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Research Article

Cut topology generated by generalized triangular fuzzy numbers

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ABSTRACT

In this study, we propose a new topology called the cut topology. We first develop a synthetic base by using the α cut family of the generalized triangular fuzzy number. Then, we generate the cut topology by the synthetic base and give subspace cut topology on a subset of any set. Moreover, the fundamental concepts such as interior, closure, and limit points of a set in the cut topology are discussed. Finally, neighborhood, continuity, and homeomorphism are analyzed with examples.

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INTRODUCTION

Fuzzy set theory has proven to be a powerful tool for handling uncertainty and imprecision. Fuzzy numbers, a fundamental concept within fuzzy set theory, extend traditional numerical representations to accommodate uncertainty in a more flexible manner. Among various types of fuzzy numbers, generalized triangular fuzzy numbers emerge as a versatile and expressive model, offering a broader scope for capturing complex relationships and uncertainties in real-world applications. The study of fuzzy numbers often involves their application in diverse fields such as decision analysis, optimization, and system modeling. One crucial aspect of leveraging fuzzy numbers lies in their ability to form topological structures that encapsulate the relationships and interactions between different elements. Due to the importance of mathematical expression of uncertain concepts, researchers study new set theories such as fuzzy sets [1] and intuitionistic fuzzy sets [2]. Zadeh [1] transforms the two-valued set (the classical set) into the infinite-valued set (the fuzzy sets) with fuzzy sets theory. He proposes the concepts of fuzzy set and fuzzy number to deal with uncertainty and gives their applications in [3]. Then Zadeh and Chang [4] study fuzzy mapping. Also, many researchers study fuzzy numbers on the set of real numbers ℝ in [5-9].

The topological characteristics of fuzzy numbers are instrumental in comprehending the nuances of imprecision and ambiguity inherent in decision-making scenarios. In addition, the researchers obtain different topologies based on the theory of fuzzy sets. For example, Chang [10] identifies fuzzy topology and denotes it by T. He shows that T is the family of fuzzy sets on the set E and satisfies three axioms similar to the classical topology. Then, Lowen [11] redefines the fuzzy topology using the semi-closure operators and the hypergraph function to show the connection between fuzzy topological spaces. Also, Onasanya and

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Hoshoka-Mayerova [12] obtain τ^* topology by cuts collection of the fuzzy set. But τ^* topology is defined on the fuzzy set, not on the universal fuzzy set. Zhang [13] obtains a natural topology for fuzzy numbers, and Font et al. [14] define convergence topology.

Padmapriya, and Thangavelu [15] define τ_A topology generated by alpha cuts of a fuzzy set having finite elements. Then, Padmapriya and Thangavelu [16] study two different topologies generated by fuzzy numbers. They [16] show that $T(A)$ is generated on the set of real numbers $\mathbb{R}$ and $T^+(A)$ is generated on (a, b). It is observed that in some cases the topologies are not valid when τ^* in [16] and $T(A)$ topologies in [12] which are studied in section 2. Therefore, in this study we show that τ^* and $T(A)$ topologies are incorrect by counterexamples. We also point out that assertion (1) of Proposition 3.3 in [15] is not true in general. We verify that the corresponding assertion in [15] is incorrect by a counterexample. After correcting it [[15], (1) of Proposition 3.3], we obtain a new topology by α cuts family of a generalized triangular fuzzy number in section 3. Therefore, we consider the α cuts family of the generalized triangular fuzzy number as a synthetic base, and we define the cut topology. We give some important properties in the cut topology. Then, we study fundamental concepts such as subspace topology, interior, exterior, and limit points for a set in the cut topology. We also analyze neighborhood, continuity, and homeomorphism in the cut topological spaces with examples. This paper is derived from the Yanardag's masters thesis in [19].

PRELIMINARY

In this section, we consider the many of the basic concepts in general topology and give the basic properties of fuzzy sets. Then we show by a counterexample that assertion (1) of Proposition 3.3 in [15] is not true in general. After we examine τ^* topology given by the authors Onasanya and Mayerova in [12] and $T(A)$ topology given by the authors Padmapriya and Thangavelu in [16], we observe that both topologies are not valid in some examples.

Definition 2.1. [18] Let (E, τ) be a topological space and subfamily $\mathfrak{B} \subseteq \tau$. If every set in τ is a union of sets from $\mathfrak{B}$, then $\mathfrak{B}$ is called a basis for τ.

Definition 2.2. [19] Let $\mathfrak{B}$ be a collection of subsets of non-empty set E. $\mathfrak{B}$ is called a synthetic base(proto-basis) for a topology on E when the following conditions hold

1. $E = \bigcup_{B \in \mathfrak{B}} B$,
2. If $B_1, B_2 \in \mathfrak{B}$, then $B_1 \cap B_2$ is a union of subsets from $\mathfrak{B}$.

Definition 2.3. [1] Let E be a non-empty set. A fuzzy set $\tilde{A}$ is defined as

$$\tilde{A} = \{\langle x, \mu_{\tilde{A}}(x)\rangle \mid x \in E\}$$

where membership function $\mu_{\tilde{A}}: E \to [0,1]$ for $x \in E$. The set of all fuzzy sets on E is denoted by $\mathcal{F}(E)$.

Definition 2.4. [8] Let $\tilde{A}, \tilde{B} \in \mathcal{F}(E)$. Then, for all $x \in E$,

1. The union of fuzzy sets $\tilde{A}$ and $\tilde{B}$ is denoted by $\tilde{A} \tilde{\cup} \tilde{B}$ and this is defined as

$$\tilde{A} \tilde{\cup} \tilde{B} = \{\langle x, \mu_{\tilde{A}\tilde{\cup}\tilde{B}}(x)\rangle \mid x \in E\}$$

where $\mu_{(\tilde{A}\tilde{\cup}\tilde{B})(x)} = max\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}$. Also supremum can be replaced with maximum for arbitrary unions of fuzzy sets $\tilde{A}_i$ for $i \in \mathbb{N}$,

$$\left(\bigcup_{i\in\mathbb{N}} \tilde{A}_i\right)(x) = \sup\{\mu_{\tilde{A}_i}(x) \mid i \in \mathbb{N}\}.$$

2. The intersection of fuzzy sets $\tilde{A}$ and $\tilde{B}$ is denoted by $\tilde{A} \tilde{\cap} \tilde{B}$ and this is defined as

$$\tilde{A} \tilde{\cap} \tilde{B} = \{\langle x, \mu_{\tilde{A}\tilde{\cap}\tilde{B}}(x)\rangle \mid x \in E\}$$

where $\mu_{(\tilde{A}\tilde{\cap}\tilde{B})(x)} = min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}$. Also infimum can be replaced with minimum for arbitrary intersections of fuzzy sets $\tilde{A}_i$ for $i \in \mathbb{N}$,

$$\left(\bigcap_{i\in\mathbb{N}} \tilde{A}_i\right)(x) = \inf\{\mu_{\tilde{A}_i}(x) \mid i \in \mathbb{N}\}.$$

3. The complement of fuzzy set $\tilde{A}$ is denoted by $\tilde{A}^c$ and this is defined as

$$\tilde{A}^c = \{\langle x, \mu^c(x) \mid x \in E\}$$

where $\mu^c_{\tilde{A}}(x) = 1 - \mu_{\tilde{A}}(x)$.

Definition 2.5. [8] Let $\tilde{A}$ be a fuzzy set defined on E and $\alpha \in [0,1]$. Then,

1. α cut of $\tilde{A}$, denoted by ${}^{\alpha}\tilde{A}$, is defined as

$${}^{\alpha}\tilde{A} = \{x \in E \mid \mu_{\tilde{A}}(x) \geq \alpha,\ \alpha \in [0,1]\};$$

2. strong α cut of $\tilde{A}$, denoted by ${}^{\alpha+}\tilde{A}$, is defined as

$${}^{\alpha+}\tilde{A} = \{x \in E \mid \mu_{\tilde{A}}(x) > \alpha,\ \alpha \in [0,1]\}.$$

Teorem 2.6. [8] Let $\tilde{A}, \tilde{B} \in \mathcal{F}(E)$. Then, the following properties hold for all $\alpha, \beta \in [0,1]$:

1. ${}^{\alpha+}\tilde{A} \subseteq {}^{\alpha}\tilde{A}$,
2. ${}^{\beta}\tilde{A} \subseteq {}^{\alpha}\tilde{A}$ and ${}^{\beta+}\tilde{A} \subseteq {}^{\alpha+}\tilde{A}$ if $\alpha \leq \beta$,
3. ${}^{\alpha}(\tilde{A} \tilde{\cap} \tilde{B}) = {}^{\alpha}\tilde{A} \cap {}^{\alpha}\tilde{B}$ and ${}^{\alpha}(\tilde{A} \tilde{\cup} \tilde{B}) = {}^{\alpha}\tilde{A} \cup {}^{\alpha}\tilde{B}$,
4. $\bigcap_{i\in\mathbb{N}} {}^{\alpha}\tilde{A}_i = {}^{\alpha}\left(\tilde{\bigcap}_{i\in\mathbb{N}} \tilde{A}_i\right)$,
5. $\bigcup_{i\in\mathbb{N}} {}^{\alpha}\tilde{A}_i \subseteq {}^{\alpha}\left(\tilde{\bigcup}_{i\in\mathbb{N}} \tilde{A}_i\right)$,
6. ${}^{\alpha}\tilde{A} \cup {}^{\beta}\tilde{A} = \begin{cases} {}^{\alpha}\tilde{A}, & \text{if } \alpha \leq \beta \\ {}^{\beta}\tilde{A}, & \text{otherwise} \end{cases}$ *and* ${}^{\alpha}\tilde{A} \cap {}^{\beta}\tilde{A} = \begin{cases} {}^{\beta}\tilde{A}, & \text{if } \alpha \leq \beta \\ {}^{\alpha}\tilde{A}, & \text{otherwise} \end{cases}$,
7. ${}^{\alpha_1}\tilde{A} \cup {}^{\alpha_2}\tilde{A} \cup {}^{\alpha_3}\tilde{A} \cup \ldots \cup {}^{\alpha_n}\tilde{A} = {}^{\alpha}\tilde{A}$ where $\alpha = \min\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$,
8. ${}^{\alpha_1}\tilde{A} \cap {}^{\alpha_2}\tilde{A} \cap {}^{\alpha_3}\tilde{A} \cap \ldots \cap {}^{\alpha_n}\tilde{A} = {}^{\beta}\tilde{A}$ where $\beta = \max\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$.

Definition 2.7. [20] Let $w \in (0,1]$ and $a \leq b \leq c$ such that $a, b, c \in \mathbb{R}$. A fuzzy set $\tilde{A}$ is called generalized triangular fuzzy number on $\mathbb{R}$, whose membership function is defined as

$$\mu_{\widetilde{A_w}}(x) = \begin{cases} \dfrac{w(x-a)}{b-a}, & a \le x \le b \\ \dfrac{w(c-x)}{c-b}, & b \le x \le c \\ 0, & \text{otherwise.} \end{cases}$$

Generalized triangular fuzzy number is denoted by by $\tilde{A}_w = [a, b, c; w]$.

Note 2.8. [21] If $w = 1$, then $\tilde{A}_1 = [a, b, c; 1]$ can be shown by $A = [a, b, c]$. Also, $\tilde{A}_1$ is called a triangular fuzzy number.

Note 2.9. [22] ${}^{\alpha}\tilde{A}_w$ is a closed interval and ${}^{\alpha+}\tilde{A}_w$ is an open interval on $\mathbb{R}$ for $\alpha \in (0, w]$. Therefore, ${}^{\alpha}\tilde{A}_w$ and ${}^{\alpha+}\tilde{A}_w$ can be given as

$${}^{\alpha}\tilde{A}_w = [x_\alpha, y_\alpha] = \left[a + \frac{\alpha(b-a)}{w}, c - \frac{\alpha(c-b)}{w}\right],$$

$${}^{\alpha+}\tilde{A}_w = (x_\alpha, y_\alpha) = \left(a + \frac{\alpha(b-a)}{w}, c - \frac{\alpha(c-b)}{w}\right).$$

Arbitrary unions of ${}^{\alpha}\tilde{A}_w$ and arbitrary intersections of ${}^{\alpha}\tilde{A}_w$ are given by the authors Padmadriya and Thangavelu in Proposition 3.3 in [15] as follow:

Let $\tilde{A}$ be a fuzzy set defined on set E. For

$\alpha_0 = \inf\{x \mid x \in \Delta\}$, $\beta_0 = \sup\{x \mid x \in \Delta\}$, $\Delta \subseteq [0,1]$ and $\alpha, \beta \in [0,1]$, the following properties hold

1. ${}^{\alpha_0}\tilde{A} = \bigcup_{\alpha\in\Delta} {}^{\alpha}\tilde{A}$,
2. ${}^{\beta_0}\tilde{A} \subseteq \bigcap_{\alpha\in\Delta} {}^{\alpha}\tilde{A}$.

But the following example shows that assertion (1) in Proposition 3.3 given by the authors Padmadriya and Thangavelu in [15] are not true in general.

Example 2.10. Let $\tilde{A} = [3,5,7]$ be a fuzzy set defined on $\mathbb{R}$ and $\Delta = (0.5,1]$. Since $\inf\{x \mid x \in \Delta\} = 0.5$ and ${}^{0.5}\tilde{A} = [2,4]$, we have $\bigcup_{\alpha\in\Delta} {}^{\alpha}\tilde{A} = [2,4]$ according to Padmapriya et al. [15]. But ${}^{\alpha}\tilde{A}$ are closed intervals and an arbitrary union of closed intervals for $\Delta = (0.5,1]$ must be in the form of an open interval. That is $\bigcup_{\alpha\in\Delta} {}^{\alpha}\tilde{A} = (2,4)$ for $\Delta = (0.5,1]$.

In this case, it is clearly that ${}^{0.5}\tilde{A} \not\subseteq \bigcup_{\alpha\in\Delta} {}^{\alpha}\tilde{A}$ for $\Delta = (0.5,1]$. Therefore ${}^{\alpha_0}\tilde{A} = \bigcup_{\alpha\in\Delta} {}^{\alpha}\tilde{A}$ given by Padmapriya et al. [15] is not verified in general.

Thus (1) Proposition 3.3 in [15] can be given as follow:

Proposition 2.11. Let $\tilde{A}$ be a fuzzy set defined on E. For $\alpha_0 = \inf\{x \mid x \in \Delta\}$, $\beta_0 = \sup\{x \mid x \in \Delta\}$, $\Delta \subseteq [0,1]$ and $\alpha, \beta \in [0,1]$, the following properties hold

Then, the authors Padmapriya and Thangavelu proposed $T(\tilde{A})$ and $T^{+}(\tilde{A})$ in [16] as follow. $T(\tilde{A})$ is collection of all α cuts of $\tilde{A}$ and $T^{+}(\tilde{A})$ is collection of all strong α cuts of $\tilde{A}$ on E.

Proposition 2.12. [16] Let $\tilde{A} = [a, b, c]$ be a triangular fuzzy number. Then, the following properties hold

1. $T(\tilde{A}) = \{[x, y]: (x-a)(c-b) = (b-a)(c-y),$ $a \le x \le b \text{ and } b \le y \le c\} \cup \{\emptyset\}$,
2. $T^{+}(\tilde{A}) = \{(x, y): (x-a)(c-b) = (b-a)(c-y),$ $a \le x \le b \text{ and } b \le y \le c\} \cup \{\mathbb{R}\}$.

Also, the authors Padmapriya and Thangavelu [16] gave that $T(\tilde{A})$ and $T^{+}(\tilde{A})$ generated the topology on $\mathbb{R}$ and (a, b), respectively. Then, they presented $T(\tilde{A})$ and $T^{+}(\tilde{A})$ generated by different types of fuzzy numbers.

In addition, a new topology τ^* was introduced in Theorem 2.1 by the authors Onasanya and Mayerova [12]. The τ^* topology is the collection of ${}^{\alpha}\tilde{A}$. It is defined on a fuzzy set $\tilde{A}$ of E rather than on non-empty set E. τ^* is given as

$$\tau^* = \{{}^{\alpha}\tilde{A} \mid \mu_{\tilde{A}}(x) \ge \alpha\}\backslash E$$

for $\alpha \in [0,1]$.

When τ^* topology given by the authors Onasanya and Mayerova in [12] and $T(A)$ topology given by the authors Padmapriya and Thangavelu in [16] are examined, it is clear that two topologies are similar. τ^* is the collection of ${}^{\alpha}\tilde{A}$ and is defined on $\tilde{A}$; $T(A)$ is the collection of ${}^{\alpha}\tilde{A}$ and is defined on the set of real numbers $\mathbb{R}$. But it is observed that topologies are not valid in the following example.

Example 2.13. Let triangular fuzzy number $\tilde{A} = [2,4,5]$ be defined on $\mathbb{R}$. Then, $T(\tilde{A})$ defined on $\mathbb{R}$ and τ^* defined on $\tilde{A}$ are as follows, respectively.

$$\begin{aligned} T(\tilde{A}) &= \{{}^{\alpha}\tilde{A} \mid \forall\alpha \in [0,1]\} \cup \{\emptyset, \mathbb{R}\} \\ &= \{[x_\alpha, y_\alpha] \mid \forall\alpha \in [0,1]\} \cup \{\emptyset, \mathbb{R}\} \\ &= \{[2+2\alpha,\ 5-\alpha] \mid \forall\alpha \in [0,1]\} \cup \{\emptyset, \mathbb{R}\} \end{aligned}$$

and

$$\begin{aligned} \tau^* &= \{{}^{\alpha}\tilde{A} \mid \forall\alpha \in [0,1]\} \cup \{\emptyset\} \\ &= \{[x_\alpha, y_\alpha] \mid \forall\alpha \in [0,1]\} \cup \{\emptyset\} \\ &= \{[2+2\alpha,\ 5-\alpha] \mid \forall\alpha \in [0,1]\} \cup \{\emptyset\}. \end{aligned}$$

According to Padmapriya et al., $T(\tilde{A})$ is a topology. In this case, an arbitrary union of elements in $T(\tilde{A})$ must be in $T(\tilde{A})$. It is clear that for all $\alpha \in (0.5, 1]$, $[x_\alpha, y_\alpha] \in T(\tilde{A})$. Thus, we get $\bigcup_{\alpha\in I}[x_\alpha, y_\alpha] = (3, 4.5)$ for $I = (0.5, 1]$ and $\alpha \in (0.5, 1]$. But $(3, 4.5) \notin T(\tilde{A})$. Therefore, $T(\tilde{A})$ is not a topology on $\mathbb{R}$.

Similarly, since $(3, 4.5) \notin \tau^*$, τ^* are not a topology on $\tilde{A}$. Therefore τ^* and $T(\tilde{A})$ are not valid in general.

RESULTS AND DISCUSSION

In this section, the α cuts family of the generalized triangular fuzzy number $\tilde{A}_w$ is considered as the synthetic base. The cut topology is generated by this synthetic base. Also, the fundamental concepts such as subspace topology, neighborhood in the cut topological space, continuity and homeomorphism are examined with examples.

The Cut Topological Space and its Properties

In this subsection, first a new topology $T(\beta_{\tilde{A}_w})$ called the cut topology is generated. Then the fundamental concepts such as interior, exterior and closure of a set in the cut topological space are given with examples.

Theorem 3.1. Let the generalized triangular fuzzy number $\tilde{A}_w = [a, b, c; w]$ be defined on $A = [a, c]$. Then

$$\beta_{\tilde{A}_w} = \{{}^{\alpha}\tilde{A}_w \mid \alpha \in [0, w],\ \exists! \, w \in (0,1]\}$$

is the synthetic base of a topology on A.

Proof. It is obvious that ${}^{\alpha}\tilde{A}_w \subseteq A$ for all $\alpha \in [0,1]$ with ${}^{\alpha}\tilde{A}_w = [x_\alpha, y_\alpha]$. By Definition 2.2,

1. $A = \bigcup_{{}^{\alpha}\tilde{A}_w \in \beta_{\tilde{A}_w}} {}^{\alpha}\tilde{A}_w$ since ${}^{0}\tilde{A}_w = A$ for $\alpha = 0$;
2. If $\alpha_1 \leq \alpha_2$ for $\alpha_1, \alpha_2 \in [0, w]$, then ${}^{\alpha_1}\tilde{A}_w \supseteq {}^{\alpha_2}\tilde{A}_w$ and ${}^{\alpha_1}\tilde{A}_w \cap {}^{\alpha_2}\tilde{A}_w = {}^{\alpha_2}\tilde{A}_w$. However if $\alpha_2 \leq \alpha_1$, then it is easily seen that ${}^{\alpha_1}\tilde{A}_w \cap {}^{\alpha_2}\tilde{A}_w \in \beta_{\tilde{A}_w}$. Therefore, for all ${}^{\alpha_1}\tilde{A}_w, {}^{\alpha_2}\tilde{A}_w \in \beta_{\tilde{A}_w}$, ${}^{\alpha_1}\tilde{A}_w \cap {}^{\alpha_2}\tilde{A}_w$ is a union of several subsets of $(\beta_{\tilde{A}_w})$. Thus $\beta_{\tilde{A}_w}$ is the synthetic base of a topology on A.

Example 3.2. Let the generalized triangular fuzzy number $\tilde{A}_{0.5} = [3,5,7; 0.5]$ be defined on $A = [3,7]$. Then,

$\beta_{\tilde{A}_{0.5}} = \{{}^{\alpha}\tilde{A}_{0.5} | \alpha \in [0,0.5]\}$ is the synthetic base of a topology on A.

It is obvious that ${}^{\alpha}\tilde{A}_{0.5} \subseteq A$ for all $\alpha \in [0,0.5]$ with ${}^{\alpha}\tilde{A}_{0.5} = [x_\alpha, y_\alpha]$. Thus, $\beta_{\tilde{A}_{0.5}}$ is the subset family of [3,7]. Also,

1. $A = \bigcup_{{}^{\alpha}\tilde{A}_{0.5} \in \beta_{\tilde{A}_{0.5}}} {}^{\alpha}\tilde{A}_{0.5}$ since ${}^{0}\tilde{A}_w = [3,7]$ for $\alpha = 0$;
2. Since sets ${}^{\alpha}\tilde{A}_{0.5}$ for $\alpha \in [0,1]$ are closed intervals, finite intersections of sets ${}^{\alpha}\tilde{A}_{0.5}$ belong to $\beta_{\tilde{A}_{0.5}}$.

Therefore, by Definition 2.2, $\beta_{\tilde{A}_{0.5}}$ is the synthetic base of a topology on set A.

Theorem 3.3. Let the generalized triangular fuzzy number $\tilde{A}_w = [a, b, c; w]$ be defined on $A = [a, c]$. Then,

$$T(\beta_{\tilde{A}_w}) = \{\left[a + \frac{\alpha(b-a)}{w}, c - \frac{\alpha(c-b)}{w}\right], \left(a + \frac{\alpha(b-a)}{w}, c - \frac{\alpha(c-b)}{w}\right) \mid \alpha \in [0, w], \exists! \, w \in (0,1]\}$$

is a topology on A generated by $(\beta_{\tilde{A}_w})$.

Proof. ${}^{\alpha}\tilde{A}_w = [x_\alpha, y_\alpha]$ for $\alpha \in (0, w]$ are closed intervals such that ${}^{\alpha}\tilde{A}_w = \left[a + \frac{\alpha(b-a)}{w}, c - \frac{\alpha(c-b)}{w}\right]$ for $0 \leq \alpha \leq w$. Since sets ${}^{\alpha}\tilde{A}_w$ are closed intervals for $0 \leq \alpha \leq w$, then $\bigcup_{\alpha \in I} {}^{\alpha}\tilde{A}_w$, the arbitrary union of sets ${}^{\alpha}\tilde{A}_w$ for $I \subseteq [0,1]$, is one of $\emptyset$, a closed interval ${}^{v}\tilde{A}_w$ and an open interval ${}^{v+}\tilde{A}_w$ such that $v = inf\{\alpha \mid \alpha \in I\}$. Since $(b, b) = \emptyset$ for $\alpha = w$, $\emptyset$ can be given as (b, b). Then,

$$T(\beta_{\tilde{A}_w}) = \{\left[a + \frac{\alpha(b-a)}{w}, c - \frac{\alpha(c-b)}{w}\right], \left(a + \frac{\alpha(b-a)}{w}, c - \frac{\alpha(c-b)}{w}\right) \mid \alpha \in [0, w],\ \exists! \, w \in (0,1]\}$$

is obtained and proof is completed.

Definition 3.4. Let the generalized triangular fuzzy number $\tilde{A}_w = [a, b, c; w]$ be defined on $A = [a, c]$. Then

$$T(\beta_{\tilde{A}_w}) = \{\left[a + \frac{\alpha(b-a)}{w}, c - \frac{\alpha(c-b)}{w}\right], \left(a + \frac{\alpha(b-a)}{w}, c - \frac{\alpha(c-b)}{w}\right) \mid \alpha \in [0, w],\ \exists! \, w \in (0,1]\}$$

where $T(\beta_{\tilde{A}_w})$ is defined on set A, is called the cut topology or the topology generated by generalized triangular fuzzy number $\tilde{A}_w$. Also, $(A, T(\beta_{\tilde{A}_w}))$ is called the cut topological space.

Definition 3.5. Let $(A, T(\beta_{\tilde{A}_w}))$ be the cut topological space and $B \subseteq A$. Then,

1. B is an open set if $B \in T(\beta_{\tilde{A}_w})$,
2. B is a closed set if $(A \backslash B) \in T(\beta_{\tilde{A}_w})$.

Definition 3.6. Let $(A, T(\beta_{\tilde{A}_w}))$ be the cut topological space and the generalized triangular fuzzy number $\tilde{A}_w = [a, b, c; w]$ be defined on $A = [a, c]$. Then, $T(\beta_{\tilde{A}_w})^k$ is called the closed set family and it is defined by

$$T(\beta_{\tilde{A}_w})^k = \{A \backslash \left[a + \frac{\alpha(b-a)}{w}, c - \frac{\alpha(c-b)}{w}\right], A \backslash \left(a + \frac{\alpha(b-a)}{w}, c - \frac{\alpha(c-b)}{w}\right) \mid \alpha \in [0, w],\ \exists! \, w \in (0,1]\}.$$

Example 3.7. Let the generalized triangular fuzzy number $\tilde{A}_{0.5} = [1, 2, 3; 0.5]$ be defined on $A = [1, 3]$. Thus, we have ${}^{\alpha}\tilde{A}_{0.5} = \left[1 + \frac{\alpha}{0.5}, 3 - \frac{\alpha}{0.5}\right]$ for $0 \leq \alpha \leq 0.5$. Since arbitrary union of sets ${}^{\alpha}\tilde{A}_{0.5}$ is one of open interval, closed interval or $\emptyset$; then

$$T(\beta_{\tilde{A}_{0.5}}) = \{\left[1 + \frac{\alpha}{0.5}, 3 - \frac{\alpha}{0.5}\right], \left(1 + \frac{\alpha}{0.5}, 3 - \frac{\alpha}{0.5}\right) \mid \alpha \in [0, 0.5]\}$$

is obtained. Hence, by Definition 3.4., $T(\beta_{\tilde{A}_{0.5}})$ is the cut topology on set A.

Definition 3.8. Let $(A, T(\beta_{\tilde{A}_w}))$ be the cut topological space and $B \subseteq A$. $T_B(\beta_{\tilde{A}_w})$ is the subspace topology on set B and is defined by

$$T_B(\beta_{\tilde{A}_w}) = \{U \cap B \mid U \in T(\beta_{\tilde{A}_w})\}.$$

Definition 3.9. Let the generalized triangular fuzzy number $\tilde{A}_{0.5} = [8, 10, 14; 0.5]$ be defined on $A = [8, 14]$ and $(A, T(\beta_{\tilde{A}_w}))$ be the cut topological space. For $B = (8,9]$,

$$\begin{aligned} T_B(\beta_{\tilde{A}_{0.5}}) &= \{U \cap B \mid U \in T(\beta_{\tilde{A}_{0.5}})\} \\ &= \{\{9\}, (8,9], (8+4\alpha, 9], [8+4\alpha, 9] \mid \alpha \in (0, 0.25]\} \end{aligned}$$

is obtained. By Definition 3.8, $T_B(\beta_{\tilde{A}_{0.5}})$ is the subspace topology on set B.

Theorem 3.10. Let the generalized triangular fuzzy numbers $\tilde{A}_{w_1} = [a, b, c; w_1]$ and $\tilde{A}_{w_2} = [a, b, c; w_2]$ be defined on $A = [a, c]$. For $\alpha \in [0, w_1]$ and $\gamma \in [0, w_2]$, if $\alpha = \frac{\gamma . w_1}{w_2}$, then ${}^{\alpha}\tilde{A}_{w_1} = {}^{\gamma}\tilde{A}_{w_2}$.

Proof. The proof is clear from Note 2.9.

Theorem 3.11. Let the generalized triangular fuzzy numbers $\tilde{A}_{w_1} = [a, b, c; w_1]$ and $\tilde{A}_{w_2} = [a, b, c; w_2]$ be defined on $A = [a, c]$. For $\alpha \in [0, w_1]$ and $\gamma \in [0, w_2]$, if $\alpha = \frac{\gamma . w_1}{w_2}$, then

$$T\left(\beta_{\tilde{A}_{w_1}}\right) = T\left(\beta_{\tilde{A}_{w_2}}\right).$$

Proof. Let $\alpha = \frac{\gamma . w_1}{w_2}$. By Theorem 3.10, ${}^{\alpha}\tilde{A}_{w_1} = {}^{\gamma}\tilde{A}_{w_2}$. Then,

$$\begin{aligned}
T\left(\beta_{\tilde{A}_{w_1}}\right) &= \{\left[a + \frac{\alpha(b-a)}{w_1}, c - \frac{\alpha(c-b)}{w_1}\right], \\
&\quad \left(a + \frac{\alpha(b-a)}{w_1}, c - \frac{\alpha(c-b)}{w_1}\right) \\
&\quad \mid \alpha \in [0, w_1],\ \exists!\, w_1 \in (0,1]\} \\
&= \{\left[a + \frac{\frac{\gamma . w_1}{w_2}(b-a)}{w_1}, c - \frac{\frac{\gamma . w_1}{w_2}(c-b)}{w_1}\right], \\
&\quad \left(a + \frac{\frac{\gamma . w_1}{w_2}(b-a)}{w_1}, c - \frac{\frac{\gamma . w_1}{w_2}(c-b)}{w_1}\right) \\
&\quad \mid \gamma \in [0, w_2],\ \exists!\, w_1, w_2 \in (0,1]\} \\
&= \{\left[a + \frac{\gamma(b-a)}{w_2}, c - \frac{\gamma(c-b)}{w_2}\right], \\
&\quad \left(a + \frac{\gamma(b-a)}{w_2}, c - \frac{\gamma(c-b)}{w_2}\right) \\
&\quad \mid \gamma \in [0, w_2],\ \exists!\, w_2 \in (0,1]\} \\
&= T\left(\beta_{\tilde{A}_{w_2}}\right).
\end{aligned}$$

Hence, $T(\beta_{\tilde{A}_{w_1}}) = T(\beta_{\tilde{A}_{w_2}})$ and proof is completed.

Definition 3.12. Let $(A, T(\beta_{\tilde{A}_w}))$ be the cut topological space and $B \subseteq A$. If there is at least one open set U such that $x \in U \subseteq B$, then x is called an interior point of B. The union of all the interior points of B is called the interior of B and is denoted by B°.

Theorem 3.13. Let $(A, T(\beta_{\tilde{A}_w}))$ be the cut topological space and $B \subseteq A$. Then

$$B^{\circ} = \bigcup_{i \in \mathbb{N}} \{U_i : U_i \subseteq B,\ U_i \in T\left(\beta_{\tilde{A}_w}\right)\}.$$

Proof. It is obvious.

Definition 3.14. Let $(A, T(\beta_{\tilde{A}_w}))$ be the cut topological space and $B \subseteq A$. Then

$$B^{\circ} \neq \emptyset \quad if \quad b \in B,$$
$$B^{\circ} = \emptyset \quad if \quad b \notin B.$$

Definition 3.15. Let $(A, T(\beta_{\tilde{A}_w}))$ be the cut topological space and $B \subseteq A$. The closure $\overline{B}$ of B is the intersection of all closed subsets of K containing A. That is

$$\overline{B} = \bigcap_{i \in \mathbb{N}} \{K_i : B \subseteq K_i,\ (A \backslash K_i) \in T\left(\beta_{\tilde{A}_w}\right)\}.$$

Note 3.16. Let $(A, T(\beta_{\tilde{A}_w}))$ be the cut topological space and $B \subseteq A$. If $b \in B$, then $\overline{B} = A$.

Definition 3.17. Let $(A, T(\beta_{\tilde{A}_w})$ be the cut topological space and $B \subseteq A$. The outside dis(B) of B is defined by

$$\text{dis(B)} = A \setminus \overline{B}.$$

Note 3.18. Let $(A, T(\beta_{\tilde{A}_w}))$ be the cut topological space and $B \subseteq A$. If $b \in B$, then dis(B) $= \emptyset$.

Definition 3.19. Let $(A, T(\beta_{\tilde{A}_w})$ be the cut topological space and $B \subseteq A$. If $(B \backslash \{x\}) \cap U \neq \emptyset$ for each $U \in T(\beta_{\tilde{A}_w})$ in the property $x \in U$, then x is called the limit point of B. The union of all the limit points of B is denoted by B'

Note 3.20. Let $(A, T(\beta_{\tilde{A}_w})$ be the cut topological space and $B \subseteq A$. If $b \in B$, then $B' = [a, b) \cup (b, c]$.

Theorem 3.21. Let $(A, T(\beta_{\tilde{A}_w}))$ be the cut topological space. Then ${}^{\alpha}\tilde{A}_w \cup ({}^{\alpha}\tilde{A}_w)'$ is a closed set.

Proof. Let $b \in ({}^{\alpha}\tilde{A}_w)$ for all $\alpha \in [0, w]$. By Note 3.20, $({}^{\alpha}\tilde{A}_w)' = A - \{b\}$. Then,

${}^{\alpha}\tilde{A}_w \cup ({}^{\alpha}\tilde{A}_w)' = {}^{\alpha}\tilde{A}_w \cup (A - \{b\}) = A$ is obtained. Since $A^t = \emptyset$ and $\emptyset \in T(\beta_{\tilde{A}_w})$, A is a closed set. Hence ${}^{\alpha}\tilde{A}_w \cup ({}^{\alpha}\tilde{A}_w)'$ is a closed set.

Definition 3.22. Let $(A, T(\beta_{\tilde{A}_w}))$ be the cut topological space. If there exists an open set $U \in T(\beta_{\tilde{A}_w})$ such that

$$B \cap U = \{x\}$$

for $x \in U$ with $B \subseteq A$, then x is called an isolated point of set B.

Note 3.23. Let $(A, T(\beta_{\tilde{A}_w}))$ be the cut topological space. If $b \in B$, then b is an isolated point of set B.

Definition 3.24. Let $(A, T(\beta_{\tilde{A}_w}))$ be the cut topological space and $B \subseteq A$. A point $x \in A$ is called a boundary point of B if x is in the closure of both sets A and $A \setminus B$. The union of all the boundary points is defined by

$$\partial(B) = \overline{B} \cap \overline{A \setminus B}.$$

Example 3.25. Let the generalized triangular fuzzy number $\tilde{A}_{0.6} = [6, 8, 10; 0.6]$ be defined on $A = [6, 10]$, $(A, T(\beta_{\tilde{A}_{0.6}}))$ be the cut topological space and $B = (7, 10)$. Let us describe the interior, closure, outside, limit points, isolated points and boundary points of B, respectively.

The cut topology $T(\beta_{\tilde{A}_{0.6}})$ is shown as

$$T(\beta_{\tilde{A}_{0.6}}) = \{\left[6+\frac{2\alpha}{0.6}, 10-\frac{2\alpha}{0.6}\right], \left(6+\frac{2\alpha}{0.6}, 10-\frac{2\alpha}{0.6}\right) \mid \alpha \in [0, 0.6]\}.$$

For $T(\beta_{\tilde{A}_{0.6}})$, by Definition 3.12, since $b = 8$ and $8 \in B$, $B^{\circ} \neq \emptyset$ and $B^{\circ} = (7,9)$;

by Definition 3.15 and Note 3.16, since $b = 8$ and $8 \in B$, $\overline{B} = A$;

by Definition 3.17 and Note 3.18, since $b = 8$ and $8 \in B$, dis(B) $= \emptyset$.

Since $8 \in \{8\}$ and $(B \backslash \{8\}) \cap \{8\} = \emptyset$ for $\{8\} \in T(\beta_{\tilde{A}_{0.6}})$, 8 is not a limit point of B. But all points in set A except $\{8\}$ are limit points of B. Because $(B \backslash \{x\}) \cap U \neq \emptyset$ for each $U \in T(\beta_{\tilde{A}_{0.6}})$ in the property $x \in U$. So $B' = [6,10] \setminus \{8\}$. By Definition 3.22 and Note 3.23, since $b = 8$ and $8 \in B$, $\{8\}$ is an isolated point of B.

By Definition 3.24,

$$\begin{aligned} \partial(B) &= \overline{B} \cap \overline{A \setminus B} \\ &= [6,10] \cap ([6,7] \cup [9,10]) \\ &= [6,7] \cup [9,10]. \end{aligned}$$

Definition 3.26. Let $(A, T(\beta_{\tilde{A}_w})$ be the cut topological space and $x \in A$. If there is $U \in T(\beta_{\tilde{A}_w})$ such that $x \in U \subseteq B$, then B is called a neighborhood of point x. The neighborhood family of point x is denoted by $\mathcal{N}_w(x)$.

Note 3.27. Let $(A, T(\beta_{\tilde{A}_w})$ be the cut topological space. The only neighborhood of points a and c is ${}^{0}\tilde{A}_w$. But all points except a and c have infinitely many neighborhoods.

Continuity and Homeomorphism In The Cut Topological Space

In this subsection, the concepts of continuity and homeomorphism are discussed with examples in view of the cut topological space.

Definition 3.28. Let generalized triangular fuzzy numbers $\tilde{A}_{w_1} = [a, b, c; w_1]$ and $\tilde{B}_{w_2} = [m, n, p; w_2]$ be defined on sets $A = [a, c]$ and $B = [m, p]$, respectively. Also, $f\colon (A, T(\beta_{\tilde{A}_{w_1}})) \rightarrow (B, T(\beta_{\tilde{B}_{w_2}}))$ be a function where $A, T(\beta_{\tilde{A}_{w_1}}))$ and $(B, T(\beta_{\tilde{B}_{w_2}}))$ are the cut topological spaces. If $f^{-1}(U) \in T(\beta_{\tilde{A}_{w_1}})$ for each $U \in T(\beta_{\tilde{B}_{w_2}})$, then f is called a continuous function on A.

Theorem 3.29. $\tilde{A}_{w_1} = [a, b, c; w_1]$ and $\tilde{A}_{w_2} = [a, b, c; w_2]$ generalized triangular fuzzy numbers be defined on $A = [a, c]$. Then, the function $f\colon (A, T(\beta_{\tilde{A}_{w_1}})) \rightarrow (A, T(\beta_{\tilde{A}_{w_2}}))$ such that $f(x) = x$ is a continuous function on A.

Proof. $\tilde{A}_{w_1} = [a, b, c; w_1]$ and $\tilde{A}_{w_2} = [a, b, c; w_2]$ are defined on $A = [a, c]$. By Theorem 3.11, we get $T(\beta_{\tilde{A}_{w_1}}) = T(\beta_{\tilde{A}_{w_2}})$. Since $f(x) = x$, then $f(U) = U$ for each $U \in T(\beta_{\tilde{A}_{w_2}})$. Also, since $U \in T(\beta_{\tilde{A}_{w_1}})$, then f is a continuous function.

Theorem 3.30. Let generalized triangular fuzzy numbers $\tilde{A}_{w_1} = [a, b, c; w_1]$ and $\tilde{B}_{w_2} = [m, n, p; w_2]$ be defined on $A = [a, c]$ and $B = [m, p]$, respectively. Then, $f\colon (A, T(\beta_{\tilde{A}_{w_1}})) \rightarrow (B, T(\beta_{\tilde{B}_{w_2}}))$ defined by

$$f(x) = \begin{cases} \dfrac{x(n-m)+mb-an}{b-a}, & a \leq x \leq b \\ \dfrac{x(p-n)+nc-pb}{c-b}, & b < x \leq c. \end{cases}$$

Then, f is an injective, surjective and continuous function.

Proof. By definition of the function f, $f(A) = B$ i.e. $f([a, c]) = [m, p]$ is obtained. That is, the preimage of all elements in B is in A. Hence, the function f is a surjective function.

Consider $f(x_1) = f(x_2)$ with $x_1, x_2 \in [a, c]$ to determine whether the function f is injective. If $f(x_1) = f(x_2)$ for $a \leq x \leq b$, then

$$\begin{aligned} f(x_1) &= f(x_2) \\ \frac{x_1(n-m)+mb-an}{b-a} &= \frac{x_2(n-m)+mb-an}{b-a} \\ x_1 &= x_2 \end{aligned}$$

by the definition of function f.

Similarly, if $f(x_1) = f(x_2)$ for $b < x \leq c$, then it is clearly that $x_1 = x_2$. Therefore, the function f is an injective function.

Now, we must shown that $f^{-1}(U) \in T(\beta_{\tilde{A}_{w_1}})$ for each $U \in T(\beta_{\tilde{B}_{w_2}})$ to determine whether the function f is continuous. If $U \in T(\beta_{\tilde{B}_{w_2}})$, then U must be form in one of $[x_\gamma, y_\gamma]$, (x_γ, y_γ) or $\emptyset$.

If $U = \emptyset$, it is clear that $f^{-1}(\emptyset) = \emptyset$ and $\emptyset \in T(\beta_{\tilde{A}_{w_1}})$.

Let consider $U = [x_\gamma, y_\gamma]$. Now, we have to show that $f^{-1}([x_\gamma, y_\gamma]) \in T(\beta_{\tilde{A}_{w_1}})$ for each $[x_\gamma, y_\gamma] \in T(\beta_{\tilde{B}_{w_2}})$. The inverse of f, denoted by f^{-1}, is defined by

$$f^{-1}(x) = \begin{cases} \dfrac{x(b-a)-mb+an}{n-m}, & m \leq x \leq n \\ \dfrac{x(c-b)-nc+pb}{p-n}, & n < x \leq p. \end{cases}$$

Since $m \leq x \leq n$ and $n < x \leq p$ by definition of the function f^{-1}, we consider as $x = x_\gamma$ and $x = y_\gamma$, respectively. Also, it is known that

$$[x_\alpha, y_\alpha] = \left[a + \frac{\alpha(b-a)}{w_1}, \; c - \frac{\alpha(c-b)}{w_1}\right]$$

and

$$[x_\gamma, y_\gamma] = \left[m + \frac{\gamma(n-m)}{w_2}, \; p - \frac{\gamma(p-n)}{w_2}\right]$$

for all $\alpha \in [0, w\ ]$ and $\gamma \in [0, w\ ]$. In this case, if $m \leq x \leq n$, then $x = m + \frac{\gamma(n-m)}{w_2}$. By definition of f^{-1},

$$f^{-1}\left(m + \frac{\gamma(n-m)}{w_2}\right) = \frac{\left(m+\frac{\gamma(n-m)}{w_2}\right)(b-a)-mb+an}{n-m} = a + \frac{\alpha(b-a)}{w_1}$$

for $\gamma = \frac{\alpha w_2}{w_1}$.

Similarly, we find $f^{-1}\left(p-\frac{\gamma(p-n)}{w_2}\right)=\frac{\left(p-\frac{\gamma(p-n)}{w_2}\right)(c-b)-nc+pb}{p-n}=c-\frac{\alpha(c-b)}{w_1}$ for $\gamma=\frac{\alpha w_2}{w_1}$. Therefore, it is clearly that $f^{-1}([x_\gamma,y_\gamma])=[x_\gamma,y_\gamma]$ for $\gamma=\frac{\alpha w_2}{w_1}$ and $f^{-1}([x_\gamma,y_\gamma])\in T\,(\beta_{\tilde{A}_{w_1}})$.

Similarly if $U=(x_\gamma,y_\gamma)$, then $f^{-1}((x_\gamma,y_\gamma))\in T\,(\beta_{\tilde{A}_{w_1}})$ Hence, the function f is continuous. It is proven.

Note 3.31. By Theorem 3.30, $\{x_\alpha, y_\alpha\}$ which are the endpoints of the closed interval ${}^{\alpha}\tilde{A}_{w_1}$ and $\{x_\gamma, y_\gamma\}$ which are the endpoints of the closed interval ${}^{\gamma}\tilde{B}_{w_2}$ corresponding for $\gamma=\frac{\alpha w_2}{w_1}$, match one-to-one. Therefore, the function f in Theorem 3.30 can be given as

$$f(x)=\begin{cases}x_{\frac{\alpha w_2}{w_1}}, & x=x_\alpha\\ y_{\frac{\alpha w_2}{w_1}}, & x=y_\alpha\end{cases}$$

Also similarly, the function f defined by

$$f(x)=\begin{cases}y_{\frac{\alpha w_2}{w_1}}, & x=x_\alpha\\ x_{\frac{\alpha w_2}{w_1}}, & x=y_\alpha\end{cases}$$

is an injective, surjective and continuous function.

Theorem 3.32. Let generalized triangular fuzzy numbers $\tilde{A}_{w_1}=[a,\,b,\,c;\,w_1]$ and $\tilde{B}_{w_2}=[m,\,n,\,p;\,w_2]$ be defined on $A=[a,c]$ and $B=[m,p]$, respectively. If $f\colon (A,T\,(\beta_{\tilde{A}_{w_1}}))\to(B,T\,(\beta_{\tilde{B}_{w_2}}))$ is an injective, surjective and continuous function, then $f(b)=n$.

Proof Let $f\colon (A,T\,(\beta_{\tilde{A}_{w_1}}))\to(B,T\,(\beta_{\tilde{B}_{w_2}}))$ be an injective, surjective and continuous function. Since f is surjective, there is at least one $x\in A$ such that $f^{-1}(n)=\{x\}$. Since f is injective, $\{x\}$ in the form $f^{-1}(n)=\{x\}$ is unique. Also, since f is continuous, then $\{x\}\in T\,(\beta_{\tilde{A}_{w_1}})$ for $\{n\}\in T\,(\beta_{\tilde{A}_{w_2}})$. But $\{b\}$ is the only open set in the form of a single point in $T\,(\beta_{\tilde{A}_{w_1}})$, that is $x=b$. In this case, we have $f^{-1}(n)=b$. Hence $f(b)=n$.

Theorem 3.33. Let the generalized triangular fuzzy numbers $\tilde{A}_{w_1}=[a,\,b,\,c;\,w_1]$, $\tilde{B}_{w_2}=[m,\,n,\,p;\,w_2]$ and $\tilde{C}_{w3}=[r,s,t;w_3]$ be defined on sets $A=[a,c]$, $B=[m,p]$ and $C=[r,t]$, respectively.

If $f\colon (A,\ T\ (\beta_{\tilde{A}_{w_1}}))\to(B,\ T\ (\beta_{\tilde{B}_{w_2}}))$ and $g\colon (B,\ T\ (\beta_{\tilde{B}_{w_2}}))\to(C,\ T\ (\beta_{\tilde{C}_{w_3}}))$ are continuous functions, then the composite function $g\circ f$ where $g\circ f\colon (A,T\,(\beta_{\tilde{A}_{w_1}}))\to(C,T\,(\beta_{\tilde{C}_{w_3}}))$ is a continuous function.

Proof. Let $f\colon (A,\ T\ (\beta_{\tilde{A}_{w_1}}))\to(B,\ T\ (\beta_{\tilde{B}_{w_2}}))$ and $g\colon (B,T\,(\beta_{\tilde{B}_{w_2}}))\to(C,T\,(\beta_{\tilde{C}_{w_3}})$ be continuous functions. Since $(g\circ f)^{-1}(U)=f^{-1}(g^{-1}(U))$ and the function g is continuous, then $(g^{-1})(U)\in T\,(\beta_{\tilde{B}_{w_2}})$ for each $U\in T\,(\beta_{\tilde{C}_{w_3}})$. Since the function f is continuous, then $f^{-1}(g^{-1}(U))\in T\,(\beta_{\tilde{A}_{w_1}})$ for each $g^{-1}(U)\in T\,(\beta_{\tilde{B}_{w_2}})$. Hence, the composite function $g\circ f$ is continuous.

Example 3.34. Generalized fuzzy numbers $\tilde{A}_{0.8}=[2,\,3,\,6;\,0.8]$ and $\tilde{B}_{0.4}=[9,\,10,\,11;\,0.4]$ be defined on $A=[2,\,6]$ and $B=[9,\,11]$, respectively. $f\colon (A,\,T(\beta_{\tilde{A}_{0.8}}))\to(B,\,T(\beta_{\tilde{B}_{0.4}}))$ is a continuous function defined by

$$f(x)=\begin{cases}x+7, & 2\le x\le 3\\ \frac{x}{3}+9, & 3<x\le 6\end{cases}$$

on A. In this case, f^{-1} is equal

$$f^{-1}(x)=\begin{cases}x-7, & 9\le x\le 10\\ 3x-27, & 10<x\le 11.\end{cases}$$

Also, we have

$$\begin{aligned}T(\beta_{\tilde{A}_{0.8}})&=\{{}^{\alpha}\tilde{A}_{0.8},{}^{\alpha+}\tilde{A}_{0.8}\mid \alpha\in[0,0.8]\}\\ &=\{\left[2+\frac{\alpha}{0.8},6-\frac{3\alpha}{0.8}\right],\left(2+\frac{\alpha}{0.8},6-\frac{3\alpha}{0.8}\right)\\ &\quad\mid \alpha\in[0,0.8]\}\end{aligned}$$

and

$$\begin{aligned}T(\beta_{\tilde{B}_{0.4}})&=\{{}^{\gamma}\tilde{B}_{0.4},{}^{\gamma+}\tilde{B}_{0.4}\mid \gamma\in[0,0.4]\}\\ &=\{\left[9+\frac{\gamma}{0.4},11-\frac{\gamma}{0.4}\right],\left(9+\frac{\gamma}{0.4},11-\frac{\gamma}{0.4}\right)\\ &\quad\mid \gamma\in[0,0.4]\}.\end{aligned}$$

By definition the function f^{-1}, we find

$$f^{-1}\left(9+\frac{\alpha/2}{0.4}\right)=2+\frac{\alpha}{0.8}$$

and

$$f^{-1}\left(11-\frac{\alpha/2}{0.4}\right)=6-\frac{3\alpha}{0.8}$$

for $\gamma=\alpha/2$

Since $2+\frac{\alpha}{0.8}\in T(\beta_{\tilde{A}_{0.8}})$ and $6-\frac{3\alpha}{0.8}\in T(\beta_{\tilde{A}_{0.8}})$ for $\alpha\in[0,0.8]$, then the function f is continuous on A.

Definition 3.35. Let the function $f\colon (A,\,T\,(\beta_{\tilde{A}_{w_1}}))\to(B,\,T\,(\beta_{\tilde{B}_{w_2}}))$ be injective and surjective. If the function f and its inverse f^{-1} are continuous functions, then f is called homeomorphism from A to B. Also these topological spaces are homeomorphic and is denoted by $(A,T\,(\beta_{\tilde{A}_{w_1}}))\cong(B,T\,(\beta_{\tilde{B}_{w_2}}))$.

Proposition 3.36. Let generalized triangular fuzzy numbers $\tilde{A}_{w_1}=[a,\,b,\,c;\,w_1]$ and $\tilde{B}_{w_2}=[m,\,n,\,p;\,w_2]$ be defined on sets $A=[a,c]$ and $B=[m,p]$, respectively. If $f\colon (A,T\,(\beta_{\tilde{A}_{w_1}}))\to(B,T\,(\beta_{\tilde{B}_{w_2}}))$ is defined by

$$f(x)=\begin{cases}x_\gamma, & x=x_\alpha\\ y_\gamma, & x=y_\alpha\end{cases}\quad\text{or}\quad f(x)=\begin{cases}y_\gamma, & x=x_\alpha\\ x_\gamma, & x=y_\alpha\end{cases}$$

for $\gamma=\frac{\alpha w_2}{w_1}$ where ${}^{\alpha}\tilde{A}_{w_1}=[x_\alpha,y_\alpha]$ and ${}^{\gamma}\tilde{B}_{w_2}=[x_\gamma,y_\gamma]$, then the function f is homeomorphism from A to B.

Proof. By Theorem 3.30, the function f is injective, surjective and continuous for $\gamma = \frac{\alpha w_2}{w_1}$. Also, by Theorem 3.30, f^{-1} is a continuous function for $\alpha = \frac{\gamma w_2}{w_1}$. Hence, the function f is homeomorphism from A to B.

Generalized fuzzy numbers $\tilde{A}_{0.6} = [1, 2, 3; 0.6]$ and $\tilde{B}_{0.4} = [6, 8, 10; 0.4]$ be defined on sets $A = [1, 3]$ and $B = [6, 10]$, respectively. Also, the function $f: (A, T(\beta_{\tilde{A}0.6})) \to (B, T(\beta_{\tilde{B}0.4}))$ be defined by

$$f(x) = \begin{cases} x_\gamma, & x = x_\alpha \\ y_\gamma, & x = y_\alpha \end{cases}$$

for $\gamma = 2\alpha/3$ such that ${}^{\alpha}\tilde{A}_{0.6} = [x_\alpha, y_\alpha]$ and ${}^{\gamma}\tilde{B}_{0.4} = [x_\gamma, y_\gamma]$.

In this case, it is clear that by Theorem 3.30, the function f is an injective, surjective and continuous on set A.

Consider ${}^{\gamma}\tilde{B}_{0.4} \in T(\beta_{\tilde{B}0.4})$where

$$T(\beta_{\tilde{A}_{0.6}}) = \{{}^{\alpha}\tilde{A}_{0.6}, {}^{\alpha+}\tilde{A}_{0.6} |\ \alpha \in [0, 0.6]\} = \{ \left[1 + \frac{\alpha}{0.6}, 3 - \frac{\alpha}{0.6}\right], \left(1 + \frac{\alpha}{0.6}, 3 - \frac{\alpha}{0.6}\right) \mid \alpha \in [0, 0.6]\},$$

and

$$T(\beta_{\tilde{B}_{0.4}}) = \{{}^{\gamma}\tilde{B}_{0.4}, {}^{\gamma+}\tilde{B}_{0.4} |\ \gamma \in [0, 0.4]\} = \{ \left[6 + \frac{2\gamma}{0.4}, 10 - \frac{2\gamma}{0.4}\right], \left(6 + \frac{2\gamma}{0.4}, 10 - \frac{2\gamma}{0.4}\right) \mid \gamma \in [0, 0.4]\}.$$

By Theorem 3.30 and $\gamma = 2\alpha/3$, $f^{-1}({}^{2\alpha/3}\tilde{B}_{0.4}) = {}^{\alpha}\tilde{A}_{0.6}$ for each $\alpha \in [0,0.6]$. Since $f^{-1}({}^{\gamma}\tilde{B}_{0.4}) \in T(\beta_{\tilde{A}0.6})$ for each ${}^{\gamma}\tilde{B}_{0.4} \in T(\beta_{\tilde{B}0.4})$, f^{-1} is a continuous function. Hence, the function f is homeomorphism from A to B and $(A, T(\beta_{\tilde{A}0.6})) \cong (B, T(\beta_{\tilde{B}0.4}))$.

CONCLUSION

In this study, we proposed a new topology called the cut topology. It is generated by using α cuts family of the generalized triangular fuzzy number. Then, we showed that τ^* topology in Onasanya et al. [12] and $T(A)$ topology in Padmapriya et al. [15] are not valid in some examples. Also, we gave the fundamental concepts such as neighborhood, interior, closure, limit points of a set in the cut topology. In addition, continuity and homeomorphism with examples in the cut topology.

In future studies compactness, path connectedness, separation axioms, net, filter and sequence in the cut topological space can be studied. Also, the cut topology can be applied on different types of fuzzy number such as trapezoid fuzzy number and L-R types. In addition, the cut topology can be generated by intuitionistic fuzzy sets which is a generalization of fuzzy set.

AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

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