Sigma J Eng Nat Sci, Vol. 43, No. 1, pp. 290–300, February, 2025

Sigma Journal of Engineering and Natural Sciences
Web page info: https://sigma.yildiz.edu.tr
DOI: 10.14744/sigma.2025.00022

Research Article

Improved maximum likelihood estimators for the parameters of the two-parameter lindley distribution

Hasan Hüseyin GÜL[1,*]

[1]*Department of Statistics, Giresun University, Giresun, 28100, Türkiye*

ARTICLE INFO

Article history
Received: 08 December 2023
Revised: 15 January 2024
Accepted: 14 February 2024

Keywords:
Bootstrap Bias-Correction; Cox-Snell Bias-Correction; Maximum Likelihood Estimators; Monte-Carlo Simulation; Two-Parameter Lindley Distribution

ABSTRACT

Two-parameter Lindley (TPL) distribution is becoming increasingly popular for modeling lifetime and survival times data, while maximum likelihood estimators (MLEs) are biased for small and moderate sample sizes. This problem has been a motivation to obtain nearly unbiased estimators for the parameters of the model. For this purpose, for the first time, two different techniques, the Cox-Snell methodology, and Efron's bootstrap method, have been used to improve modified nearly unbiased estimators for MLEs of the unknown parameters of the TPL distribution. A Monte Carlo simulation study has been performed to compare the performance of these proposed techniques with different sample sizes and parameter values. In the simulation study, bias and mean square error (MSE) criteria were taken into consideration as evaluation criteria. In addition, a real example is given to demonstrate the applicability of the techniques. The numerical results show that the bias-corrected estimators outperform the other estimators in terms of biases and mean square errors.

Cite this article as: GÜL HH. Improved maximum likelihood estimators for the parameters of the two-parameter lindley distribution. Sigma J Eng Nat Sci 2025;43(1):290–300.

INTRODUCTION

The Lindley distribution with one parameter was proposed by Lindley [1] as an alternative to the exponential distribution in the context of fiducial distributions and the Bayes theorem. It is also known that the one-parameter Lindley distribution is a mixture of exponential (θ) and gamma (2, θ) distributions. Its probability density function (pdf) and cumulative distribution function (CDF) are given by

$$f(x;\theta) = \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}, \qquad x > 0, \theta > 0, \tag{1}$$

$$F(x;\theta) = 1 - \left(1 + \frac{\theta x}{\theta+1}\right)e^{-\theta x}, \ \ x > 0, \theta > 0. \tag{2}$$

The Lindley distribution, which was overshadowed by the exponential distribution for many years, has been studied by many authors as a lifetime model in recent years. Ghitany et al. [2] discussed various statistical properties such as moments, failure rate function, entropies, stochastic ordering, maximum likelihood (ML), and method of moments (MoM) estimations. Using a real data set, they also showed that the Lindley distribution can be a better model than the exponential distribution. Mazucheli

***Corresponding author.**
*E-mail address: hasan.huseyin@giresun.edu.tr

This paper was recommended for publication in revised form by Editor-in-Chief Ahmet Selim Dalkilic

Published by Yıldız Technical University Press, İstanbul, Turkey

and Achcar [3] used the Lindley distribution within the simple competing risks distribution as a possible alternative to the exponential or Weibull distributions. Krishna and Kumar [4] derived and studied model properties and reliability measures of Lindley distribution. They used progressively Type-II censored sample data in the estimation process. Al-Mutairi et al. [5] studied the estimation of stress-strength parameter $R = P(Y < X)$ when X and Y are independent Lindley random variables. Some researchers have studied parameter estimation of Lindley distribution hybrid censored data. See, for example, Gupta and Singh [6], Al-Zahrani and Ali [7], and Jia and Song [8].

The pdf and cdf of the two-parameter Lindley (TPL) distribution proposed by Shanker et al. [9] are given by

$$f(x;\theta) = \frac{\theta^2}{\theta+\alpha}(1+\alpha x)e^{-\theta x},\ \ x>0, \theta>0, \alpha>-\theta, \quad (3)$$

$$F(x;\theta) = 1-\left(\frac{\theta+\alpha+\alpha\theta x}{\theta+\alpha}\right)e^{-\theta x},\ \ x>0, \theta>0, \alpha>-\theta. \quad (4)$$

At $\alpha = 1$, the TPL distribution is the one-parameter Lindley distribution, and at $\alpha = 0$, the TPL distribution is the exponential distribution.

This article deals with deriving modified maximum likelihood estimators (MLEs) that are analytic second-order biases for the parameters of TPL distribution. The choice of estimation method for estimating the parameters of any probability distribution is a very important issue. Among the estimation methods, the MLE is the most widely used method due to its important appealing. For instance, they are asymptotically unbiased and normally distributed, efficient, consistent, etc. However, it is to be noted that most of these properties depend on the large sample size condition. Therefore, especially the unbiasedness property may not be applicable for small and moderate sample sizes. For this reason, it is important to develop nearly unbiased estimators for TPL distribution.

In this article, two different methods are used to reduce the bias of the MLE from order $\mathcal{O}(n^{-1})$ to order $\mathcal{O}(n^{-2})$ for the TPL distribution. The first is the analytical methodology proposed by Cox and Snell [10] which is called the "corrective" approach to derive "bias-corrected" MLEs of second order. This analytical method means that bias correction is done by subtracting the bias (estimated at the MLE of the parameter) from the original MLE. The second is Efron's [11] bootstrap resampling method which is called "parametric bootstrap". This method, unlike the "bias-correction" method introduced by [10], is performed numerically without deriving an analytical expression for the bias function. In the literature, several authors have studied to develop nearly unbiased estimators for the parameters of several distributions. Readers may refer to Corderio et al. [12], Cribari-Neto and Vasconcellos [13], Saha and Paul [14], Lemonte et al. [15], Lemonte [16], Giles [17], Giles et al. [18], Ling and Giles [19], Giles et al. [20], Schwartz and Giles [21], Wang and Wang [22], Reath et al. [23], Mazucheli and Dey [24], Mazuheli et al. [25], Mazucheli et al. [26], Menezes and Mazucheli [27], Menezes et al. [28], Tsai et al. [29], Dey and Wang [30].

The main objective of this study is to obtain almost unbiased estimators for the parameters of the TPL distribution. In the literature, bias-corrected estimators for the TPL distribution have not yet been investigated and no comprehensive study has been conducted. This paper addresses this issue using both analytical and simulation-based methods. For this purpose, two different methods are considered, one analytical method suggested by [10] and the other bootstrap-based bias-corrected. Then, a Monte-Carlo simulation study is performed to compare the performance of the proposed estimators with respect to the bias and mean square error (MSE) criteria. In addition, a real data application is presented to demonstrate the applicability of the methods.

In the next sections 2 and 3, point estimation by the maximum likelihood method and its two different bias-corrected versions for the TPL distribution is discussed. Section 4 reports a Monte-Carlo simulation experiment that evaluates the performance among the improved modified Cox-Snell bias-corrected estimator (BCE) and bootstrap-based bias-corrected estimator (PBE). For illustrative purposes, a real data set is presented in Section 5. Finally, some remarks in Section 6 closes the paper.

Maximum Likelihood Estimation

Suppose that $y = (y_1,\ldots,y_n)$ be a random sample of size n from the TPL distribution with pdf (1). The log-likelihood function is:

$$l(\Theta|y) = n log(\theta^2) - n log(\theta+\alpha) + \sum_{x=1}^{n} log(1+\alpha y) - \theta\sum_{x=1}^{n} y_i \quad (5)$$

where $\Theta = (\theta, \alpha)$. The MLEs of $\hat{\theta}$ and $\hat{\alpha}$ of the unknown parameters θ and α are obtained by solving the non-linear equations:

$$\frac{\partial l}{\partial\theta} = \frac{2n}{\theta} - \frac{\alpha}{\alpha+\theta} - \sum_{x=1}^{n} y_i\,, \quad (6)$$

$$\frac{\partial l}{\partial\alpha} = -\frac{n}{\alpha+\theta} + n\sum_{x=1}^{n}(y_i)(1+\alpha y_i)^{-1}. \quad (7)$$

The expected Fisher information matrix is given in the Appendix. Equations (6) and (7) do not seem to be solved directly. Therefore, a suitable numerical algorithm must be used. The Nelder-Mead optimization method (Nelder and Mead, [31]) in MATLAB software is used to obtain the estimated parameters.

BIAS-CORRECTED MLES

Cox-Snell Method

For a p-dimensional parameter vector Θ, [10] demonstrated that when the sample data are independent, but not necessarily identically distributed, the bias of the s-th element of the MLE Θ, $\hat{\Theta}$ is calculated as

$$Bias\left(\widehat{\Theta}_s\right)=\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{l=1}^{p}\kappa^{si}\kappa^{jl}\left[0.5\kappa_{ijl}+\kappa_{ij,l}\right]+\mathcal{O}(n^{-2}) \quad (8)$$

where $s = 1,\dots, p$, $K = [-\kappa^{ij}]$ is the (i,j)-th element of the inverse of the expected Fisher information,

$$\kappa_{ijl}=\mathbb{E}\left[\frac{\partial^3}{\partial\Theta_i\partial\Theta_j\partial\Theta_l}l(\Theta|y)\right],$$

and

$$\kappa_{ij,l}=\mathbb{E}\left[\frac{\partial^2}{\partial\Theta_i\partial\Theta_j}l(\Theta|y)\frac{\partial}{\partial\Theta_l}l(\Theta|y)\right].$$

Eq. (8) can be rewritten in the following form:

$$Bias\left(\widehat{\Theta}_s\right)=\sum_{s=1}^{p}\kappa^{si}\sum_{j=1}^{p}\sum_{l=1}^{p}\left[\kappa_{ij}^{(l)}-\frac{1}{2}\kappa_{ijl}\right]+\mathcal{O}(n^{-2}),\ s=1,2,\dots,p. \quad (9)$$

Now, let $a_{ij}^{(l)}=\kappa_{ij}^{(l)}-\left(\kappa_{ijl}/2\right)$, for $i,j,l=1,2,\dots,p$ and define the following matrices,

$$A^{(l)}=\left\{a_{ij}^{(l)}\right\},\quad i,j,l=1,2,\dots,p,$$
$$A=\left[A^{(1)}|A^{(2)}|\cdots|A^{(p)}\right].$$

They also showed that the $\mathcal{O}(n^{-1})$ the bias of the MLE of Θ in Eq. (9) can be re-expressed as:

$$Bias\left(\widehat{\Theta}_s\right)=K^{-1}Avec(K^{-1})+\mathcal{O}(n^{-2}),$$

where *vec* (.) is an operator that creates a column vector from a matrix by stacking the column vectors below one another. Then, a "bias-corrected" MLE for Θ can be obtained as:

$$\widetilde{\Theta}=\widehat{\Theta}-\widehat{K}^{-1}\hat{A}vec\left(\widehat{K}^{-1}\right), \quad (11)$$

where $\widehat{K}=K|_{\widehat{\Theta}}$ and $\hat{A}=A|_{\widehat{\Theta}}$, and it can be shown that the bias of $\widetilde{\Theta}$ will be $\mathcal{O}(n^{-2})$.

For the TPL distribution, after extensive algebraic manipulation, see the Appendix, the following equations are obtained:

$$\kappa_{111}=\frac{4n}{\theta^3}-\frac{2\alpha}{(\alpha+\theta)^3}$$

$$\kappa_{112}=\kappa_{121}=\kappa_{211}=-\frac{2n}{(\alpha+\theta)^3}$$

$$\kappa_{222}=-\frac{2n}{(\alpha+\theta)^3}+\frac{2n^2}{(\alpha+\theta)}\left(\frac{3\theta^2e^{\frac{\theta}{\alpha}}\Gamma\left(1,\frac{\theta}{\alpha}\right)\alpha-2\alpha^2\theta+\theta^3e^{\frac{\theta}{\alpha}}\Gamma\left(1,\frac{\theta}{\alpha}\right)+\alpha^3-\alpha\theta^2}{\alpha^5}\right)$$

$$\kappa_{122}=\kappa_{212}=\kappa_{221}=-\frac{2n}{(\alpha+\theta)^3}$$

$$\kappa_{11,1}=\frac{4n}{\theta^3}-\frac{2\alpha}{(\alpha+\theta)^3}$$

$$\kappa_{12,1}=\kappa_{12,2}=-\frac{2n}{(\alpha+\theta)^3}$$

$$\kappa_{22,1}=-\frac{2n}{(\alpha+\theta)^3}+\frac{\left(-\alpha\theta+\theta^2e^{\frac{\theta}{\alpha}}\Gamma\left(1,\frac{\theta}{\alpha}\right)+\alpha^2\right)}{(\alpha+\theta)^2\alpha^3}-\frac{\left(-\alpha+2\theta e^{\frac{\theta}{\alpha}}\Gamma\left(1,\frac{\theta}{\alpha}\right)+\frac{\theta^2e^{\frac{\theta}{\alpha}}\Gamma\left(1,\frac{\theta}{\alpha}\right)}{\alpha}-\theta e^{\frac{\theta}{\alpha}}e^{-\frac{\theta}{\alpha}}\right)}{(\alpha+\theta)\alpha^3}$$

$$\kappa_{11,2}=\frac{1}{(\alpha+\theta)^2}-\frac{2\alpha}{(\alpha+\theta)^3}$$

$$\kappa_{22,2}=-\frac{2n}{(\alpha+\theta)^3}+\frac{\left(-\alpha\theta+\theta^2e^{\frac{\theta}{\alpha}}\Gamma\left(1,\frac{\theta}{\alpha}\right)+\alpha^2\right)}{(\alpha+\theta)^2\alpha^3}-\frac{\left(-\theta-\frac{\theta^3e^{\frac{\theta}{\alpha}}\Gamma\left(1,\frac{\theta}{\alpha}\right)}{\alpha^2}+\frac{\theta^2e^{\frac{\theta}{\alpha}}e^{-\frac{\theta}{\alpha}}}{\alpha}-2\alpha\right)}{(\alpha+\theta)\alpha^3}$$
$$+\frac{3\left(-\alpha\theta+\theta^2e^{\frac{\theta}{\alpha}}\Gamma\left(1,\frac{\theta}{\alpha}\right)+\alpha^2\right)}{(\alpha+\theta)\alpha^4}.$$

The analytic expressions are given in the Appendix. The elements of $A^{(1)}$ are obtained as follows:

$$a_{11,1}=\kappa_{11,1}-0.5\kappa_{111}=\frac{1}{2}\left(\frac{4n}{\theta^3}-\frac{2\alpha}{(\alpha+\theta)^3}\right)$$

$$a_{12,1}=\kappa_{12,1}-0.5\kappa_{112}=-\frac{n}{(\alpha+\theta)^3}$$

$$a_{22,1}=\kappa_{22,1}-0.5\kappa_{122}=-\frac{2n}{(\alpha+\theta)^3}+\frac{\left(-\alpha\theta+\theta^2e^{\frac{\theta}{\alpha}}\Gamma\left(1,\frac{\theta}{\alpha}\right)+\alpha^2\right)}{(\alpha+\theta)^2\alpha^3}$$
$$-\frac{\left(-\alpha+2\theta e^{\frac{\theta}{\alpha}}\Gamma\left(1,\frac{\theta}{\alpha}\right)+\frac{\theta^2e^{\frac{\theta}{\alpha}}\Gamma\left(1,\frac{\theta}{\alpha}\right)}{\alpha}-\theta e^{\frac{\theta}{\alpha}}e^{-\frac{\theta}{\alpha}}\right)}{(\alpha+\theta)\alpha^3}$$
$$+\frac{n}{(\alpha+\theta)^3}.$$

The elements of $A^{(2)}$ are:

$$a_{11,2}=\kappa_{11,2}-0.5\kappa_{112}=\frac{1}{(\alpha+\theta)^2}-\frac{2\alpha}{(\alpha+\theta)^3}+\frac{n}{(\alpha+\theta)^3}$$

$$a_{12,2}=\kappa_{12,2}-0.5\kappa_{122}=-\frac{2n}{(\alpha+\theta)^3}+\frac{n}{(\alpha+\theta)^3}$$

$$a_{22,2}=\kappa_{22,2}-0.5\kappa_{222}=-\frac{2n}{(\alpha+\theta)^3}+\frac{\left(-\alpha\theta+\theta^2e^{\frac{\theta}{\alpha}}\Gamma\left(1,\frac{\theta}{\alpha}\right)+\alpha^2\right)}{(\alpha+\theta)^2\alpha^3}$$
$$-\frac{\left(-\theta-\frac{\theta^3e^{\frac{\theta}{\alpha}}\Gamma\left(1,\frac{\theta}{\alpha}\right)}{\alpha^2}+\frac{\theta^2e^{\frac{\theta}{\alpha}}e^{-\frac{\theta}{\alpha}}}{\alpha}-2\alpha\right)}{(\alpha+\theta)\alpha^3}$$
$$+\frac{3\left(-\alpha\theta+\theta^2e^{\frac{\theta}{\alpha}}\Gamma\left(1,\frac{\theta}{\alpha}\right)+\alpha^2\right)}{(\alpha+\theta)\alpha^4}-0.5\left(-\frac{2n}{(\alpha+\theta)^3}\right.$$
$$\left.+\frac{2n^2}{(\alpha+\theta)}\left(\frac{3\theta^2e^{\frac{\theta}{\alpha}}\Gamma\left(1,\frac{\theta}{\alpha}\right)\alpha-2\alpha^2\theta+\theta^3e^{\frac{\theta}{\alpha}}\Gamma\left(1,\frac{\theta}{\alpha}\right)+\alpha^3-\alpha\theta^2}{\alpha^5}\right)\right).$$

Using Corderio and Klein's [32] modification of [10]'s method, the bias of $\hat{\Theta}$ can be written as follows:

$$Bias\left(\widehat{\Theta}_s\right)=Bias\begin{pmatrix}\hat{\theta}\\ \hat{\alpha}\end{pmatrix}=K^{-1}Avec(K^{-1}) \quad (12)$$

where $\widehat{K} = K|_{\theta=\widehat{\theta};\alpha=\widehat{\alpha}}$ and $\hat{A} = A|_{\theta=\widehat{\theta};\alpha=\widehat{\alpha}}$.

Parametric Bootstrap Method

An alternative approach considered to obtain modified nearly unbiased estimators for the TPL distribution is based on the parametric bootstrap resampling method ([11], Efron and Tibshirani [33], Davison and Hinkley [34]). In this method bias correction is performed numerically without deriving an analytical expression for the bias function. It uses the empirical distribution of the sample as an approximation of the population distribution. The estimated bias of $\hat{\Theta}_s$ for a parameter vector Θ is given by

$$Bias(\widehat{\Theta}) = \frac{1}{B}\sum_{j=1}^{B}\widehat{\Theta}_j - \widehat{\Theta}, \tag{13}$$

where $\hat{\Theta}_j$ is the MLE of Θ obtained from the j-th Bootstrap sample. Thus, the bootstrap bias-corrected estimator (PBE) is

$$\widehat{\Theta}_{PBE} = 2\widehat{\Theta} - \frac{1}{B}\sum_{j=1}^{B}\widehat{\Theta}_j . \tag{14}$$

Although it does not involve any analytical derivatives, it provides a second-order unbiased estimator, which is a very important and remarkable feature. For more details, see [33].

NUMERICAL EVALUATIONS

In this section, a Monte Carlo simulation study is carried out to compare the performance of the finite-sample behavior of the MLEs, BCEs, and PBEs of the TPL distribution. The Monte Carlo experiments are evaluated by selecting sample sizes n=10, 30, and 50, θ=1.0, 1.5, 2.0, 3.0, and 5.0, α=0.5, 1.0, 1.5, and 3.0. The pseudo-random samples are simulated using the inverse transform method from TPL distribution, that is $x = (x_1,\ldots, x_n)$ is generated from:

$$x_i = -\left(\left(\frac{\alpha}{W\left((u_i-1)\theta\alpha e^{\left(-\frac{\alpha}{\theta+\alpha}\right)}\right)}\right)\alpha+\theta+\alpha\right)\backslash(\theta\alpha),$$

where u_i are random numbers from a uniform distribution and $W(.)$ is the Lambert W function.

The Monte-Carlo and Bootstrap replication numbers are taken as M=10,000 and B=1000, respectively. The simulation results are reported in Tables 1-4.

It is observed from Table 1 that, the biases of α_{MLE}, α_{BCE} and α_{PBE} is smaller than the biases of θ_{MLE}, θ_{BCE} and θ_{PBE}. In most cases, the biases and MSEs of all estimators of θ and α approach zero as n increases. This shows that all estimators are consistent. The bias-corrected estimators α_{BCE} and θ_{BCE} clearly outperform the other estimators according to both the bias and MSE criteria. Similar results are observed in Tables 2-4. These Monte-Carlo simulation results reveal that the bias-corrected estimators perform favorably in bringing the estimates closer to their true values.

In the case of α = 0.5, the plots for the θ and α parameters are given in Figures 1-4 for the estimated bias and MSE.

Table 1. Estimated bias (mean square errors) for θ and α, (α = 0.5)

		Estimator of θ			Estimator of α		
θ	n	MLE	BCE	PBE	MLE	BCE	PBE
1.0	10	0.266(0.181)	0.007(0.009)	0.096(0.047)	0.191(0.147)	0.002(0.003)	0.040(0.023)
	30	0.087(0.053)	0.005(0.008)	0.047(0.023)	0.061(0.047)	0.001(0.002)	0.029(0.018)
	50	0.048(0.046)	0.001(0.003)	0.022(0.019)	0.024(0.048)	0.002(0.003)	0.015(0.013)
1.5	10	0.325(0.195)	0.003(0.007)	0.044(0.034)	0.246(0.175)	0.003(0.004)	0.021(0.021)
	30	0.133(0.064)	0.003(0.005)	0.032(0.022)	0.089(0.053)	0.003(0.003)	0.014(0.014)
	50	0.071(0.051)	0.001(0.005)	0.020(0.017)	0.052(0.045)	0.001(0.001)	0.013(0.015)
2.0	10	0.358(0.212)	0.004(0.003)	0.052(0.038)	0.257(0.169)	0.000(0.002)	0.026(0.019)
	30	0.163(0.077)	0.003(0.006)	0.031(0.021)	0.103(0.056)	0.001(0.002)	0.011(0.014)
	50	0.088(0.059)	0.004(0.004)	0.019(0.019)	0.065(0.048)	0.000(0.001)	0.014(0.014)
3.0	10	0.390(0.244)	0.003(0.001)	0.073(0.041)	0.314(0.214)	0.002(0.004)	0.037(0.022)
	30	0.197(0.102)	0.001(0.003)	0.037(0.027)	0.132(0.071)	0.001(0.001)	0.021(0.019)
	50	0.116(0.063)	0.000(0.002)	0.024(0.020)	0.078(0.052)	0.001(0.002)	0.016(0.016)
5.0	10	0.466(0.301)	0.010(0.006)	0.117(0.075)	0.343(0.219)	0.007(0.007)	0.059(0.037)
	30	0.234(0.115)	0.008(0.006)	0.083(0.049)	0.157(0.083)	0.002(0.004)	0.038(0.029)
	50	0.132(0.076)	0.006(0.003)	0.045(0.038)	0.088(0.067)	0.003(0.006)	0.023(0.025)

Table 2. Estimated bias (mean square errors) for θ and α, (α = 1.0)

		Estimator of θ			**Estimator of α**		
θ	n	**MLE**	**BCE**	**PBE**	**MLE**	**BCE**	**PBE**
1.0	10	0.184(0.138)	0.003(0.006)	0.072(0.055)	0.166(0.122)	0.001(0.003)	0.038(0.028)
	30	0.076(0.049)	0.002(0.003)	0.044(0.037)	0.058(0.043)	0.001(0.001)	0.022(0.019)
	50	0.038(0.043)	0.000(0.001)	0.021(0.018)	0.024(0.044)	0.000(0.000)	0.017(0.012)
1.5	10	0.292(0.193)	0.006(0.007)	0.108(0.080)	0.194(0.156)	0.004(0.003)	0.061(0.043)
	30	0.194(0.097)	0.002(0.002)	0.082(0.061)	0.063(0.049)	0.000(0.001)	0.033(0.029)
	50	0.128(0.079)	0.001(0.002)	0.037(0.039)	0.039(0.044)	0.001(0.000)	0.019(0.014)
2.0	10	0.424(0.270)	0.007(0.006)	0.146(0.103)	0.247(0.178)	0.007(0.006)	0.094(0.066)
	30	0.263(0.121)	0.005(0.004)	0.106(0.077)	0.116(0.063)	0.005(0.005)	0.048(0.041)
	50	0.161(0.079)	0.001(0.002)	0.064(0.042)	0.054(0.049)	0.001(0.002)	0.025(0.024)
3.0	10	0.479(0.326)	0.012(0.006)	0.165(0.126)	0.290(0.195)	0.009(0.006)	0.123(0.094)
	30	0.317(0.170)	0.009(0.007)	0.109(0.092)	0.131(0.067)	0.004(0.005)	0.076(0.065)
	50	0.215(0.097)	0.003(0.003)	0.075(0.065)	0.065(0.052)	0.001(0.001)	0.039(0.038)
5.0	10	0.540(0.385)	0.011(0.009)	0.199(0.148)	0.362(0.228)	0.008(0.007)	0.151(0.117)
	30	0.359(0.192)	0.008(0.005)	0.121(0.095)	0.173(0.081)	0.009(0.004)	0.088(0.073)
	50	0.258(0.116)	0.005(0.003)	0.080(0.056)	0.095(0.059)	0.003(0.005)	0.057(0.056)

Table 3. Estimated bias (mean square errors) for θ and α, (α = 1.5)

		Estimator of θ			**Estimator of α**		
θ	n	**MLE**	**BCE**	**PBE**	**MLE**	**BCE**	**PBE**
1.0	10	0.175(0.129)	0.005(0.007)	0.094(0.066)	0.159(0.117)	0.006(0.006)	0.079(0.055)
	30	0.043(0.024)	0.001(0.002)	0.039(0.024)	0.036(0.024)	0.001(0.002)	0.045(0.031)
	50	0.030(0.022)	0.002(0.002)	0.016(0.011)	0.022(0.025)	0.001(0.002)	0.020(0.019)
1.5	10	0.220(0.149)	0.007(0.006)	0.125(0.087)	0.198(0.140)	0.007(0.005)	0.101(0.073)
	30	0.092(0.038)	0.006(0.004)	0.063(0.044)	0.063(0.032)	0.002(0.003)	0.046(0.034)
	50	0.039(0.028)	0.001(0.001)	0.030(0.021)	0.022(0.024)	0.002(0.001)	0.014(0.016)
2.0	10	0.294(0.169)	0.009(0.007)	0.159(0.109)	0.255(0.158)	0.009(0.006)	0.146(0.096)
	30	0.148(0.052)	0.008(0.005)	0.081(0.063)	0.116(0.045)	0.002(0.005)	0.077(0.053)
	50	0.065(0.038)	0.003(0.003)	0.031(0.028)	0.057(0.029)	0.004(0.003)	0.041(0.030)
3.0	10	0.378(0.217)	0.011(0.009)	0.192(0.137)	0.376(0.208)	0.012(0.007)	0.185(0.121)
	30	0.201(0.072)	0.006(0.005)	0.104(0.062)	0.188(0.066)	0.007(0.002)	0.109(0.077)
	50	0.124(0.049)	0.007(0.005)	0.066(0.046)	0.091(0.041)	0.003(0.003)	0.052(0.042)
5.0	10	0.471(0.303)	0.013(0.009)	0.248(0.165)	0.459(0.257)	0.011(0.009)	0.263(0.158)
	30	0.286(0.136)	0.004(0.006)	0.155(0.098)	0.224(0.090)	0.002(0.004)	0.146(0.088)
	50	0.212(0.089)	0.006(0.003)	0.094(0.056)	0.118(0.049)	0.005(0.005)	0.072(0.043)

Illustrative Example

In this section, to demonstrate the applicability of the improved bias-corrected estimators for the TPL distribution, two different data sets are used.

Data Set 1: The following data set represents the failure times (in minutes) for a sample of 15 electronic components in an accelerated life test by Lawless [35]. The data are given in Table 5.

Descriptive statistics of the failure times of electronic components data are given in Table 6.

Data Set 2: The following data set represents the number of cycles to failure for 25 100-cm specimens of yarn, tested at a particular strain level, by [35]. Both data sets were recently used as an illustrative example for the TPL distribution by [9].

Table 4. Estimated bias (mean square errors) for θ and α, (α = 3)

		Estimator of θ			Estimator of α		
θ	n	MLE	BCE	PBE	MLE	BCE	PBE
1.0	10	0.287(0.242)	0.007(0.006)	0.124(0.082)	0.262(0.235)	0.005(0.006)	0.088(0.074)
	30	0.054(0.033)	0.002(0.001)	0.049(0.033)	0.035(0.028)	0.001(0.001)	0.037(0.039)
	50	0.038(0.027)	0.001(0.002)	0.022(0.018)	0.020(0.026)	0.001(0.000)	0.021(0.016)
1.5	10	0.211(0.158)	0.009(0.005)	0.143(0.097)	0.196(0.141)	0.007(0.007)	0.111(0.095)
	30	0.082(0.039)	0.001(0.003)	0.067(0.056)	0.059(0.030)	0.002(0.003)	0.057(0.047)
	50	0.056(0.031)	0.003(0.001)	0.031(0.024)	0.033(0.026)	0.003(0.001)	0.034(0.028)
2.0	10	0.254(0.176)	0.009(0.007)	0.165(0.112)	0.227(0.159)	0.009(0.007)	0.143(0.108)
	30	0.110(0.054)	0.005(0.004)	0.092(0.068)	0.074(0.037)	0.006(0.006)	0.077(0.067)
	50	0.051(0.042)	0.004(0.002)	0.044(0.035)	0.036(0.029)	0.002(0.002)	0.032(0.030)
3.0	10	0.325(0.197)	0.013(0.007)	0.183(0.119)	0.262(0.168)	0.009(0.011)	0.159(0.119)
	30	0.156(0.072)	0.008(0.003)	0.105(0.086)	0.108(0.045)	0.007(0.004)	0.075(0.082)
	50	0.094(0.045)	0.003(0.001)	0.056(0.042)	0.061(0.031)	0.003(0.003)	0.042(0.038)
5.0	10	0.423(0.268)	0.017(0.009)	0.202(0.142)	0.352(0.208)	0.011(0.009)	0.188(0.135)
	30	0.248(0.109)	0.007(0.005)	0.127(0.088)	0.163(0.069)	0.009(0.007)	0.103(0.086)
	50	0.140(0.071)	0.005(0.004)	0.074(0.051)	0.089(0.037)	0.004(0.003)	0.049(0.041)

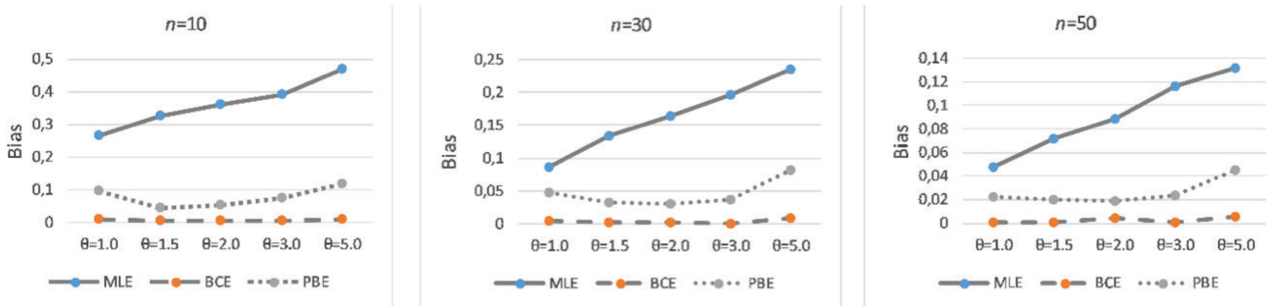

Figure 1. Estimated bias for θ for α=0.5.

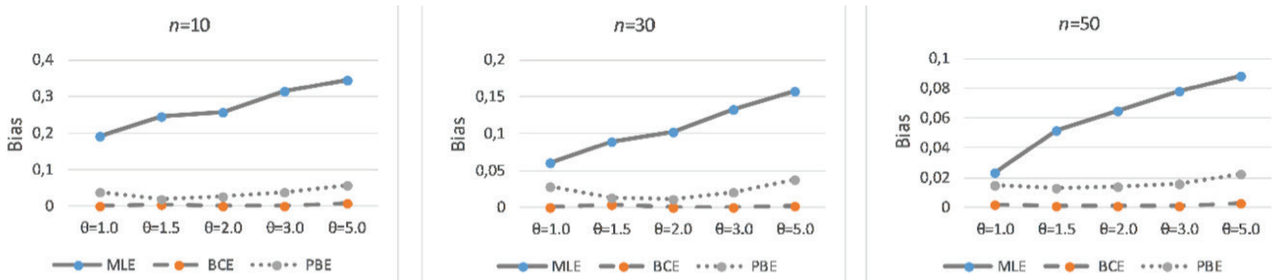

Figure 2. Estimated bias for α for α=0.5.

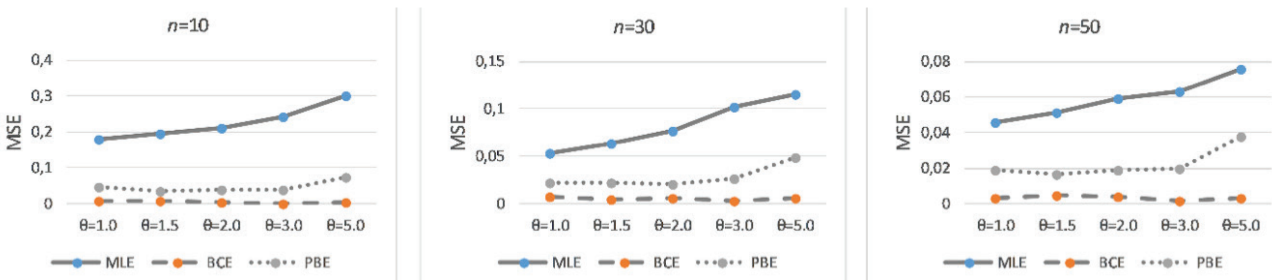

Figure 3. Estimated MSE for θ for α=0.5.

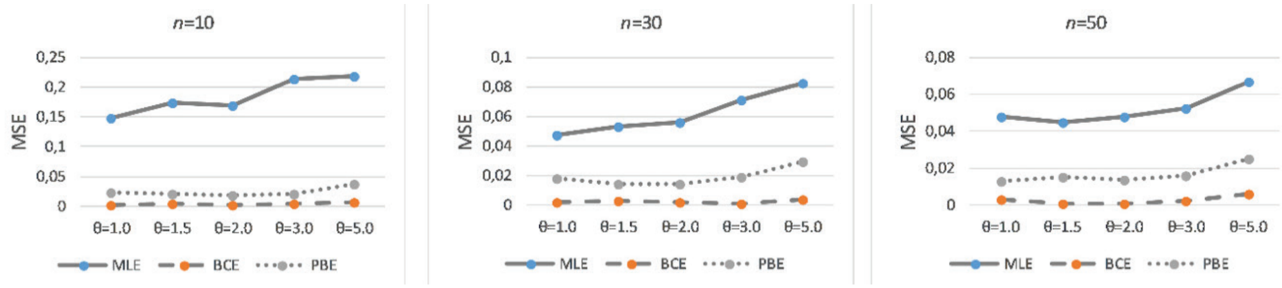

Figure 4. Estimated MSE for α for α=0.5.

Table 5. Failure times of electronic components

1.4	5.1	6.3	10.8	12.1	18.5	19.7	22.2	23.0
30.6	37.3	46.3	53.9	59.8	66.2			

Table 6. Summary of the descriptive statistics for the failure times of electronic components data

Mean	Min	Max	SD	Range
27.5467	1.4	66.2	20.7634	64.8

Table 7. Number of cycles to failure for 25 100-cm specimens of yarn

15	20	38	42	61	76	86	98	121
146	149	157	175	176	180	180	198	220
224	251	264	282	321	325	653		

Table 8. Summary of the descriptive statistics for the Number of cycles to failure for 25 100-cm specimens of yarn data.

Mean	Min	Max	SD	Range
178.32	15	653	133.8008	638

Table 9. MLEs, BCEs, and PBEs (Bootstrap standard errors)

	Data Set 1		Data Set 2	
Estimators	θ	α	θ	α
MLE	0.1559(0.6482)	0.0622(0.0327)	0.0109(0.0851)	0.1288(0.2336)
BCE	0.1547(0.6344)	0.0809(0.0264)	0.0098(0.0779)	0.1255(0.2283)
PBE	0.1553(0.6365)	0.0798(0.0277)	0.0095(0.0842)	0.1247(0.2326)

The point estimates of θ and α along with standard errors with MLEs, BCEs, and PBEs of the TPL distribution for both data sets are presented in Table 9.

In comparing the three considered estimation methods for the TPL distribution, the smallest standard error is taken as a criterion. It is observed that bias-corrected estimates have smaller standard errors for both parameters θ and α for both data sets. Moreover, the BCEs and PBEs gave similar bootstrap standard errors. In addition, Table 9 shows that the MLEs overestimates θ and α for both data sets.

CONCLUSION

In the literature, there are no studies on the bias reduction of MLE estimates of the parameters of the two-parameter Lindley distribution. This gap in the literature is very important for improving the parameter estimation of this distribution. In this paper, a "corrective" approach to derive analytical expressions for bias-corrected maximum likelihood estimator suggested by [10] for the parameters of the two-parameter Lindley distribution has been adopted. Besides, Efron's bootstrap resampling technique, which is not an analytical method, is considered an alternative bias-correction technique. In addition, to demonstrate the applicability of the above-mentioned techniques, a real data application is performed. The results show that the proposed bias-corrected estimators are preferred in terms of bias and mean square error for the two-parameter Lindley distribution over the maximum likelihood and bootstrap bias-corrected estimators.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

REFERENCES

[1] Lindley DV. Fiducial distributions and Bayes' theorem. J Royal Stat Soc Series B (Methodological) 1958;20:102–107. [CrossRef]

[2] Ghitany ME, Atieh B, Nadarajah S. Lindley distribution and its application. Mathematics and computers in simulation 2008;78:493–506. [CrossRef]

[3] Mazucheli J, Achcar JA. The Lindley distribution applied to competing risks lifetime data. Computer methods and programs in biomedicine 2011;104:188–192. [CrossRef]

[4] Krishna H, Kumar K. Reliability estimation in Lindley distribution with progressively type II right censored sample. Math Comput Simul 2011;82:281–294. [CrossRef]

[5] Al-Mutairi DK, Ghitany ME, Kundu D. Inferences on stress-strength reliability from Lindley distributions. Commun Stat Theory Methods 2013;42:1443–1463. [CrossRef]

[6] Gupta PK, Singh B. Parameter estimation of Lindley distribution with hybrid censored data. Int J Syst Assur Eng Manag 2013;4:378–385. [CrossRef]

[7] Al-Zahrani B, Ali MA. Recurrence relations for moments of multiply type-II censored order statistics from Lindley distribution with applications to inference. Stat Optim Inform Comput 2014;2:147–160. [CrossRef]

[8] Jia J, Song H. Parameter estimation of lindley distribution under generalized first-failure progressive hybrid censoring schemes. IAENG Int J Appl Math 2022;52:799.

[9] Shanker R, Kamlesh KK, Fesshaye H. A two parameter Lindley distribution: Its properties and applications. Biostat Biom Open Access J 2017;1:85–90. [CrossRef]

[10] Cox DR, Snell EJ. A general definition of residuals. Journal of the Royal Statistical Society: Series B (Methodological) 1968;30:248–265. [CrossRef]

[11] Efron B. The jackknife, the bootstrap and other resampling plans. CBMS-NSF Regional Conference Series in Applied Mathematics, Monograph 38, SIAM, Philadelphia.

[12] Cordeiro GM, Da Rocha EC, Da Rocha JGC, Cribari-Neto F. Bias-corrected maximum likelihood estimation for the beta distribution. J Stat Comput Simul 1997;58:21–35. [CrossRef]

[13] Cribari-Neto F, Vasconcellos KL. Nearly unbiased maximum likelihood estimation for the beta distribution. J Stat Comput Simul 2002;72:107–118. [CrossRef]

[14] Saha K, Paul S. Bias-corrected maximum likelihood estimator of the negative binomial dispersion parameter. Biometrics 2005;61:179–185. [CrossRef]

[15] Lemonte AJ, Cribari-Neto F, Vasconcellos KL. Improved statistical inference for the two-parameter Birnbaum-Saunders distribution. Comput Stat Data Anal 2007;51:4656–4681. [CrossRef]

[16] Lemonte AJ. Improved point estimation for the Kumaraswamy distribution. J Stat Comput Simul 2011;81:1971–1982. [CrossRef]

[17] Giles DE. Bias reduction for the maximum likelihood estimators of the parameters in the half-logistic distribution. Commun Stat Theory Methods 2012;41:212–222. [CrossRef]

[18] Giles DE, Feng H, Godwin RT. On the bias of the maximum likelihood estimator for the two-parameter Lomax distribution. Commun Stat Theory Methods 2013;42:1934–1950. [CrossRef]

[19] Ling X, Giles DE. Bias reduction for the maximum likelihood estimator of the parameters of the generalized Rayleigh family of distributions. Commun Stat Theory Methods 2014;43:1778–1792. [CrossRef]

[20] Giles DE, Feng H, Godwin RT. Bias-corrected maximum likelihood estimation of the parameters of the generalized Pareto distribution. Commun Stat Theory Methods 2016;45:2465–2483. [CrossRef]

[21] Schwartz J, Giles DE. Bias-reduced maximum likelihood estimation of the zero-inflated Poisson distribution. Commun Stat Theory Methods 2016;45:465–478. [CrossRef]
[22] Wang M, Wang W. Bias-corrected maximum likelihood estimation of the parameters of the weighted Lindley distribution. Commun Stat Simul Comput 2017;46:530–545. [CrossRef]
[23] Reath J, Dong J, Wang M. Improved parameter estimation of the log-logistic distribution with applications. Comput Stat 2018;33:339–356. [CrossRef]
[24] Mazucheli J, Dey S. Bias-corrected maximum likelihood estimation of the parameters of the generalized half-normal distribution. J Stat Comput Simul 2018;88:1027–1038. [CrossRef]
[25] Mazucheli J, Menezes AFB, Dey S. Improved maximum-likelihood estimators for the parameters of the unit-gamma distribution. Commun Stat Theory Methods 2018;47:3767–3778. [CrossRef]
[26] Mazucheli J, Menezes AFB, Dey S. Bias-corrected maximum likelihood estimators of the parameters of the inverse Weibull distribution. Commun Stat Simul Comput 2019;48:2046–2055. [CrossRef]
[27] Menezes AFB, Mazucheli J. Improved maximum likelihood estimators for the parameters of the Johnson SB distribution. Commun Stat Simul Comput 2020;49:1511–1526. [CrossRef]
[28] Menezes A, Mazucheli J, Alqallaf F, Ghitany ME. Bias-corrected maximum likelihood estimators of the parameters of the unit-weibull distribution. Austrian Journal of Statistics 2021;50:41–53. [CrossRef]
[29] Tsai TR, Xin H, Fan YY, Lio Y. Bias-Corrected Maximum Likelihood Estimation and Bayesian Inference for the Process Performance Index Using Inverse Gaussian Distribution. Stats 2022;5:1079–1096. [CrossRef]
[30] Dey S, Wang L. Methods of estimation and bias corrected maximum likelihood estimators of unit burr III distribution. Am J Math Manag Sci 2022;41:316–333. [CrossRef]
[31] Nelder JA, Mead R. A simplex method for function minimization. Comput J 1965;7:308–313. [CrossRef]
[32] Cordeiro GM, Klein R. Bias correction in ARMA models. Stat Probab Lett 1994;19:169–176. [CrossRef]
[33] Efron B, Tibshirani RJ. An introduction to the bootstrap. Boca Raton London New York Washington, D.C.: Chapman & Hall/CRC; 1993.
[34] Davison AC, Hinkley DV. Bootstrap methods and their application (No. 1). Cambridge: Cambridge University Press; 1997. [CrossRef]
[35] Lawless JF. Statistical models and methods for lifetime data. Hoboken, New Jersey: John Wiley & Sons; 2011.

APPENDIX

Analytic bias correction for the TPL distribution with parameters θ and α.

The log-likelihood function for the TPL distribution is given by

$$l(\Theta|y) = nlog(\theta^2) - nlog(\theta+\alpha) + \sum_{x=1}^{n} log(1+\alpha y) - \theta \sum_{x=1}^{n} y_i.$$

The derivatives of the log-likelihood function up to the third order are given:

$$\frac{\partial l}{\partial \theta} = \frac{2n}{\theta} - \frac{\alpha}{\alpha+\theta} - \sum_{x=1}^{n} y_i,$$

$$\frac{\partial l}{\partial \alpha} = -\frac{n}{\alpha+\theta} + n\sum_{x=1}^{n}(y_i)(1+\alpha y_i)^{-1},$$

$$\frac{\partial^2 l}{\partial \theta \partial \alpha} = \frac{n}{(\alpha+\theta)^2},$$

$$\frac{\partial^2 l}{\partial \theta^2} = -\frac{2n}{\theta^2} + \frac{\alpha}{(\alpha+\theta)^2},$$

$$\frac{\partial^3 l}{\partial \theta^3} = \frac{4n}{\theta^3} - \frac{2\alpha}{(\alpha+\theta)^3},$$

$$\frac{\partial^2 l}{\partial \alpha^2} = \frac{n}{(\alpha+\theta)^2} - n\sum_{x=1}^{n}(y_i)^2\,(1+\alpha y_i)^{-2},$$

$$\frac{\partial^3 l}{\partial \alpha^3} = -\frac{2n}{(\alpha+\theta)^3} - n2\sum_{x=1}^{n}(y_i)^3\,(1+\alpha y_i)^{-3},$$

$$\frac{\partial^3 l}{\partial \theta^2 \partial \alpha} = -\frac{2n}{(\alpha+\theta)^3},$$

$$\frac{\partial^3 l}{\partial \theta \partial \alpha^2} = -\frac{2n}{(\alpha+\theta)^3}.$$

The expected Fisher information matrix is given:

$$I(\Theta|y) = \begin{bmatrix} -\frac{2n}{\theta^2} + \frac{\alpha}{(\alpha+\theta)^2} & \frac{n}{(\alpha+\theta)^2} \\ \frac{n}{(\alpha+\theta)^2} & \frac{n}{(\alpha+\theta)^2} - n\left(\frac{1}{(\alpha+\theta)}\right)\left(\frac{n\theta^2 e^{\frac{\theta}{\alpha}}\Gamma\left(1,\frac{\theta}{\alpha}\right)+\alpha(\alpha-\theta)}{\alpha^3}\right) \end{bmatrix}.$$

The following equations are obtained:

$$\kappa_{111} = \frac{4n}{\theta^3} - \frac{2\alpha}{(\alpha+\theta)^3},$$

$$\kappa_{112} = \kappa_{121} = \kappa_{211} = -\frac{2n}{(\alpha+\theta)^3},$$

$$\kappa_{222} = -\frac{2n}{(\alpha+\theta)^3} + \frac{2n^2}{(\alpha+\theta)}\left(\frac{3\theta^2 e^{\frac{\theta}{\alpha}}\Gamma\left(1,\frac{\theta}{\alpha}\right)\alpha - 2\alpha^2\theta + \theta^3 e^{\frac{\theta}{\alpha}}\Gamma\left(1,\frac{\theta}{\alpha}\right) + \alpha^3 - \alpha\theta^2}{\alpha^5}\right),$$

$$\kappa_{122} = \kappa_{212} = \kappa_{221} = -\frac{2n}{(\alpha+\theta)^3},$$

$$\kappa_{11,1} = \frac{4n}{\theta^3} - \frac{2\alpha}{(\alpha+\theta)^3},$$

$$\kappa_{12,1} = \kappa_{12,2} = -\frac{2n}{(\alpha+\theta)^3},$$

$$\kappa_{22,1} = -\frac{2n}{(\alpha+\theta)^3} + \frac{\left(-\alpha\theta+\theta^2 e^{\frac{\theta}{\alpha}}\Gamma\left(1,\frac{\theta}{\alpha}\right)+\alpha^2\right)}{(\alpha+\theta)^2\alpha^3} - \frac{\left(-\alpha+2\theta e^{\frac{\theta}{\alpha}}\Gamma\left(1,\frac{\theta}{\alpha}\right)+\frac{\theta^2 e^{\frac{\theta}{\alpha}}\Gamma\left(1,\frac{\theta}{\alpha}\right)}{\alpha}-\theta e^{\frac{\theta}{\alpha}} e^{-\frac{\theta}{\alpha}}\right)}{(\alpha+\theta)\alpha^3},$$

$$\kappa_{11,2} = \frac{1}{(\alpha+\theta)^2} - \frac{2\alpha}{(\alpha+\theta)^3},$$

$$\kappa_{22,2} = -\frac{2n}{(\alpha+\theta)^3} + \frac{\left(-\alpha\theta+\theta^2 e^{\frac{\theta}{\alpha}}\Gamma\left(1,\frac{\theta}{\alpha}\right)+\alpha^2\right)}{(\alpha+\theta)^2\alpha^3} - \frac{\left(-\theta-\frac{\theta^3 e^{\frac{\theta}{\alpha}}\Gamma\left(1,\frac{\theta}{\alpha}\right)}{\alpha^2}+\frac{\theta^2 e^{\frac{\theta}{\alpha}} e^{-\frac{\theta}{\alpha}}}{\alpha}-2\alpha\right)}{(\alpha+\theta)\alpha^3} + \frac{3\left(-\alpha\theta+\theta^2 e^{\frac{\theta}{\alpha}}\Gamma\left(1,\frac{\theta}{\alpha}\right)+\alpha^2\right)}{(\alpha+\theta)\alpha^4}.$$

The elements of $A^{(1)}$ are obtained as follows:

$$a_{11,1} = \kappa_{11,1} - 0.5\kappa_{111} = \frac{1}{2}\left(\frac{4n}{\theta^3} - \frac{2\alpha}{(\alpha+\theta)^3}\right),$$
$$a_{12,1} = \kappa_{12,1} - 0.5\kappa_{112} = -\frac{n}{(\alpha+\theta)^3},$$
$$a_{22,1} = \kappa_{22,1} - 0.5\kappa_{122} = -\frac{2n}{(\alpha+\theta)^3} + \frac{\left(-\alpha\theta+\theta^2 e^{\frac{\theta}{\alpha}}\Gamma\left(1,\frac{\theta}{\alpha}\right)+\alpha^2\right)}{(\alpha+\theta)^2\alpha^3} - \frac{\left(-\alpha+2\theta e^{\frac{\theta}{\alpha}}\Gamma\left(1,\frac{\theta}{\alpha}\right)+\frac{\theta^2 e^{\frac{\theta}{\alpha}}\Gamma\left(1,\frac{\theta}{\alpha}\right)}{\alpha}-\theta e^{\frac{\theta}{\alpha}}e^{-\frac{\theta}{\alpha}}\right)}{(\alpha+\theta)\alpha^3} + \frac{n}{(\alpha+\theta)^3}.$$

The elements of $A^{(2)}$ are:

$$a_{11,2} = \kappa_{11,2} - 0.5\kappa_{112} = \frac{1}{(\alpha+\theta)^2} - \frac{2\alpha}{(\alpha+\theta)^3} + \frac{n}{(\alpha+\theta)^3},$$
$$a_{12,2} = \kappa_{12,2} - 0.5\kappa_{122} = -\frac{2n}{(\alpha+\theta)^3} + \frac{n}{(\alpha+\theta)^3},$$
$$a_{22,2} = \kappa_{22,2} - 0.5\kappa_{222} = -\frac{2n}{(\alpha+\theta)^3} + \frac{\left(-\alpha\theta+\theta^2 e^{\frac{\theta}{\alpha}}\Gamma\left(1,\frac{\theta}{\alpha}\right)+\alpha^2\right)}{(\alpha+\theta)^2\alpha^3} - \frac{\left(-\theta-\frac{\theta^3 e^{\frac{\theta}{\alpha}}\Gamma\left(1,\frac{\theta}{\alpha}\right)}{\alpha^2}+\frac{\theta^2 e^{\frac{\theta}{\alpha}}e^{-\frac{\theta}{\alpha}}}{\alpha}-2\alpha\right)}{(\alpha+\theta)\alpha^3}$$
$$+\frac{3\left(-\alpha\theta+\theta^2 e^{\frac{\theta}{\alpha}}\Gamma\left(1,\frac{\theta}{\alpha}\right)+\alpha^2\right)}{(\alpha+\theta)\alpha^4} - 0.5\left(-\frac{2n}{(\alpha+\theta)^3} + \frac{2n^2}{(\alpha+\theta)}\left(\frac{3\theta^2 e^{\frac{\theta}{\alpha}}\Gamma\left(1,\frac{\theta}{\alpha}\right)\alpha-2\alpha^2\theta+\theta^3 e^{\frac{\theta}{\alpha}}\Gamma\left(1,\frac{\theta}{\alpha}\right)+\alpha^3-\alpha\theta^2}{\alpha^5}\right)\right).$$

Finally, using Corderio and Klein (1994) modification of Coxx and Snell's (1968) method, the bias of $\widehat{\Theta}$ can be written as follows:

$$Bias\left(\widehat{\Theta}_s\right) = Bias\begin{pmatrix}\hat{\theta}\\ \hat{\alpha}\end{pmatrix} = K^{-1}Avec(K^{-1}).$$