2)|), \sigma_{max}((M_1 + M_2)^{-1}(M_1 - M_2))\} < 1$ satisfy, then HLCP (2) has a unique solution.

Proof. The proof is directly hold by the AVE (19) and Lemma (2.4).

Based on Lemma (2.5) and Lemma (2.6) with AVE (18), we get the following necessary and sufficient conditions for the unique solution of HLCP (2).

Theorem 3.26. The HLCP(2) has a unique solution if and only if matrix $(M_1 + M_2) - (M_1 - M_2)\overline{D}$ is nonsingular for any diagonal matrix $\overline{D} \in [-I, I]$.

Theorem 3.27. If $det(M_1 + M_2) \neq 0$ and $\rho\{(M_1 + M_2)^{-1}(M_1 - M_2)\overline{D}\} < 1$ for any diagonal matrix $\overline{D} \in [-I, I]$, then the HLCP (2) has a unique solution.

Now, we discuss the following examples:

Example 3.2. Consider the matrices A and B as follows

$$A_{m\times m} = a_{ij} = \begin{cases} 1.0 & if\ i = j \\ 0 & if\ i > j\ or\ i < j \end{cases} \text{ and}$$

$$B_{m\times m} = b_{ij} = \begin{cases} 0.786 & if\ i = j \\ 0 & if\ i > j \\ 0.111 & if\ i < j \end{cases}$$

with m=50 and n=25, the result is as follows: $\sigma_{max}(B)^{25} = 0.00242976 < 1 = \sigma_{min}(A)^{25}$.

This fulfills the condition specified in Theorem 3.3. Consequently, the AVE (10) possesses a unique solution for each b.

Example 3.3. Let see the Example (3.1) again, where A and B is given below

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} -0.9 & 0.2 \\ -0.4 & -0.8 \end{bmatrix}.$$

Clearly, $\sigma_{min}(A) = 0, \sigma_{max}(B) = 1.0132, \sigma_{max}(|B|) = 1.16$ and $\sigma_{max}(A^{-1}B) = 1.0132, \sigma_{min}(A^{-1}B) = 0.7896$.

Here AVE (3) has unique solution x = $[6.6667, -38.3334]^T$ for the $b = [5, -5]^T$, but conditions (7) , (8) and (9) are not satisfying. Also n-AVE (10) for n=2 has unique solution x = $[11.1429, 5.8571]^T$ for the $b = [5, -5]^T$, but condition, $\sigma_{max}(B^2) < \sigma_{min}(A^2)$ is not satisfying. In the future, it is possible to modify these conditions.

CONCLUSION

The research of the unique solution is a vital branch of theoretical analysis of the LCP, HLCP and AVE. Due to the numerous applications of the LCP and HLCP in different fields, the study of unique solvability has been given a lot of attention. By equivalence relation between LCP and AVE, the study of the unique solvability conditions of the AVE is also interesting. In the literature, many unique solvability conditions are established for AVE/HLCP/LCP (see, e.g., [7, 11, 14, 20-21, 25, 29, 38-40, 48-49]). Some conditions did not satisfy some instances (see Example 3.1), so revision is needed for such conditions.

In this study, we have obtained the unique solvability condition for LCP (1), HLCP (2), and n-AVE (10). This is a generalization of the works that were previously established in [20, 38, 47-48] for AVE. We obtained unique solvability results for LCP and HLCP, which are superior to those of Achache et al. [14] and Li et al. [11]. By using Example (3.3), we raise an open problem for the future: "Can existing conditions be revised for the unique solution of AVE (3)?" Further investigation is necessary for this. Furthermore, future research on numerical methods for LCP, HLCP, and n-AVE offers intriguing and promising avenues to explore.

The following is the outline of the findings of this paper:

1. The unique solvability conditions for n-AVE are discussed.

2. New unique solvability conditions have been suggested for the HLCP and LCP.

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AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw

data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

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