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Research Article

Estimation of the Gompertz distribution's parameters under folded ranked set sampling

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ABSTRACT

In the current paper, estimation of the unknown parameters of the Gompertz distribution is considered using different methods of estimation based on simple random sampling (SRS), ranked set sampling (RSS), and folded ranked set sampling (FRSS). Methods of estimation will be considered including maximum likelihood (ML) and Bayes estimation. A comprehensive Monte Carlo simulation study is carried out to compare the resultant estimators via their biases and relative efficiencies (RE). A real data example is presented for illustration. The uniqueness of this study is that there is no parameter estimation study based on RSS and FRSS under the Gompertz distribution in the literature. The results indicate that FRSS outperforms both estimators and all sampling schemes in terms of bias. According to the relative efficiency, the ML and Bayes estimators are quite competitive.

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INTRODUCTION

Ranked set sampling (RSS) is a sampling scheme commonly used as an efficient alternative to simple random sampling (SRS) to estimate the population parameters in situations where actual measurements of study variables are expensive, difficult, or time-consuming. Furthermore, RSS is a practical sampling technique for improving the precision and increasing the efficiency of estimators of the population parameters. Therefore, RSS technique is used in many different fields such as medical, biological, ecological, physical, and social sciences.

The procedure of using RSS is as follows:

- *m* sets of size m are selected from the population based on SRS.
- The *m* units of each sample are ranked visually or using auxiliary information with respect to the variable of interest.
- The smallest observation unit from the first set was chosen. From the second set, the second smallest observation unit was chosen. The procedure was continued until the element with the largest rank from the m^{th} sample was chosen.
- Repeat steps 1-3 *r* times (cycles) until obtaining a sample of size $n=mr$. These n-measured units constitute the

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ranked set sample, which is denoted by $X_{(i)j}$; i=1, 2,…, m; j=1, 2,…, r.

Bani-Mustafa et al. [1] suggested the folded ranked set sampling (FRSS). This sampling scheme can be summarized in the following. In (FRSS) m random samples should be selected for each of size m, where m is typically small to reduce ranking error. FRSS scheme consists of the following steps:

- Select $[(m + 1)/2]$ random samples each of size m.
- Rank the units within each sample with respect to a variable of interest by a visual inspection or using auxiliary information.
- Select the 1st and the m^{th} units from the first sample for actual measurement.
- Select the first 2nd and the $(m - 1)^{th}$ units of the second sample for the actual measurement.
- The procedure is continued until the m^{th} unit is selected from the sample.
- The cycle may be repeated r times to obtain the desired sample size.

Suppose that the cycle is repeated once, selected FRSS for different sample sizes is shown in Figure 1.

This article is concerned with the ML and Bayes estimators of the unknown parameters of Gompertz distribution based on SRS, RSS, and FRSS. The Gompertz distribution was introduced by Gompertz [2] for modeling human mortality and fitting actuarial tables. This distribution is a generalization of exponential distribution and is associated with some well-known distributions such as exponential, Weibull, Gumbel, double exponential, or generalized logistic, (see Willekens [3]). Gompertz distribution is a very flexible distribution that can be skewed left and right by adjusting the parameters, (see Pollard and Valkovics [4]).

The probability density function (pdf) and cumulative distribution function (cdf) are given, respectively, by

$$f(x) = \beta e^{\left(\lambda x - \frac{\beta}{\lambda}(e^{\lambda x}-1)\right)}, \quad x > 0, \ \lambda, \beta > 0 \tag{1}$$

and

$$F(x) = 1 - e^{\left(-\frac{\beta}{\lambda}(e^{\lambda x}-1)\right)} \tag{2}$$

for m=3

$\boldsymbol{X_1}$	X_2	$\boldsymbol{X_3}$
X_1	$\boldsymbol{X_2}$	X_3

for m=4

$\boldsymbol{X_1}$	X_2	X_3	$\boldsymbol{X_4}$
X_1	$\boldsymbol{X_2}$	$\boldsymbol{X_3}$	X_4

for m=5

$\boldsymbol{X_1}$	X_2	X_3	X_4	$\boldsymbol{X_5}$
X_1	$\boldsymbol{X_2}$	X_3	$\boldsymbol{X_4}$	X_5
X_1	X_2	$\boldsymbol{X_3}$	X_4	X_5

for m=6

$\boldsymbol{X_1}$	X_2	X_3	X_4	X_5	$\boldsymbol{X_6}$
X_1	$\boldsymbol{X_2}$	X_3	X_4	$\boldsymbol{X_5}$	X_6
X_1	X_2	$\boldsymbol{X_3}$	$\boldsymbol{X_4}$	X_5	X_6

Figure 1. Selection scheme of FRSS for different sample sizes.

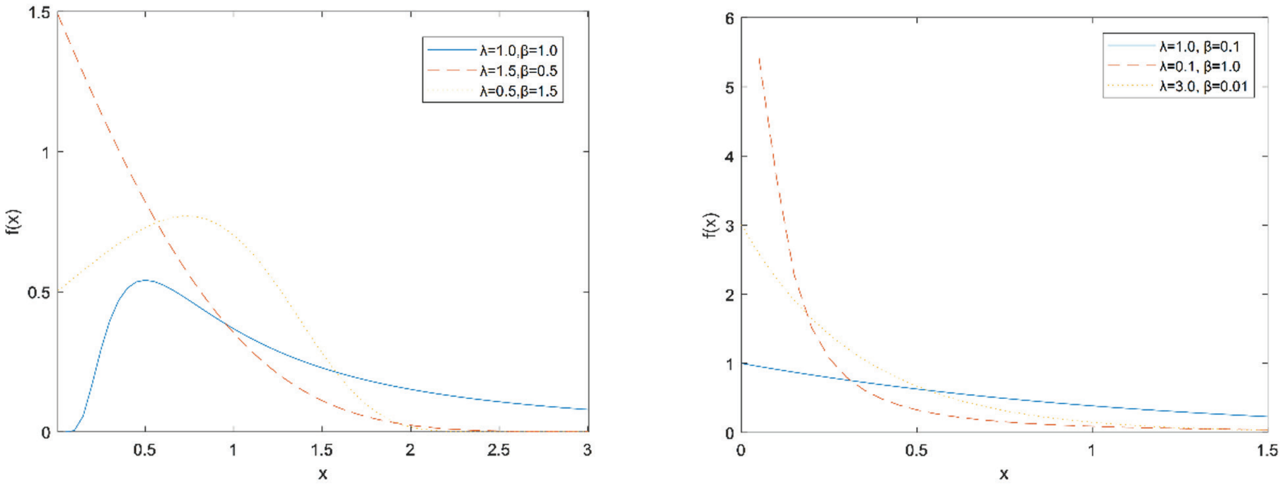

Figure 2. The graphs of density for different values of the parameters.

where λ is the shape parameter and β is the scale parameter. The Gompertz distribution will tend to an exponential distribution when $\lambda \to 0$. The pdf graphs of the Gompertz distribution for different shape and scale parameters are given in Figure 2.

Literature Review

The RSS was first suggested by McIntyre [5] as an efficient alternative to simple random sampling (SRS). Takahasi and Wakimoto [6] proved that RSS mean is an unbiased estimator for the population mean with a smaller variance compared to the SRS mean. Dell and Clutter [7] showed that RSS is more efficient than SRS whether the ranking is perfect or not. Various modification of the RSS have been suggested in the statistic literature, see extreme ranked set sampling (ERSS) by Samawi et al. [8], median ranked set sampling (MRSS) by Muttlak [9], double ranked set sampling (DRSS) by Al-Saleh and Al-Kadiri [10], moving ERSS (MERSS) by Al-Saleh and Al-Hadrami [11], bivariate ERSS (BERSS) by Al-Saleh and Samawi [12], L RSS (LRSS) by Al-Nasser [13], robust ERSS (RERSS) by Al-Nasser and Mustafa [14], folded RSS (FRSS) by Bani-Mustafa et al. [1].

Parameter estimation for any statistical distribution based on RSS and its modifications have been very attractive in recent years. The literature review about the parameter estimation based on the RSS schemes is summarized in Table 1.

In addition, RSS and its modifications can also be used in multi-criteria decision-making problems. Some recently proposed alternative techniques are Interval-valued Q-Rung Orthopair Hesitant fuzzy sets by Özlü [32], single-valued neutrosophic type-2 hesitant fuzzy sets (SVNT2HFS) by [33], Q-rung orthopair probabilistic hesitant fuzzy hybrid aggregating operators by Özlü [34], new q-rung orthopair fuzzy Aczel–Alsina weighted geometric operators by Özlü [35], and bipolar valued probabilistic hesitant fuzzy sets (BVPHFSs) by Özlü et al. [36].

Motivation and Contribution

The following contributions were made in this article.

- The most important reason for using FRSS design in this article is that it has been never preferred for parameter estimation studies in the literature.
- In addition, the Gompertz distribution is almost never used in estimation based on RSS and its modifications.
- It is clear from the literature on RSS sampling techniques and all their modifications that they waste sampling units; the FRSS design was preferred to overcome this problem in data collection. Because the purpose of RSS designs is to estimate the population mean and to reduce the amount of wasted sampling units.
- The shape and scale parameters of Gompertz distribution are estimated under SRS, RSS, and FRSS schemes.
- The estimation is made using ML and Bayesian estimation methods.
- The Bayes estimators under squared error loss function are discussed assuming gamma priors for both the shape and scale parameters.
- We evaluated the performance of ML and Bayes estimators through an extensive Monte Carlo simulation study. We compared the biases, mean square errors (MSEs), and relative efficiencies of estimators under

Table 1. The summarize of the literature about parameter estimation based on the RSS schemes

Author	RSS scheme	Distribution	Estimation method
Stokes [15]	RSS	Bivariate normal	ML
Shaibu and Muttlak [16]	ERSS, Median RSS	Normal, exponential, gamma	ML
Modares and Zheng [17]	RSS	Bivariate normal, bivariate extreme value	ML
Abu-Deyyah and Al-Sawi [18]	Moving ERSS	Exponential	ML
Helu et al. [19]	RSS, Modified RSS	Weibull	ML, Mom, Bayes
Al-Omari and Al-Hadhrami [20]	ERSS	Modified Weibull	ML
Omar and Ibrahim [21]	RSS, Median RSS, ERSS	Pareto	ML, Mom, MML, Ad hoc
Hassan [22]	RSS	Exponentiated exponential	ML, Bayes
Hussian [23]	RSS	Kumaraswamy	ML, Bayes
Khamnei and Abusalah [24]	RSS	Generalized logistic	ML
Dey et al. [25]	RSS, Median RSS, Modified RSS	Rayleigh	ML, Bayes
Esemen and Gürler [26]	RSS, Median RSS, ERSS	Generalized Rayleigh	ML
He et al. [27]	RSS	Log-logistic	ML
Samuh et al. [28]	RSS, Median RSS, ERSS	Weibull-Pareto	ML
Taconeli and Giolo [29]	RSS	Power Lindley, weighted Lindley	ML
Joukar et al. [30]	RSS, Maximum RSS	Exponential-Poisson	ML, Bayes
Khamnei et al. [31]	RSS	Exponentiated Pareto	ML

SRS, RSS, and FRSS with different sample sizes and parameter values.

- An illustrative example is given in order to evaluate and compare the performances of the proposed estimators.
- The results also showed that FRSS design is more accurate and effective than the SRS and RSS designs in estimating the parameters of the Gompertz distribution.

The remainder of this article is organized as follows. ML and Bayesian methods of estimation of unknown parameters are respectively discussed based on SRS, RSS, and FRSS in Sections 2 and 3. Section 4 presents a Monte Carlo simulation study to compare the performances of the proposed estimators and sampling schemes. In Section 5, a real data example is given to illustrate the computations of our proposed estimators. The final Section briefly displays summary and conclusion of the findings.

MAXIMUM LIKELIHOOD ESTIMATION (MLE)

MLE Based On SRS

Let $X_1, X_2, ..., X_n$ be independent and identically distributed random variables from Gompertz distribution with pdf given in Eq. (1). The likelihood function of λ and β is given by

$$L_S(\lambda,\beta;x)=\beta^n e^{\left(\lambda\sum_{i=1}^n x_i-\frac{\beta}{\lambda}\left(e^{\lambda\sum_{i=1}^n x_i}-1\right)\right)}$$

and the log-likelihood function is

$$l_{SRS}(\lambda,\beta;x)=nlog(\beta)+\lambda\sum_{i=1}^n x_i-\frac{\beta}{\lambda}\left(e^{\lambda\sum_{i=1}^n x_i}-1\right) \tag{3}$$

The normal equations become:

$$\begin{aligned}\frac{dl}{d\lambda}&=\sum_{i=1}^n x_i-\frac{\beta}{\lambda}\sum_{i=1}^n x_i e^{\lambda\sum_{i=1}^n x_i}\\&+\frac{\beta}{\lambda^2}\sum_{i=1}^n e^{\lambda x_i}-\frac{n\beta}{\lambda^2}=0\end{aligned} \tag{4}$$

$$\frac{dl}{d\beta}=\frac{n}{\beta}-\frac{\sum_{i=1}^n\left(e^{\lambda x_i}-1\right)}{\lambda}=0. \tag{5}$$

Obviously, there is no closed-form solution for the two non-linear equations (4) and (5), therefore numerical methods must be applied to solve these equations.

MLE Based On RSS

Let $\{X_{(i)j},\ i = 1, 2,..., m;\ j = 1, ..., r\}$ be a ranked set sample of size $n=mr$ drawn from the Gompertz distribution where m is the set size and r is the number of cycles, respectively. The likelihood function of the RSS sample for Gompertz data is given by:

$$\begin{aligned}L_r(\lambda,\beta;x)&=\prod_{j=1}^r\prod_{i=1}^m C_1 f(x_{ij};\lambda,\beta)\left[F(x_{ij};\lambda,\beta)\right]^{i-1}\left[1-F(x_{ij};\lambda,\beta)\right]^{m-i}\\&=\beta^{mr}{C_1}^{mr}\prod_{j=1}^r\prod_{i=1}^m e^{\lambda x_{ij}}\left[e^{-\frac{\beta}{\lambda}\left(e^{\lambda x_{ij}}-1\right)}\right]^{m-i+1}\left[1-e^{-\frac{\beta}{\lambda}\left(e^{\lambda x_{ij}}-1\right)}\right]^{i-1},\end{aligned} \tag{6}$$

where $C_1=\frac{m!}{(i-1)!(m-i)!}$. Then, the log-likelihood function is

$$\begin{aligned}l_{RSS}(\lambda,\beta)&=mrlogC_1+mrlog\beta+\lambda\sum_{j=1}^r\sum_{i=1}^m x_{ij}\\&-\frac{\beta}{\lambda}\sum_{j=1}^r\sum_{i=1}^m(m-i+1)\left(e^{\lambda x_{ij}}-1\right)\\&+\sum_{j=1}^r\sum_{i=1}^m(i-1)log\left(1-e^{-\frac{\beta}{\lambda}\left(e^{\lambda x_{ij}}-1\right)}\right),\end{aligned} \tag{7}$$

and the likelihood normal equations become

$$\begin{aligned}\frac{dl}{d\lambda}&=\sum_{j=1}^r\sum_{i=1}^m x_{ij}+\frac{\beta}{\lambda^2}\sum_{j=1}^r\sum_{i=1}^m(m-i+1)\left(e^{\lambda x_{ij}}-1\right)\\&-\frac{\beta}{\lambda}\sum_{j=1}^r\sum_{i=1}^m(m-i+1)x_{ij}\left(e^{\lambda x_{ij}}-1\right)\\&+\sum_{j=1}^r\sum_{i=1}^m(i-1)\frac{\frac{\beta}{\lambda}\sum_{j=1}^r\sum_{i=1}^m x_{ij}e^{\lambda x_{ij}}-\frac{\beta}{\lambda^2}\sum_{j=1}^r\sum_{i=1}^m\left(e^{\lambda x_{ij}}-1\right)}{\left(1-e^{-\frac{\beta}{\lambda}\left(e^{\lambda x_{ij}}-1\right)}\right)}=0,\end{aligned}$$

$$\begin{aligned}\frac{dl}{d\beta}&=\frac{mr}{\beta}-\frac{1}{\lambda}\sum_{j=1}^r\sum_{i=1}^m(m-i+1)\left(e^{\lambda x_{ij}}-1\right)\\&+\sum_{j=1}^r\sum_{i=1}^m(i-1)\frac{\frac{\left(e^{\lambda x_{ij}}-1\right)}{\lambda}e^{-\frac{\beta}{\lambda}\left(e^{\lambda x_{ij}}-1\right)}}{\left(1-e^{-\frac{\beta}{\lambda}\left(e^{\lambda x_{ij}}-1\right)}\right)}=0.\end{aligned}$$

We will use numerical methods to solve these equations.

MLE Based On FRSS

The required FRSS will be Let $X=\left\{X_{(i)j,}\ i=1,2,\dots,\frac{m+1}{2};\ j=1,\dots,r\right\}\cup\left\{X_{(m-i+1)j,}\ i=1,2,\dots,\frac{m+1}{2};\ j=1,\dots,r\right\}$. The likelihood function of the FRSS sample for Gompertz data is given by

$$\begin{aligned}L_f(\lambda,\beta;x)&=\prod_{j=1}^r\prod_{i=1}^{\frac{m+1}{2}}C_1 f(x_{ij};\lambda,\beta)\left[F(x_{ij};\lambda,\beta)\right]^{i-1}\\&\times\left[1-F(x_{ij};\lambda,\beta)\right]^{m-i}\prod_{j=1}^r\prod_{\substack{i=1\\i<m-i+1}}^{\frac{m+1}{2}}C_1 f(x_{ij(m-i+1)};\lambda,\beta)\\&\times\left[F(x_{ij(m-i+1)};\lambda,\beta)\right]^{m-i}\left[1-F(x_{ij(m-i+1)};\lambda,\beta)\right]^{i-1},\\&=\beta^{mr}{C_2}^{mr}\prod_{j=1}^r\prod_{i=1}^{\frac{m+1}{2}}e^{\lambda x_{ij}}\left[e^{-\frac{\beta}{\lambda}\left(e^{\lambda x_{ij}}-1\right)}\right]^{m-i+1}\\&\times\left[1-e^{-\frac{\beta}{\lambda}\left(e^{\lambda x_{ij}}-1\right)}\right]^{i-1}\prod_{j=1}^r\prod_{\substack{i=1\\i<m-i+1}}^{\frac{m+1}{2}}e^{\lambda x_{ij(m-i+1)}}\\&\times\left[e^{-\frac{\beta}{\lambda}\left(e^{\lambda x_{ij(m-i+1)}}-1\right)}\right]^{i}\left[1-e^{-\frac{\beta}{\lambda}\left(e^{\lambda x_{ij(m-i+1)}}-1\right)}\right]^{m-i},\end{aligned} \tag{8}$$

where $C_2=(C_1)^2=\left(\frac{m!}{(i-1)!(m-i)!}\right)^2$. The log-likelihood function is then given by

$$\begin{aligned}l_{FRSS}(\lambda,\beta)&=mrlogC_2+mrlog\beta+\lambda\sum_{j=1}^r\sum_{i=1}^{\frac{m+1}{2}}x_{ij}\\&-\frac{\beta}{\lambda}\sum_{j=1}^r\sum_{i=1}^{\frac{m+1}{2}}(m-i+1)\left(e^{\lambda x_{ij}}-1\right)\\&+\sum_{j=1}^r\sum_{i=1}^{\frac{m+1}{2}}(i-1)log\left(1-e^{-\frac{\beta}{\lambda}\left(e^{\lambda x_{ij}}-1\right)}\right)\\&+\lambda\sum_{j=1}^r\sum_{\substack{i=1\\i<m-i+1}}^{\frac{m+1}{2}}x_{ij(m-i+1)}\\&-\frac{\beta}{\lambda}\sum_{j=1}^r\sum_{\substack{i=1\\i<m-i+1}}^{\frac{m+1}{2}}i\left(e^{\lambda x_{ij(m-i+1)}}-1\right)\\&+\sum_{j=1}^r\sum_{\substack{i=1\\i<m-i+1}}^{\frac{m+1}{2}}(m-i)log\left(1-e^{-\frac{\beta}{\lambda}\left(e^{\lambda x_{ij(m-i+1)}}-1\right)}\right)\end{aligned} \tag{9}$$

and the likelihood normal equations become

$$\frac{dl}{d\lambda} = \sum_{j=1}^{r}\sum_{i=1}^{\frac{m+1}{2}} x_{ij} + \frac{\beta}{\lambda^2}\sum_{j=1}^{r}\sum_{i=1}^{\frac{m+1}{2}}(m-i+1)\left(e^{\lambda x_{ij}}-1\right)$$
$$-\frac{\beta}{\lambda}\sum_{j=1}^{r}\sum_{i=1}^{\frac{m+1}{2}}(m-i+1)x_{ij}\left(e^{\lambda x_{ij}}-1\right)$$
$$+\sum_{j=1}^{r}\sum_{i=1}^{\frac{m+1}{2}}(i-1)\frac{\frac{\beta}{\lambda}\sum_{j=1}^{r}\sum_{i=1}^{m}x_{ij}e^{\lambda x_{ij}}-\frac{\beta}{\lambda^2}\sum_{j=1}^{r}\sum_{i=1}^{m}\left(e^{\lambda x_{ij}}-1\right)}{\left(1-e^{-\frac{\beta}{\lambda}\left(e^{\lambda x_{ij}}-1\right)}\right)}$$
$$+\sum_{j=1}^{r}\sum_{\substack{i=1\\ i<m-i+1}}^{\frac{m+1}{2}} x_{ij(m-i+1)} + \frac{\beta}{\lambda^2}\sum_{j=1}^{r}\sum_{\substack{i=1\\ i<m-i+1}}^{\frac{m+1}{2}} i\left(e^{\lambda x_{ij(m-i+1)}}-1\right)$$
$$-\frac{\beta}{\lambda}\sum_{j=1}^{r}\sum_{\substack{i=1\\ i<m-i+1}}^{\frac{m+1}{2}} i x_{ij(m-i+1)}\left(e^{\lambda x_{ij(m-i+1)}}-1\right)$$
$$+\sum_{j=1}^{r}\sum_{\substack{i=1\\ i<m-i+1}}^{\frac{m+1}{2}}(m-i)\frac{\frac{\beta}{\lambda}\sum_{j=1}^{r}\sum_{i=1}^{m}x_{ij}e^{\lambda x_{ij(m-i+1)}}-\frac{\beta}{\lambda^2}\sum_{j=1}^{r}\sum_{i=1}^{m}\left(e^{\lambda x_{ij(m-i+1)}}-1\right)}{\left(1-e^{-\frac{\beta}{\lambda}\left(e^{\lambda x_{ij(m-i+1)}}-1\right)}\right)} = 0,$$

$$\frac{dl}{d\beta} = \frac{mr}{\beta} - \frac{1}{\lambda}\sum_{j=1}^{r}\sum_{i=1}^{\frac{m+1}{2}}(m-i+1)\left(e^{\lambda x_{ij}}-1\right) + \sum_{j=1}^{r}\sum_{i=1}^{\frac{m+1}{2}}(i-1)\frac{\frac{\left(e^{\lambda x_{ij}}-1\right)}{\lambda}e^{-\frac{\beta}{\lambda}\left(e^{\lambda x_{ij}}-1\right)}}{\left(1-e^{-\frac{\beta}{\lambda}\left(e^{\lambda x_{ij}}-1\right)}\right)}$$
$$-\frac{1}{\lambda}\sum_{j=1}^{r}\sum_{\substack{i=1\\ i<m-i+1}}^{\frac{m+1}{2}} i\left(e^{\lambda x_{ij(m-i+1)}}-1\right)$$
$$+\sum_{j=1}^{r}\sum_{\substack{i=1\\ i<m-i+1}}^{\frac{m+1}{2}}(m-i)\frac{\frac{\left(e^{\lambda x_{ij(m-i+1)}}-1\right)}{\lambda}e^{-\frac{\beta}{\lambda}\left(e^{\lambda x_{ij(m-i+1)}}-1\right)}}{\left(1-e^{-\frac{\beta}{\lambda}\left(e^{\lambda x_{ij(m-i+1)}}-1\right)}\right)} = 0.$$

The solutions are not in closed form, to obtain estimates for λ and β, the normal equations are needed to solve them.

BAYESIAN ESTIMATION

Bayes Estimation Based On SRS

The Bayes estimators of shape parameter λ and scale parameter β are considered to be random variables with a joint prior distribution. It is assumed that λ and β have the following gamma prior distributions;

$$\pi_1(\lambda) = \frac{b^a}{\Gamma(a)}\lambda^{a-1}e^{-b\lambda},\ \pi_2(\beta) = \frac{d^c}{\Gamma(c)}\beta^{c-1}e^{-d\beta} \quad (10)$$

where λ, $\beta > 0$ all the hyperparameters a, b, c, and d are assumed to known and non-negative. Using the priors given in Eq. (10), the joint posterior distribution of (λ, β) can be expressed as

$$\pi\left(\lambda,\beta|\underline{x}\right) = \frac{L\left(\underline{x}|\lambda,\beta\right)\pi_1(\lambda)\pi_2(\beta)}{\iint_0^\infty L\left(\underline{x}|\lambda,\beta\right)\pi_1(\lambda)\pi_2(\beta)d\lambda d\beta} = \frac{T_1}{\iint_0^\infty T_1 d\lambda d\beta},$$

where

$$T_1 = \frac{b^a d^c}{\Gamma(a)\Gamma(c)}\lambda^{a-1}\beta^{n+c-1}e^{-b\lambda-d\beta+\lambda\sum_{i=1}^{n}x_i-\frac{\beta}{\lambda}e^{-\sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)}}.$$

Then the Bayes estimation of λ and β are obtained by maximization of the joint posterior density function

$$\underset{\lambda,\beta}{\operatorname{argmax}}\,\pi_{SRS}\left(\lambda,\beta|\underline{x}\right),$$

with respect to the parameters λ and β.

Bayes Estimation Based On RSS

Let λ and β be independent random variables with prior gamma distributions given in Eq. (10) and the likelihood function given in Eq. (6), we get the joint posterior pdf of λ and β by

$$\pi\left(\lambda,\beta|\underline{x}\right) = \frac{L\left(\underline{x}|\lambda,\beta\right)\pi_1(\lambda)\pi_2(\beta)}{\iint_0^\infty L\left(\underline{x}|\lambda,\beta\right)\pi_1(\lambda)\pi_2(\beta)d\lambda d\beta} = \frac{T_2}{\iint_0^\infty T_2 d\lambda d\beta},$$

where

$$T_2 = K_1{}^{mr}C_1{}^{mr}\lambda^{a-1}\beta^{mr+c-1}e^{-b\lambda-d\beta+\lambda\sum_{j=1}^{r}\sum_{i=1}^{m}x_{ij}}$$
$$\times\prod_{j=1}^{r}\prod_{i=1}^{m}e^{\lambda x_{ij}}\left[e^{-\frac{\beta}{\lambda}\left(e^{\lambda x_{ij}}-1\right)}\right]^{m-i+1}\left[1-e^{-\frac{\beta}{\lambda}\left(e^{\lambda x_{ij}}-1\right)}\right]^{i-1},$$

and

$$K_1 = \frac{b^a d^c}{\Gamma(a)\Gamma(c)}.$$

Then the Bayes estimation of λ and β are obtained by maximization of the joint posterior density function

$$\underset{\lambda,\beta}{\operatorname{argmax}}\,\pi_{RSS}\left(\lambda,\beta|\underline{x}\right), \quad (12)$$

with respect to the parameters λ and β.

Bayes Estimation Based On FRSS

The Bayes estimators of λ and β are obtained similar to the procedure used in subsections (3.1) and (3.2). Based on prior distributions presented in Eq. (10) and the likelihood function presented in Eq. (8), the joint posterior distribution of (λ, β) is

$$\pi\left(\lambda,\beta|\underline{x}\right) = \frac{L\left(\underline{x}|\lambda,\beta\right)\pi_1(\lambda)\pi_2(\beta)}{\iint_0^\infty L\left(\underline{x}|\lambda,\beta\right)\pi_1(\lambda)\pi_2(\beta)d\lambda d\beta} = \frac{T_3}{\iint_0^\infty T_3 d\lambda d\beta},$$

where

$$T_3 = K_1{}^{mr}C_2{}^{mr}\lambda^{a-1}\beta^{mr+c-1}e^{-b\lambda-d\beta+\lambda\sum_{j=1}^{r}\sum_{i=1}^{\frac{m+1}{2}}x_{ij}}$$
$$\times\prod_{j=1}^{r}\prod_{i=1}^{\frac{m+1}{2}}e^{\lambda x_{ij}}\left[e^{-\frac{\beta}{\lambda}\left(e^{\lambda x_{ij}}-1\right)}\right]^{m-i+1}$$
$$\times\left[1-e^{-\frac{\beta}{\lambda}\left(e^{\lambda x_{ij}}-1\right)}\right]^{i-1}\prod_{j=1}^{r}\prod_{\substack{i=1\\ i<m-i+1}}^{\frac{m+1}{2}}e^{\lambda x_{ij(m-i+1)}}$$
$$\times\left[e^{-\frac{\beta}{\lambda}\left(e^{\lambda x_{ij(m-i+1)}}-1\right)}\right]^{i}\left[1-e^{-\frac{\beta}{\lambda}\left(e^{\lambda x_{ij(m-i+1)}}-1\right)}\right]^{m-i}.$$

Then the Bayes estimation of λ and β are obtained by maximization of the joint posterior density function

$$\underset{\lambda,\beta}{\operatorname{argmax}}\,\pi_{FRSS}\left(\lambda,\beta|\underline{x}\right), \quad (13)$$

with respect to the parameters λ and β.

SIMULATION STUDY

It is very difficult to compare the theoretical performances of the MLE and Bayes estimators proposed in the previous sections. Therefore, we carry out a simulation study in order to compare the performance of MLE and Bayes estimators based on SRS, RSS, and FRSS approaches. The comparison study is made based on biases and relative efficiency. The Monte Carlo simulation study is made using MATLAB software and is based on 10.000 replications for different set sizes, different numbers of cycles, and different parameter values. The simulation procedures are described as follows.

1. Generate SRS samples of size n=12,24,36 and also RSS and FRSS samples are generated in the number of cycles

Table 2. Biases of the estimator of λ

		SRS			RSS		FRSS	
$\lambda; \beta$	n	$\hat{\lambda}_{mle}$	$\hat{\lambda}_{bayes}$	$m; r$	$\hat{\lambda}_{mle}$	$\hat{\lambda}_{bayes}$	$\hat{\lambda}_{mle}$	$\hat{\lambda}_{bayes}$
1.0;1.0	12	0.4952	0.4993	3;4	0.4176	0.4224	0.4022	0.4002
				4;3	0.3922	0.3939	0.3997	0.3962
				6;2	0.3398	0.3404	0.3382	0.3368
	24	0.2399	0.2291	3;8	0.2077	0.2039	0.2036	0.1983
				4;6	0.1967	0.1912	0.1888	0.1916
				6;4	0.1797	0.1724	0.1722	0.1718
	36	0.1534	0.1485	3;12	0.1507	0.1343	0.1279	0.1330
				4;9	0.1318	0.1269	0.1272	0.1207
				6;6	0.1150	0.1114	0.1116	0.1191
1.5;0.5	12	0.4033	0.3899	3;4	0.3282	0.3281	0.3223	0.3182
				4;3	0.2960	0.2913	0.3020	0.3226
				6;2	0.2628	0.2565	0.2608	0.2527
	24	0.1824	0.1816	3;8	0.1569	0.1597	0.1551	0.1526
				4;6	0.1408	0.1498	0.1500	0.1577
				6;4	0.1286	0.1254	0.1263	0.1242
	36	0.1136	0.1177	3;12	0.1012	0.1063	0.1000	0.1024
				4;9	0.1033	0.0966	0.1004	0.0997
				6;6	0.0885	0.0843	0.0867	0.0887
0.5;1.5	12	0.5655	0.5720	3;4	0.4915	0.4911	0.4997	0.4831
				4;3	0.4906	0.4768	0.4854	0.4743
				6;2	0.4143	0.4241	0.4054	0.4069
	24	0.2793	0.2597	3;8	0.2412	0.2409	0.2368	0.2359
				4;6	0.2325	0.2257	0.2320	0.2237
				6;4	0.2025	0.2169	0.2110	0.2146
	36	0.1775	0.1725	3;12	0.1611	0.1667	0.1540	0.1581
				4;9	0.1530	0.1553	0.1591	0.1502
				6;6	0.1453	0.1427	0.1436	0.1389
1.0;0.1	12	0.1764	0.1765	3;4	0.1426	0.1391	0.1350	0.1388
				4;3	0.1253	0.1277	0.1299	0.1305
				6;2	0.1081	0.1065	0.1067	0.1116
	24	0.0828	0.0842	3;8	0.0679	0.0642	0.0675	0.0694
				4;6	0.0615	0.0614	0.0645	0.0664
				6;4	0.0516	0.0556	0.0548	0.0534
	36	0.0525	0.0507	3;12	0.0469	0.0406	0.0432	0.0417
				4;9	0.0415	0.0395	0.0440	0.0421
				6;6	0.0324	0.0381	0.0353	0.0364
0.1;1.0	12	0.3354	0.3569	3;4	0.2966	0.3018	0.2844	0.3094
				4;3	0.2833	0.2936	0.2852	0.2849
				6;2	0.2609	0.2573	0.2572	0.2572
	24	0.1544	0.1604	3;8	0.1492	0.1469	0.1418	0.1463
				4;6	0.1468	0.1426	0.1344	0.1413
				6;4	0.1294	0.1308	0.1289	0.1262
	36	0.1062	0.1048	3;12	0.1002	0.0972	0.0966	0.0949
				4;9	0.0945	0.0944	0.0961	0.0937
				6;6	0.0906	0.0894	0.0891	0.0874
3.0;0.01	12	0.3876	0.3944	3;4	0.3036	0.2923	0.2989	0.3056
				4;3	0.2607	0.2739	0.2838	0.2902
				6;2	0.2317	0.2269	0.2218	0.2239
	24	0.1959	0.1870	3;8	0.1489	0.1473	0.1402	0.1397
				4;6	0.1303	0.1331	0.1325	0.1368
				6;4	0.1037	0.1145	0.1116	0.1032
	36	0.1209	0.1262	3;12	0.0901	0.0893	0.0927	0.0995
				4;9	0.0791	0.0828	0.0911	0.0858
				6;6	0.0735	0.0752	0.0720	0.0736

Table 3. Biases of the estimator of β

		SRS			RSS		FRSS	
$\lambda;\beta$	n	$\hat{\beta}_{mle}$	$\hat{\beta}_{bayes}$	$m;r$	$\hat{\beta}_{mle}$	$\hat{\beta}_{bayes}$	$\hat{\beta}_{mle}$	$\hat{\beta}_{bayes}$
1.0;1.0	12	0.0975	0.0956	3;4	0.0869	0.0884	0.0748	0.0701
				4;3	0.0924	0.0853	0.0718	0.0706
				6;2	0.0861	0.0846	0.0724	0.0685
	24	0.0570	0.0517	3;8	0.0467	0.0487	0.0452	0.0435
				4;6	0.0498	0.0504	0.0387	0.0394
				6;4	0.0503	0.0460	0.0407	0.0427
	36	0.0436	0.0421	3;12	0.0318	0.0304	0.0291	0.0295
				4;9	0.0354	0.0324	0.0284	0.0198
				6;6	0.0324	0.0330	0.0250	0.0301
1.5;0.5	12	0.0417	0.0484	3;4	0.0347	0.0355	0.0327	0.0315
				4;3	0.0364	0.0341	0.0230	0.0293
				6;2	0.0369	0.0326	0.0291	0.0287
	24	0.0302	0.0307	3;8	0.0189	0.0218	0.0161	0.0173
				4;6	0.0183	0.0200	0.0146	0.0158
				6;4	0.0189	0.0188	0.0155	0.0137
	36	0.0196	0.0169	3;12	0.0109	0.0133	0.0107	0.0120
				4;9	0.0154	0.0133	0.0100	0.0104
				6;6	0.0148	0.0128	0.0112	0.0126
0.5;1.5	12	0.1799	0.1684	3;4	0.1452	0.1445	0.1450	0.1372
				4;3	0.1653	0.1566	0.1329	0.1245
				6;2	0.1488	0.1478	0.1242	0.1683
	24	0.0870	0.0925	3;8	0.0799	0.0839	0.0759	0.0795
				4;6	0.0834	0.0848	0.0685	0.0672
				6;4	0.0797	0.0851	0.0738	0.0775
	36	0.0678	0.0793	3;12	0.0561	0.0629	0.0550	0.0523
				4;9	0.0592	0.0571	0.0532	0.0466
				6;6	0.0620	0.0594	0.0538	0.0543
1.0;0.1	12	0.0062	0.0054	3;4	0.0045	0.0040	0.0038	0.0043
				4;3	0.0045	0.0055	0.0036	0.0034
				6;2	0.0050	0.0054	0.0038	0.0041
	24	0.0034	0.0037	3;8	0.0026	0.0023	0.0021	0.0033
				4;6	0.0027	0.0031	0.0022	0.0025
				6;4	0.0026	0.0030	0.0020	0.0026
	36	0.0024	0.0026	3;12	0.0020	0.0011	0.0012	0.0011
				4;9	0.0023	0.0017	0.0017	0.0016
				6;6	0.0014	0.0023	0.0015	0.0018
0.1;1.0	12	0.1223	0.1194	3;4	0.1133	0.1144	0.1016	0.1091
				4;3	0.1108	0.1189	0.0973	0.0942
				6;2	0.1093	0.1129	0.0975	0.1014
	24	0.0739	0.0681	3;8	0.0617	0.0618	0.0539	0.0596
				4;6	0.0662	0.0653	0.0500	0.0545
				6;4	0.0640	0.0631	0.0531	0.0547
	36	0.0476	0.0462	3;12	0.0433	0.0436	0.0402	0.0413
				4;9	0.0450	0.0456	0.0381	0.0391
				6;6	0.0460	0.0459	0.0395	0.0416
3.0;0.01	12	0.0023	0.0023	3;4	0.0014	0.0015	0.0013	0.0013
				4;3	0.0015	0.0012	0.0014	0.0010
				6;2	0.0010	0.0009	0.0010	0.0010
	24	0.0010	0.0010	3;8	0.0007	0.0007	0.0007	0.0007
				4;6	0.0006	0.0005	0.0007	0.0006
				6;4	0.0005	0.0003	0.0005	0.0006
	36	0.0006	0.0007	3;12	0.0005	0.0005	0.0004	0.0002
				4;9	0.0005	0.0004	0.0005	0.0005
				6;6	0.0002	0.0002	0.0003	0.0003

Table 4. The efficiencies of the estimators

			RSS				FRSS			
$\lambda;\beta$	n	$m;r$	$\hat{\lambda}_{mle}$	$\hat{\lambda}_{bayes}$	$\hat{\beta}_{mle}$	$\hat{\beta}_{bayes}$	$\hat{\lambda}_{mle}$	$\hat{\lambda}_{bayes}$	$\hat{\beta}_{mle}$	$\hat{\beta}_{bayes}$
1.0;1.0	12	3;4	1.2958	1.2742	5.6584	1.7599	1.4789	1.3779	5.2510	1.5737
		4;3	1.4700	1.4163	6.8508	2.0083	1.3994	1.3850	4.7587	1.4448
		6;2	1.8459	1.7940	8.8800	2.5976	1.8099	1.7850	6.3543	1.9278
	24	3;8	1.1905	1.2065	1.6898	1.7171	1.3116	1.2463	1.5309	1.5489
		4;6	1.3207	1.2667	1.9876	2.0173	1.3184	1.2170	1.5147	1.4377
		6;4	1.6160	1.5239	2.5617	2.5216	1.5440	1.4371	1.8883	1.8565
	36	3;12	1.0253	1.1591	1.4836	1.6601	1.3242	1.2321	1.6112	1.5322
		4;9	1.2570	1.2470	2.0234	2.0023	1.2717	1.2032	1.4684	1.4418
		6;6	1.5316	1.5014	2.5352	2.5598	1.4617	1.3847	1.9739	1.8708
1.5;0.5	12	3;4	1.2789	1.2732	1.7848	1.7766	1.4070	1.4261	1.6564	1.7070
		4;3	1.4997	1.4590	2.1172	2.0890	1.4541	1.3629	1.5583	1.5224
		6;2	1.8664	1.8316	2.7168	2.6754	1.9114	1.8287	2.0512	2.0469
	24	3;8	1.2409	1.2741	1.6667	1.7469	1.3237	1.3187	1.5760	1.5507
		4;6	1.3659	1.3163	1.9863	1.9633	1.2524	1.2986	1.4500	1.4965
		6;4	1.6467	1.6305	2.5144	2.4318	1.5715	1.5973	1.9078	1.8854
	36	3;12	1.1991	1.1862	1.6342	1.6416	1.3037	1.2908	1.5459	1.5519
		4;9	1.3142	1.3137	1.9724	1.9452	1.2554	1.2419	1.4591	1.4489
		6;6	1.6309	1.5413	2.5535	2.4482	1.5372	1.4746	1.9066	1.8684
0.5;1.5	12	3;4	1.2689	1.2581	1.7965	1.9410	1.3229	1.3940	1.6394	1.7715
		4;3	1.3143	1.3979	2.0744	2.2476	1.3186	1.3867	1.5166	1.5698
		6;2	1.6903	1.6719	2.6170	2.7951	1.7125	1.7260	2.0051	2.0658
	24	3;8	1.2133	1.1885	1.7695	1.7701	1.3373	1.3003	1.6174	1.6534
		4;6	1.2832	1.3165	2.0507	2.0751	1.2366	1.2431	1.4725	1.4828
		6;4	1.6199	1.4282	2.6591	2.5673	1.4849	1.4302	1.9470	1.9484
	36	3;12	1.1486	1.1578	1.6366	1.7365	1.2552	1.2366	1.5132	1.5802
		4;9	1.2872	1.2667	1.9681	2.0811	1.1669	1.2136	1.4345	1.4695
		6;6	1.4379	1.4199	2.4484	2.5553	1.3451	1.3747	1.8219	1.9044
1.0;0.1	12	3;4	1.2698	1.1590	1.6875	1.7058	1.4305	1.1415	1.6363	1.6571
		4;3	1.4621	1.1705	2.0000	2.0714	1.5134	1.2098	1.5882	1.6111
		6;2	1.9272	1.0175	2.5714	2.7619	1.9914	1.2239	2.0769	2.1481
	24	3;8	1.2805	1.3600	1.6250	1.6875	1.3545	1.4299	1.5294	1.6875
		4;6	1.4676	1.5263	2.0000	2.0769	1.4259	1.4384	1.5294	1.5882
		6;4	1.7315	1.7990	2.3636	2.4545	1.7230	1.7813	2.0000	2.0769
	36	3;12	1.2140	1.2546	1.6363	1.5454	1.3018	1.3067	1.5000	1.5454
		4;9	1.3939	1.3633	2.0000	1.8889	1.3059	1.3322	1.6363	1.5454
		6;6	1.6299	1.6294	2.5714	2.4285	1.6428	1.6102	2.0000	1.8888
0.1;1.0	12	3;4	1.2845	1.3328	1.8565	1.7762	1.4867	1.4457	1.7236	1.6277
		4;3	1.4394	1.4074	2.1720	2.1061	1.3337	1.5700	1.5618	1.5391
		6;2	1.7298	1.8369	2.7185	2.6632	1.6839	1.7827	2.0488	1.9884
	24	3;8	1.1513	1.2021	1.7254	1.7165	1.2373	1.2578	1.5582	1.5525
		4;6	1.1939	1.2964	2.0277	2.0877	1.2500	1.3881	1.4851	1.4953
		6;4	1.4759	1.5084	2.5579	2.6422	1.4000	1.4747	1.8499	1.9539
	36	3;12	1.1372	1.1600	1.6839	1.7041	1.1713	1.2028	1.5259	1.5746
		4;9	1.2415	1.2193	2.0100	1.9498	1.1776	1.3118	1.4679	1.4987
		6;6	1.3726	1.3780	2.5224	2.5080	1.3154	1.3474	1.8899	1.9079
3.0;0.01	12	3;4	1.3233	1.4114	1.5000	1.5000	1.5054	1.4552	3.0000	1.5000
		4;3	1.5879	1.5859	3.0000	3.0000	1.5686	1.6203	1.5000	3.0000
		6;2	2.0008	2.0005	3.0000	3.0000	2.1724	2.1613	3.0000	3.0000
	24	3;8	1.3113	1.2148	1.0000	3.0000	1.4620	1.3468	3.0000	1.0000
		4;6	1.5165	1.4407	1.0000	3.0000	1.5184	1.4556	3.0000	1.0000
		6;4	1.8421	1.8165	1.0000	3.0000	1.9489	1.8336	3.0000	1.0000
	36	3;12	1.2121	1.3130	1.0000	3.0000	1.3110	1.4044	3.0000	1.0000
		4;9	1.3773	1.4590	1.0000	3.0000	1.3222	1.4109	3.0000	1.0000
		6;6	1.6788	1.7884	1.0000	3.0000	1.7173	1.7532	3.0000	1.0000

using different set sizes from Gompertz distribution different parameters.

2. Calculate MLE and Bayesian estimates given in Sections 2 and 3 using SRS, RSS, and FRSS samples for chosen each sample size of n.
3. Repeat steps 1-2 N times where N=10 000. Then calculate the bias and mean squared error (MSE) of all the estimates.
4. Compute the efficiency of estimators, defined as $RE_{RSS} = \frac{MSE_{SRS}}{MSE_{RSS}}$ and $RE_{FRSS} = \frac{MSE_{SRS}}{MSE_{FRSS}}$.

The results are reported in Tables 2-4. Tables 2-3 contain the biases of the estimators for Gompertz distribution under SRS, RSS, and FRSS for different values of sample sizes and different values of parameters. Table 4 contains the efficiency of estimators for SRS relative to RSS and FRSS.

The following findings are summarized as follows:

- Based on SRS, the biases of ML and Bayes estimators of λ and β are greater than the corresponding in RSS and FRSS.
- For all methods of estimation, it can be seen that bias decreases and RE increases as the sample sizes increase for fixed value of parameters. It verifies the consistency properties of all the estimators.
- For RSS and FRSS, biases decrease as set sizes increase for fixed sample sizes.
- In terms of the bias, the ML and Bayes estimators of λ and β based on FRSS are lower than RSS in almost most cases.
- In terms of the RE, the ML and Bayes estimators of λ parameter are quite competitive, while the ML and Bayes estimators of β based on RSS have higher efficiency values.
- In general, as λ gets smaller, the biases obtained with FRSS for $\hat{\lambda}_{mle}$ get smaller.
- According to all criteria, bias, and RE, it is noted that the ML and Bayes estimators based on the RSS design have close competition.

REAL DATA APPLICATION

In this section, we present a real data analysis to illustrate the usefulness of the RSS and FRSS compared to the traditional SRS scheme. The data set includes the life of fatigue fracture of Kevlar 373/epoxy that is subject to constant pressure at the %90 stress level until all had failed (Andrews and Herzberg [37], Barlow et al. [38]). The data set is given in Table 5.

The Kolmogorov-Smirnov test is used to assess whether this data set is well-modeled by a Gompertz distribution. The ML estimation of the parameters λ, β and the p-value of the Kolmogorov-Smirnov test are 0.1216, 0.4116, and 0.1592, respectively. The Bayes estimation of the parameters λ, β and the p-value of the Kolmogorov-Smirnov test are 0.1216, 0.4115, and 0.1594, respectively. These results show that the Gompertz distribution seems to fit the data very well for both estimators. For the analysis, a sample based on SRS of size 36 is drawn from the data set, and in RSS and FRSS to obtain the same sample size m=6 and r=6

Table 5. The life of fatigue fracture of Kevlar 373/epoxy

0.0251	0.0886	0.0891	0.2501	0.3113	0.3451	0.4763	0.5650	0.5671	0.6566
0.6748	0.6751	0.6753	0.7696	0.8375	0.8391	0.8425	0.8645	0.8851	0.9113
0.9120	0.9836	1.0483	1.0596	1.0773	1.1733	1.2570	1.2766	1.2985	1.3211
1.3503	1.3551	1.4595	1.4880	1.5728	1.5733	1.7083	1.7263	1.7460	1.7630
1.7749	1.8275	1.8375	1.8503	1.8808	1.8878	1.8881	1.9316	1.9558	2.0048
2.0408	2.0903	2.1093	2.1330	2.2100	2.2460	2.2878	2.3203	2.3470	2.3513
2.4951	2.5260	2.9911	3.0256	3.2678	3.4045	3.4846	3.7433	3.7455	3.9143
4.8073	5.4005	5.4435	5.5295	6.5541	9.0960				

Table 6. ML and Bayes estimations, AIC and BIC values for Gompertz distribution

Method	Sampling	Parameter estimates $(\hat{\lambda}, \hat{\beta})$	AIC	BIC
ML	SRS	(0.1751, 0.3883)	128.0447	132.7062
	RSS	(0.1720, 0.3766)	108.2883	112.9498
	FRSS	(0.1741, 0.3802)	119.1949	123.8564
Bayes	SRS	(0.1815, 0.3859)	812.6950	817.8365
	RSS	(0.1801, 0.3766)	807.0914	811.7529
	FRSS	(0.1811, 0.3817)	807.3621	812.0236

are taken. For the purpose of comparison, the values of the MLE and Bayes parameter estimates, the Akaike information criteria (AIC), and Bayesian information criteria (BIC) are reported in Table 6.

We also observe that the data obtained under RSS is the best fitted by the Gompertz distribution for both estimators.

CONCLUSION

In this article, to estimate the unknown parameters of the Gompertz distribution, we have used two different estimation methods, namely, ML and Bayes methods based on the three different sampling schemes (SRS, RSS, and FRSS). Bayes estimates are obtained under the squared loss function. The performances of the estimators were evaluated based on the two criteria bias and RE. Under RSS designs, the biggest challenges in real-time sampling are the computational difficulties in the estimation phase and the complexity of the sample selection process. Of course, this situation is also related to the complexity of the pdf of the distribution and the number of parameters. In this case, the solution is to increase the sample size, change the parameter estimation method, or use different numerical techniques in the simulation study and real time example. The Gompertz distribution considered in this study provided both favourable bias values for small sample sizes and no computational or simulation difficulties in parameter estimation and real-time example processing. We can conclude from the simulation study that, according to the bias, FRSS outperforms both estimators and all sampling schemes in almost all cases. Furthermore, ML and Bayes estimators are competitive for RSS and FRSS in general in terms of bias. In addition, for the β parameter, the efficiency values of ML and Bayes estimators were higher in RSS scheme. This study reveals that the biases obtained with FRSS for λ and β are lower than RSS, while the efficiencies obtained with RSS for λ and β are higher than FRSS. Also, the real data set confirmed the simulation results. We see from the result, that estimates based on FRSS are closer to the given value of both parameters λ and β. This result is consistent with the numerical study in the previous section. This indicates that estimation under the FRSS method is more efficient than estimation under the SRS and RSS approaches.

AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

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