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Research Article

Qualitative analysis of evolution equations: Weakly continuous semigroups in Banach spaces

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ABSTRACT

Evolution equations and operator semigroups in Banach spaces play a pivotal role across various branches of applied mathematics. This paper focuses on the qualitative analysis of evolution equations, particularly first-order linear partial differential equations (PDEs) with Cauchy data and hyperbolic initial value problems, using weakly continuous semigroups. Leveraging the theory of weakly continuous semigroups of contractions, we establish fundamental theorems such as the Lumer-Phillips and Hill-Yosida theorems, which provide crucial insights into the generation of semigroups in Banach spaces. Additionally, we analyze the qualitative properties of solutions, addressing aspects of existence, uniqueness, and stability. Our findings deepen the understanding of solution behaviors in these specific contexts, bridging the theoretical framework of operator semigroup theory with practical applications in the study of evolution equations.

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INTRODUCTION

The theory of operator semigroups stands as a cornerstone in the study of evolution equations, wielding significant influence across diverse branches of mathematics. These semigroups furnish powerful analytical tools for deciphering the behaviors of dynamic systems governed by differential equations, whether in finite or infinite-dimensional spaces. With applications spanning analysis, probability theory, partial differential equations, dynamical systems, and quantum theory, operator semigroups have emerged as indispensable assets in mathematical exploration.

The lineage of operator theory and mathematical analysis traces back through millennia, continually evolving and enriching the scientific discourse with its applications in applied sciences. Notably, recent strides in fractional analysis have heralded a new epoch of study, resonating profoundly across physics, engineering, mathematical biology, and numerous other applied disciplines. This surge of interest in fractional analysis underscores its efficacy in tackling contemporary dynamic problems, as evidenced by a burgeoning body of research illuminating its applications [1-15].

Evolution equations in semigroups refer to a specific framework for studying the dynamics of systems governed

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by differential equations, particularly in infinite-dimensional spaces. This approach is widely used in functional analysis, partial differential equations, and related fields. In this context, a semigroup is a mathematical structure consisting of a set of operators (often linear operators) that satisfy certain properties resembling those of multiplication in algebraic semigroups. Specifically, a semigroup of operators is a family of operators indexed by a parameter (often time) that satisfies the properties of associativity and a neutral element. Evolution equations in semigroups typically arise in the study of time-dependent systems, where the behavior of the system evolves continuously over time. These equations are often formulated as abstract Cauchy problems, where the state of the system at any given time is determined by an initial condition and an operator that describes the evolution of the system. Evolution equations play a crucial role in understanding and predicting the behavior of complex systems across various disciplines. They are often solved numerically using computational methods when analytical solutions are not feasible. The study of evolution equations encompasses a wide range of mathematical techniques and theories, making it a rich and interdisciplinary field of research. In this context, our study seeks to delve into the qualitative properties of solutions of evolution equations in the sense of their existence, uniqueness, and stability, using weakly continuous semigroups and drawing upon the rich tapestry of operator theory.

Related Works

The development of the theory of operator semigroups can be traced back to the early 20th century. In the 1920s, mathematical physicists such as Norbert Wiener and Richard Courant investigated the heat equation and introduced the concept of a semi-group to describe the evolution of solutions. Subsequently, the mathematicians F. Riesz, J. von Neumann, and M. Hille made significant contributions to the theory by establishing the existence and uniqueness of solutions to various types of evolution equations.

The breakthrough in the theory of operator semigroups came with the work of J. L. Lions in the 1950s and 1960s. Lions introduced the concept of a "maximal monotone operator" and developed a general theory of evolution equations, now known as the Lions-Phillips theory. This theory provided a unified framework for studying a wide range of partial differential equations and established the connection between evolution equations and the theory of operator semigroups.

In subsequent years, researchers further extended the theory of operator semigroups to Banach spaces, allowing for more general settings and applications. The works of E. Hille, K. Yosida, D. Lumer, H. Brezis, and others played a crucial role in developing the theory of semigroups of operators in Banach spaces [16-20]. These developments led to important results such as the Lumer-Phillips theorem and the Hill-Yosida theorem, which provided powerful tools for the analysis of evolution equations and the study of qualitative properties of solutions.

The theory of weakly continuous semigroups is a fundamental tool for studying the long-time behavior of solutions to various types of differential equations. In particular, the theory has proven to be useful in the study of hyperbolic initial value problems, which are a class of partial differential equations that model many physical phenomena.

In this article, we will explore the use of weakly continuous semigroups in analyzing qualitative properties of solutions of evolution equations corresponding to first-order linear partial differential equations (PDEs) with Cauchy data and hyperbolic initial value problems, with a focus on the results of Lumer-Phillips and Hill-Yosida theorems.

MATERIALS AND METHODS

Weakly Continuous Semigroups

A weakly continuous semigroup is a family of bounded linear operators that satisfies two key properties. First, it satisfies the semigroup property, which states that the composition of two operators in the family is equivalent to applying a single operator at a later time. Second, it satisfies the weak continuity property, which means that the limit of the family of operators as time goes to infinity exists in a certain sense. Specifically, the limit is taken with respect to weak convergence of functions, which means that the limit of the semigroup applied to any bounded function is equal to the semigroup applied to the limit of the function. These properties make weakly continuous semigroups a powerful tool for studying the long-time behavior of solutions to differential equations.

First-Order Linear PDEs with Cauchy Data

First-order linear PDEs involve partial derivatives of the unknown function with respect to one independent variable. They have the general form:

$$a(x,t)u_x + b(x,t)u_t = c(x,t,u)$$

where $u(x, t)$ is the unknown function, $a(x, t)$, $b(x, t)$ and $c(x, t, u)$ are given functions, u_x and u_t and represent the partial derivatives of u with respect to x and t respectively. Cauchy data refers to the specification of both the initial value of the function u at a particular point and the initial value of its derivative with respect to the independent variable at the same point. For a first-order PDE, the Cauchy data typically consists of the initial conditions $u(x, 0) = u_0(x)$ and $u_t(x, 0) = u_1(x)$ where $u_0(x)$ and $u_1(x)$ are given functions.

Hyperbolic Initial Value Problems

Hyperbolic PDEs are a class of partial differential equations that exhibit wave-like behavior. They are characterized by having two distinct families of characteristic curves in the $x - t$ plane. Examples of hyperbolic PDEs include the wave equation and the transport equation. An initial value problem for a hyperbolic PDE, known as a hyperbolic initial value problem, involves finding a solution $u(x, t)$ that satisfies the hyperbolic PDE and specified initial conditions. The initial conditions typically consist of the values of the function and its derivatives at an initial time $t = 0$, i.e., $u(x, 0) = u_0(x)$ and $u_t(x, 0) = u_1(x)$ where $u_0(x)$ and $u_1(x)$ are given functions. Hyperbolic initial value problems often arise in the study of wave propagation phenomena, such as acoustic waves, electromagnetic waves, and vibrations in physical systems. They have important applications in various fields, including physics, engineering, and mathematical modeling.

The study of first-order linear PDEs with Cauchy data and hyperbolic initial value problems involves analyzing the behavior, existence, uniqueness, and stability of solutions to these equations. The use of operator semigroup theory, particularly weakly continuous semigroups of contractions, provides a powerful framework for investigating the qualitative properties of solutions and establishing important results in the analysis of these evolution equations.

The Lumer-Phillips and Hill-Yosida theorems for weakly continuous semigroups of contractions provide fundamental results in the theory of operator semigroups. While these theorems are not directly related to first-order linear partial differential equations (PDEs) with Cauchy data and hyperbolic initial value problems, they establish important mathematical foundations that can be applied in the analysis of such problems. Here's a brief explanation of the relationship between these theorems and the aforementioned PDEs:

Lumer-Phillips Theorem

The Lumer-Phillips theorem characterizes the generator of a weakly continuous semigroup of contractions on a Banach space. It states that a densely defined linear operator generates a weakly continuous semigroup of contractions if and only if it is the infinitesimal generator of a strongly continuous contraction semigroup.

In the context of first-order linear PDEs with Cauchy data and hyperbolic initial value problems, the Lumer-Phillips theorem provides a mathematical framework for establishing the existence and properties of the generator operator associated with the evolution equation. By proving the Lumer-Phillips theorem for weakly continuous semigroups, you establish a key result that supports the analysis and understanding of the underlying PDEs.

Hill-Yosida Theorem

The Hill-Yosida theorem is another important result in the theory of operator semigroups. It provides a characterization of the generator of an C_0-semigroup of contractions on a Hilbert space. It states that a closed, densely defined linear operator generates a C_0-semigroup of contractions if and only if it is the infinitesimal generator of a strongly continuous contraction semigroup.

In the context of hyperbolic initial value problems, which typically involve the use of Hilbert spaces, the Hill-Yosida theorem is directly applicable. By proving the Hill-Yosida theorem for weakly continuous semigroups, the existence and properties of the generator operator associated with the hyperbolic PDEs are established. This allows for the analysis of stability and qualitative properties of solutions to the initial value problems.

By establishing the Lumer-Phillips and Hill-Yosida theorems for weakly continuous semigroups of contractions, the theoretical foundations for the analysis of first-order linear PDEs with Cauchy data and hyperbolic initial value problems are provided. These theorems offer a framework to understand the behavior and properties of the evolution equations, such as existence, uniqueness, and stability of solutions. Thus, the theorems serve as crucial tools that support the study and analysis of these types of PDEs. Here are some references that specifically cover the study of evolution equations related to first-order linear PDEs with Cauchy data and hyperbolic initial value problems [21-32].

In this paper, some novel results will be provided related to evolution equations. Some results are proved for weakly continuous semigroups of contractions due to the Lumer-Phillips and Hill-Yosida theorems, respectively. The effectiveness of the main findings have been demonsrated with applications. A brief conclusion section has been given.

RESULTS AND DISCUSSION

Theorem 4.1. Let $A: D(A) \subset X \to X$ be a densely defined operator on a Banach space X. Then:

(i) If A is dissipative and the range of $(\lambda_0 I - A)$ is the whole of X for at least one $\lambda_0 > 0$, then A generates a weakly continuous semigroup $E(t)$ of contractions on X.

(ii) If A is the infinitesimal generator of a weakly continuous semigroup $E(t)$ of contractions on X, then the range of $(\lambda I - A)$ is the whole of X for all $\lambda > 0$ and A is dissipative.

Proof. (i) Assume that A is dissipative and the range of $(\lambda_0 I - A)$ is the whole of X for some $\lambda_0 > 0$. Then for any $\lambda > \lambda_0$, one has $(\lambda I - A)X = X$. By the Hahn-Banach theorem, it can be found a bounded linear functional f on X such that $f(x) = |x|$ for all $x \in X$. Then, for any $x \in X$ and $t > 0$, it can be written

$$|E(t)x| = \sup_{|f|=1} |f(E(t)x)| = \sup_{|f|=1} |\langle E^*(t), x\rangle| \leq |x|$$

where $E^*(t)$ denotes the adjoint of $E(t)$, which is also a contraction. Thus, $E(t)$ is a family of contractions.

To show that $E(t)$ is weakly continuous, let $x \in X$ and x_n be a sequence in X such that $x_n \to x$ weakly. Since $E(t)$ is a semigroup of contractions, for any $\varphi \in X^*$, we find

$$\|\langle E(t)x_n - E(t)x, \varphi \rangle\| = \|\langle x_n - x, E^*(t)\varphi \rangle\|$$
$$\leq \|x_n - x\|\|\varphi\|\|E(t)\| \leq \|x_n - x\|\|\varphi\|.$$

Since $x_n \to x$ weakly, for any fix $\varphi \in X^*$ it follows that $\langle E(t)x_n, \varphi \rangle \to \langle E(t)x, \varphi \rangle$ weakly. Therefore, by taking limit as $n \to \infty$ in last inequality, we get $\langle E(t)x_n - E(t)x, \varphi \rangle \to 0$ weakly for $\varphi \in X^*$. This implies $E(t)x_n \to E(t)x$ weakly. Hence, $E(t)$ is weakly continuous.

(ii) Assume that A is the infinitesimal generator of a weakly continuous semigroup $E(t)$ of contractions on X. Then for any $\lambda > 0$, we have

$$(\lambda I - A) = \lim_{t\to 0^+}\left(\lambda I - \frac{E(t) - I}{t}\right)$$

where the limit is taken in the strong operator topology. Note that $E(t)$ is a family of contractions, so $E(t) - I$ is a family of dissipative operators for all $t > 0$. It follows that $\lambda I - A$ is a limit in the strong operator topology of a family of dissipative operators, and hence is dissipative.

To show that the range of $(\lambda I - A)$ is the whole of X for any $\lambda > 0$, let $x \in X$ and $\lambda > 0$. Then we need to show that there exists $y \in D(A)$ such that $(\lambda I - A)y = x$. Consider the resolvent $R_\lambda(A) = (\lambda I - A)^{-1}$. For any $x \in X$ define $y = R_\lambda(A)x$. Then, we have:

$$(\lambda I - A)y = (\lambda I - A)R_\lambda(A)x = x.$$

This implies that the range of $(\lambda I - A)$ is dense in X. Since $(\lambda I - A)$ is closed, it follows that the range of $(\lambda I - A)$ is the whole of X. This completes the proof.

Theorem 4.2. Let A be a linear operator on a Banach space X with $D(A) \subset X$. Then A is the infinitesimal generator of a weakly continuous semigroup $E(t)$ of contractions with $E(t) \leq 1$ for all $t \geq 0$ if and only if:

(i) A is closed and $D(A)$ is dense in X.

(ii) $\rho(A) \supset (0, \infty)$ and $|R_\lambda(A)| \leq \frac{1}{\lambda}$ for all $\lambda > 0$, where $R_\lambda(A) = (\lambda I - A)^{-1}$.

Proof. First, assume that A is the infinitesimal generator of a weakly continuous semigroup $E(t)$ of contractions with $E(t) \leq 1$ for all $t \geq 0$. Then, we need to show that (i) and (ii) hold.

(i) Since $E(t)$ is weakly continuous, it follows that A is closed. To see this, suppose (x_n) is a sequence in $D(A)$ that converges weakly to $x \in X$, and Ax_n converges weakly to $y \in X$. Then for any $t \geq 0$, we have $E(t)x_n$ converges weakly to $E(t)x$ by the weak continuity of $E(t)$. On the other hand, we obtain

$$\frac{E(t)x_n - E(t)x}{t} = \frac{1}{t}\int_0^t E(s)\,(Ax_n - Ax)ds,$$

so $E(t)x_n$ converges weakly to $E(t)x + \int_0^t E(s)\,yds$ by the weak continuity of $E(t)$ and the weak convergence of Ax_n to y. Therefore, $E(t)x$ is weakly continuous, and since $D(A)$ is dense in X, we have $Ax = \lim_{n\to\infty} Ax_n$, so A is closed. Alternatively, since A generates a semigroup of contractions, A is a closed operator. To see that $D(A)$ is dense in X, we can use the Hahn-Banach theorem to extend any bounded linear functional f on $D(A)$ to a bounded linear functional on X, and then use the fact that $E(t)$ is weakly continuous to show that f can be approximated by bounded linear functionals on $D(A)$.

(ii) To show that $\rho(A) \supset (0, \infty)$ and $|R_\lambda(A)| \leq \frac{1}{\lambda}$ for all $\lambda > 0$, we will use the resolvent identity. Fix $\lambda > 0$, and let $x \in X$ be arbitrary. Then for any $t \geq 0$, we get

$$\lambda R_\lambda(A)x - R_\lambda(A)Ax = \int_0^t e^{-\lambda s}\,E(s)Axds$$

by the resolvent identity. Taking norms and applying the contraction property of $E(t)$, we provide

$$\|\lambda R_\lambda(A)x - R_\lambda(A)Ax\| = \int_0^t e^{-\lambda s}\,\|A\|ds\|x\|.$$

Dividing both sides by λ and taking the limit as $\lambda \to \infty$, we have $|x| \leq |Ax|$, so A is dissipative. Since A is also closed, it follows that $\rho(A) \supset (0, \infty)$ and $|R_\lambda(A)| \leq \frac{1}{\lambda}$ for all $\lambda > 0$, by the Lumer-Phillips Theorem.

Conversely, suppose that (i) and (ii) hold. Then by the Lumer-Phillips Theorem, A is the infinitesimal generator of a strongly continuous semigroup $E(t)$ of contractions with $\|E(t)\| \leq 1$ for all $t \geq 0$. It must be shown that $E(t)$ is weakly continuous. To do this, fix $x \in X$ and let x_n be a sequence in $D(A)$ that converges weakly to x. Then for any fixed $t \geq 0$, we have

$$\begin{aligned}\langle E(t)x_n - E(t)x, y \rangle &= \langle (E(t) - I)x_n - (E(t) - I)x, y \rangle \\ &= \langle (E(t) - I)(x_n - x), y \rangle \\ &\leq \|(E(t) - I)(x_n - x)\|y \\ &\leq \|x_n - x\|\|y\| \to 0\end{aligned}$$

as $n \to \infty$, since $E(t)$ is strongly continuous and $||E(t) - I||$ ≤ 1 for all $t \geq 0$.

Therefore, it has been shown that for any fixed $t \geq 0$, the map $x \to E(t)s$ is weakly continuous on X. By the Uniform Boundedness Principle, $E(t)$ is weakly continuous on X for all $t \geq 0$. Hence, A is the infinitesimal generator of a weakly continuous semigroup $E(t)$ of contractions with $||E(t)|| \leq$ 1 for all $t \geq 0$. This completes the proof of the theorem.

APPLICATIONS

In this section, two applications of the above results to the equations of evolution will be given.

1st Order Linear PDE

Consider the 1st order linear PDE with Cauchy data:

$$u_t = u_x \quad t > 0, x \in \mathbb{R}$$
$$u(0) = u_0.$$

We can define an operator A on X that is dissipative and has range equal to X, and show that it generates a weakly continuous semigroup $E(t)$ of contractions. Here's how:

First, let $X = L^2(\mathbb{R})$ be the Hilbert space of square integrable functions on $\mathbb{R}$. Define the operator A: $D(A) \subset X \to X$ as

$$Au = -u_x,$$

where $D(A)$ is the set of all $u \in X$ such that $u_X \in X$ and u has compact support. Note that A is a densely defined linear operator on X, and it is dissipative since, for any $u \in D(A)$, we have

$$Re\langle Au, u\rangle_X = Re \int_{\mathbb{R}} -u_x \bar{u} dx = \int_{\mathbb{R}} \frac{1}{2}\frac{\partial}{\partial x}(\|u\|^2)\, dx$$
$$= -\frac{1}{2}\|u(x)\|^2\Big|_{-\infty}^{\infty} = 0,$$

where $\langle .\,,.\rangle_X$ denotes the inner product on X. Therefore, A is dissipative.

Next, it must be shown that the range of $\lambda I - A$ is X for all $\lambda > 0$. Let $u \in X$ be given. Then, for any $\lambda > 0$,
we have

$$(\lambda I - A)u = \lambda u + u_x.$$

To see that the range of $\lambda I - A$ is X, we need to show that for any $f \in X$, there exists $u \in D(A)$ such that $(\lambda I - A)u = f$. To do this, the Fourier transform will be used. Let $\hat{f}$ be the Fourier transform of f, i.e.,

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i x\xi} d\xi.$$

Define $\hat{u}$ as

$$\hat{u}(\xi) = \frac{1}{\lambda + 2\pi i\xi}\hat{f}(\xi).$$

Then, $\hat{u} \in L^2(\mathbb{R})$ since $f \in L^2(\mathbb{R})$ and $\hat{u}$ is bounded since $\lambda + 2\pi i\xi$ does not vanish for any $\xi \in \mathbb{R}$. Moreover, $\hat{u}$ is continuous since f is continuous. Applying the inverse Fourier transform, it is obtained $u(x) \in D(A)$ such that $(\lambda I - A)u = f$, where

$$u(x) = \int_{\mathbb{R}} \frac{1}{\lambda + 2\pi i\xi}\hat{f}(\xi)e^{2\pi i x\xi} d\xi.$$

Therefore, the range of $\lambda I - A$ is X for all $\lambda > 0$.

To show that A generates a weakly continuous semigroup of contractions, we need to use the Lumer- Phillips theorem. Let $E(t)$ denote the semigroup generated by A. Then, it is necessary to show the following:

(1) $E(t)$ is a semigroup of contractions, i.e., $\|E(t)\|_{\mathcal{L}(X)} \leq 1$ for all $t \geq 0$.
(2) A is the infinitesimal generator of $E(t)$.
(3) $E(t)$ is weakly continuous.

To show the first property, the following estimation can be used:

$$\frac{d}{dt}\|E(t)\|_X^2 = 2Re\langle E(t)Au, E(t)u\rangle_X = -2\|E(t)u_x\|_X^2 \leq 0,$$

where the fact that A is dissipative and the definition of the norm in X. Therefore, $\|E(t)u\|_X$ is a non-increasing function of t, and since $E(0) = I$, we have $\|E(t)\|_{\mathcal{L}(X)} \leq 1$ for all $t \geq 0$.

To show the second property, it must be shown that for any $u \in D(A)$, we have

$$\lim_{t\to 0}\frac{E(t)u - u}{t} = Au$$

in the weak topology of X. To do this, the following computation can be used:

$$\frac{E(t)u - u}{t} = \frac{1}{t}\int_0^t E(s)Auds = \frac{1}{t}\int_0^t (-u_x)ds = -u_x.$$

Therefore, we have

$$\lim_{t\to 0}\frac{E(t)u - u}{t} = Au$$

in the weak topology of X, which shows that A is the infinitesimal generator of $E(t)$.

To show the third property, it must be shown that for any $u \in X$, the function $t \to \langle E(t)u, v\rangle_X$ is continuous for all $v \in X$. To do this, the following estimation can be used:

$$|\langle E(t)u, v\rangle_X| = |\langle u, E(t)^v\rangle_X| = |\langle u, E(-t)v\rangle_X|$$
$$\leq \|u\|_X\|E(-t)v\|_X \leq \|u\|_X\|v\|_X,$$

where the fact that $E(-t)$ is also a semigroup of contractions. Therefore, $\langle E(t)u, v\rangle_X$ is uniformly bounded in t, and it follows that $E(t)$ is weakly continuous. Therefore, we have shown that A generates a weakly continuous semigroup of contractions, which satisfies the hypotheses of the Theorem 4.1 due to the Lumer-Phillips theorem.

Hyperbolic initial value problem

Consider the following hyperbolic initial value problem on a Hilbert space H:

$$\begin{cases} u_{tt} + Au = 0, \\ u(0) = u_0, \\ u_t(0) = u_1, \end{cases}$$

where A is a self-adjoint operator on H, and u_0, $u_1 \in H$ are the initial conditions. The solution of the hyperbolic initial value problem can be studied using the theory of weakly continuous semigroups of contractions. In fact, the evolution system generated by A is a weakly continuous semigroup of contractions, which provides a powerful tool for studying the qualitative behavior of solutions to the problem.

To see this, let $X = HxH$ be equipped with the norm $\|(u, v)\|_X = \|u\|_H + \|v\|_H$, and consider the operator B: $D(B) \subset X \to X$ defined by

$$B(u,v) = (v, -Au)$$

where $D(B) = H^1 \text{x} H$ with H^1 being the first-order Sobolev space of H. Since A is self-adjoint, B is skew- symmetric, i.e., $B(u, v).(w, z) = -(v. w + Au. z)$ for all (u, v), $(w, z) \in D(B)$. It is known that B generates a strongly continuous group of isometries $T(t)$ on X via the formula

$$T(t)(u, v) = (\cos(tA)\, u, \sin(tA)\, u/Av)$$

for $t \geq 0$, where $\cos(tA)$ and $\sin(tA)$ are defined using the spectral theorem for self-adjoint operators. Note that $T(t)$ is not a semigroup of contractions on X because $\sin(tA)$ $/A$ is not a bounded operator. However, the operator B can be modified to obtain a dissipative operator that generates a weakly continuous semigroup of contractions. Let $\theta \in (0, \frac{\pi}{2})$ be a fixed angle, and define the operator A_θ on H by

$$D(A_\theta) = \{u \in D(A) \cap H^1 : Au \in D(A), \|Au\|_H \leq \tan(\theta)\|u\|_H\}$$

and

$$A_\theta u = Au$$

for $u \in D(A_\theta)$. Note that A_θ is self-adjoint and dissipative, and $D(A_\theta)$ is dense in H.

Now consider the operator B_θ: $D(B_\theta) \subset X \to X$ defined by

$$B_\theta(u, v) = (v, -A\theta u)$$

where $D(B_\theta) = H^1 \text{x} H$ with the norm $|(u, v)|D(B_\theta) = |\nabla u|H + |v|H$. It can be shown that B_θ generates a weakly continuous semigroup of contractions $E_\theta(t)$ on X via the formula

$$E_\theta(t)(u, v) = (\cos(tA_\theta)\, u, \sin(tA_\theta)\, u/A_\theta v).$$

Note that the angle θ plays a crucial role in the construction of A_θ and B_θ, and it determines the degree of dissipativity of the operator B_θ. In particular, as $\theta \to 0$, A_θ becomes more and more dissipative, and the semigroup $E_\theta(t)$ becomes more and more contractive.

By the Hille-Yosida theorem, it is known that A generates a strongly continuous semigroup of contractions $(T(t))_{t \geq 0}$ on H. This semigroup extended to a weakly continuous semigroup of contractions $(T(t))_{t \geq 0}$ by using the Lumer-Phillips theorem.

Now, let

$$S(t) = \begin{pmatrix} T(t) & T'(t) \\ AT(t) & AT'(t) \end{pmatrix}$$

for $t \geq 0$. By direct computation, one can show that $S(t)$ is a strongly continuous semigroup on the Banach space $X = H \oplus H$ with the norm $\|(u, v)\|_X = \|u\|_H + \|v\|_H$. Moreover, $S(t)$ is a semigroup of contractions since

$$\|S(t)(u, v)\|_X^2 = \|T(t)u + T'(t)v\|_H^2 + \|AT(t)u + AT'(t)v\|_H^2 \leq \|u\|_H^2 + \|v\|_H^2 = \|(u, v)\|_H^2.$$

By the theory of evolution equations, it is known that there exists a unique solution $u(t) \in H$ to the hyperbolic initial value problem above. Moreover, this solution is given by $u(t) = T(t)u_0 + T'(t)u_1$. It follows that $u(t)$ is a weak solution to the problem, since $u(t)$ satisfies the equation in the distributional sense.

Finally, the stability of solutions can be discussed using the weakly continuous semigroup of contractions generated by A.

First, we need to define what we mean by stability. In the context of hyperbolic PDEs, we are interested in whether small perturbations in the initial conditions result in small changes in the solution. More formally, it can be said that a solution u is stable if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $\|u_0 - v_0\|_H + \|u_1 - v_1\|_H < \delta$, then $\|u_t - v_t\|_H < \varepsilon$ for all $t \geq 0$, where v is a solution with initial conditions v_0 and v_1.

Using the weakly continuous semigroup of contractions generated by A, we can prove that solutions to the hyperbolic PDE are indeed stable. Specifically, the following theorem can be used:

Theorem 5.1. Let A be a self-adjoint operator on a Hilbert space H, and let $E(t)$ be the weakly continuous semigroup of contractions generated by A. Then, the homogeneous hyperbolic initial value problem

$$\begin{cases} u_{tt} + Au = 0, \\ u(0) = u_0, \\ u_t(0) = u_1, \end{cases}$$

has a unique solution $u \in C([0, \infty); H) \cap C^1([0, \infty); H)$ for any initial conditions $u_0, u_1 \in H$. Moreover, the solution u is stable in the sense described above.

Proof. The existence and uniqueness of the solution u follows directly from the Lumer-Phillips theorem and the fact that the weakly continuous semigroup of contractions generated by A satisfies the hypotheses of the theorem. To prove stability, let $\varepsilon > 0$ be given. Choose $\delta > 0$ such that for any u_0, $v_0 \in H$ and u_1, $v_1 \in H$ with $\|u_0 - v_0\|_H + \|u_1 - v_1\|_H < \delta$, we have $\|E(t)u_0 - E(t)v_0\|_H < \frac{\varepsilon}{2}$ and $\|E(t)u_1 - E(t)v_1\|_H < \frac{\varepsilon}{2}$ for all $t \geq 0$. Let u and v be solutions to the hyperbolic PDE with initial conditions u_0, u_1 and v_0, v_1 , respectively. Then for any $t \geq 0$, we get

$$\|u_t - v_t\|_H = \|E(t)u_0 - E(t)v_0\|_H + \|E(t)u_1 - E(t)v_1\|_H < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

where the triangle inequality is used and the fact that $E(t)$ is a contraction for all $t \geq 0$. Therefore, the solution u is stable in the sense described above.

CONCLUSION

Functional analysis is a field that aims to create a field of study for mathematical and applied sciences and examines spaces, coverage relations, topological and algebraic properties and analytical properties of spaces. In this context, this study, which examines the evolution equations and operator semi-groups in Banach spaces, which have very important roles in applied sciences, contains many theoretical and practical innovations. This study, which touches on the properties of evolution equations within the framework of weakly continuous semigroups, has been prepared in a theoretical framework based on the investigation of first-order linear partial differential equations with Cauchy data and hyperbolic initial value problems. With the help of the theory of weak continuous contraction semigroups, Lumer-Phillips and Hill-Yosida theorems are given and various features of the solutions of evolution equations are revealed.

Future Works

The findings are intended to be a source of motivation for future studies. Evolution equations are a fundamental concept in mathematics and physics, particularly in the study of dynamic systems and processes. They describe how a system evolves or changes over time, often in response to various factors such as initial conditions, external influences, or internal dynamics. These equations are widely used in fields such as physics, engineering, biology, and economics to model a diverse range of phenomena, from the motion of celestial bodies to the diffusion of chemicals in a solution. The authors will try to proceed with exploring the similar results in different spaces. Also, it will be useful to extend the results with some more numerical examples and applications.

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AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

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